Nonuniform degree models

Let $\lambda_i$ be the fraction of vertices of degree $i$.
A giant component exists, if $\sum_{i=1}^{n} i(i-2)\lambda_i > 0$.
The probability of getting a vertex is proportional to the degree of the vertex.
The net gain of the frontier is the degree of the vertex minus two.
Check that this is true for $G(n,p)$:

We want to choose $\lambda_i$ such that the above sum (giant component) goes from negative to positive.
The probability of a vertex of degree $i$ is $\binom{n}{i} p^i (1-p)^{n-i}$.

\[
\lim_{n \to \infty} \sum i(i-2) \left(\binom{n}{i} p^i (1-p)^{n-i}\right)
\]
substituted distribution for $\lambda$\[
= \lim_{n \to \infty} \sum i(i-2) \frac{n(n-1) \cdots (n-i+1)}{i! n^i} (1 - \frac{1}{n})^{n-1}
\]
\[
\leq \lim_{n \to \infty} \sum i(i-2) \frac{1}{i!} e \left(1 - \frac{1}{n}\right)^{-i}
\]
use $e$ as an upper bound\[
\leq \sum_{i=0}^{\infty} i(i-2) \frac{1}{i!}
\]
Note that
\[
\sum_{i=0}^{\infty} \frac{i}{i!} = 1 + \sum_{i=1}^{\infty} \frac{i}{i!} = \sum_{i=1}^{\infty} \frac{1}{(i-1)!} = \sum_{i=0}^{\infty} \frac{1}{i!}
\]
similarly,
\[
\sum_{i=0}^{\infty} \frac{i^2}{i!} = 2 \sum_{i=0}^{\infty} \frac{1}{i!}
\]
and
\[
\sum i(i-1)\frac{1}{i!} = \sum \frac{i^2}{i!} - 2 \sum \frac{i}{i!} = 2 \sum_{i=0}^{\infty} \frac{1}{i!} - 2 \sum_{i=0}^{\infty} \frac{1}{i!} = 0
\]

Grow a graph

**model 1:** Start with a single vertex. At each unit of time, add a new vertex plus $\delta$ edges.

The resulting degree distribution is:
\[
\lambda_i = \frac{(2\delta)^i}{(1+2\delta)^i}
\]
For example, pick $\delta = \frac{1}{4}$:
\[
\lambda_i = \frac{2^i}{3^i}
\]

**model 2:** Consider a random graph with above degree distribution. This is subtly different from the growing graph. The existence of high-degree vertices in the random graph helps connect things (there have to be low-degree ones to compensate to keep the distribution the same). For example, look at the graph of airport connections. There exist a lot of airports of low degree (such as the Ithaca airport). However, a few airports of very high degree make the air-port connection graph well-connected.
Giant component phase transition

$G(n,p)$: phase transition expected degree is 1.
model 2: expected degree $= \frac{1}{2}$
for a grown graph (e.g. model 1): even earlier.

model 2: random graph power law degree distribution

$$\sum_{i=1}^{\infty} i(i-2)\lambda_i = \frac{2}{3} \sum_{i=1}^{\infty} i(i-2)(\frac{1}{3})^i$$

recall
$$\sum_{i=1}^{\infty} ia^i = \frac{a}{(1-a)^2}$$
$$\sum_{i=1}^{\infty} i^2 a^i = \frac{a(1+\frac{a}{(1-a)^2})}{(1-a)^4}$$
$$\sum_{i=1}^{\infty} i(i-2)a^i = \frac{a(3a-1)}{(1-a)^3}$$
this equals 0 when $a = \frac{1}{3}$

Branching process

Is a given tree going to be finite or infinite?
The size of the component is for a graph that is

**finite**: proportional to $\log n$

**infinite**: proportional to $n$

Assume the probability of $i$ children is the same at every node:

$$P_0, P_1, P_2, P_3, \ldots$$

$P_i$ is the probability of $i$ children. Generating function

$$f(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots$$

Let $f_j(x)$ be the generating function for the number of children at the $j^{th}$ level.

$$f_1(x) = f(x)$$
\[
f_2(x) = f_1(f(x)) \\
\ldots \\
f_{j+1}(x) = f_j(f(x))
\]

Note that \(f_j(f(x)) = f(f_j(x))\).

Now, to find the expected number of children:

\[
f'(x) = p_1 + 2p_2 x + 3p_3 x^2 + \cdots \\
f'(1) = p_1 + 2p_2 + 3p_3 + \cdots
\]

**Generating functions**

(more on this in a later lecture...)

1. A generating function for the sum of two identically distributed random variables \(x_1\) and \(x_2\) is the square of their generating functions.

\[
f(x) = p_0 + p_1 x + p_2 x^2 + \cdots \\
\]

squaring, \(p_0 + p_1 x + p_2 x^2 + \cdots\) yields \(p_0^2 + (p_0 p_1 + p_1 p_0) x + (p_0 p_2 + p_1 p_1 + p_2 p_0) x^2 + \cdots\)

Thus,

\[
f^2(x) = p_0^2 + (p_0 p_1 + p_1 p_0) x + \cdots
\]

2. The coefficient of \(x^i\) in \(f_j(x)\) is the probability of \(i\) children in the \(j^{th}\) generation and these children contribute \(f^i(x)\) to \(j+1^{st}\) generation.

\[
f_{j+1}(x) = f_j(f(x)) = f(f_j(x))
\]