Problem Set 5

Due Date: Thurs, Feb 27

Reading

Please study Smullyan, Chapter XI, p. 101-108, and skim Chapter IV, p. 43-51 for Thurs, Feb 27.

Problems

1. Give a top-down Gentzen proof of formulas (2), (4), (6), and (8) on page 24.

2. Recall the lecture presentation of Smullyan’s definition of a tree. A tree is a 4-tuple \( < s, a, p, f > \) where \( S \) is a set of nodes, \( a \in S \), \( p \) maps \( \{ x : \) \( S | x \neq a \} \) into \( S \); it computes the predecessor of a node, the function \( f \) maps \( S \) to \( \mathbb{N}^+ = \{1, 2, 3, \ldots \} \). The two axioms are:
   - Ax 1. For all \( x \) in \( S \), \( f(x) = 1 \) iff \( x = a \).
   - Ax 2. For all \( x \) in \( S \), \( f(x) = f(p(x)) + 1 \).

   Define \( L(i) = \{ x : S | f(x) = i \} \).

   Prove carefully that \( L(i + 1) = \{ x : S | p(x) \in L(i) \} \) and describe the result graphically.

3. Recall that Refinement Logic is a single conclusion (top down) Gentzen system in which the rule \( \frac{H \vdash P \land Q}{H \vdash P, Q} \) is replaced by \( \frac{H \vdash P \lor Q}{H \vdash P} \) or \( \frac{H \vdash P \lor Q}{H \vdash Q} \) and the rule \( \frac{H, X \vdash \sim P}{H, X \vdash \sim \sim P} \) for any formula \( x \).

   Prove the following formulas in Refinement Logic:
   (a) \( (P \supset Q) \supset P \)
   (b) \( (P \supset Q) \supset \sim Q \supset \sim P \)
   (c) \( \sim Q \supset (P \supset Q) \)
   (d) \( \sim (P \lor Q) \supset \sim P \lor \sim \sim Q \)

4. Write down the rules for a Gentzen system based on the Sheffer stroke and one based on joint denial (see p. 14 of Smullyan and p.30).

5. Produce Tableau rules and Refinement rules for a logic with the constants \( t, f \) (Smullyan, p. 13), but without \( \sim \). Define \( \sim P \) as \( P \supset f \) and show how to replace any deduction using the \( \sim \) rules by one using \( P \supset f \) instead.