More on Second Moment Method

Let $x$ be a non-negative random variable. Then:

$$P(x = 0) \leq \frac{\sigma^2}{E^2(X)} = \frac{E(X^2)}{E^2(X)} - 1$$

The emergence of cycles in a graph $G(n, p)$:

- Occurs when $p = \Theta(1/n)$ e.g. $p = 1/1000n$
- Doesn’t occur if $\lim_{n \to \infty} \frac{p(n)}{1/n} = 0$

Let $x$ be the number of cycles in graph $G$:

$$E(x) = \sum_{k=3}^{n} \binom{n}{k} \frac{(k-1)!}{2} p^k$$

$$E(x) \leq \sum_{k=3}^{n} \frac{n(n-1)...(n-k)(k-1)!}{k!} p^k$$

$$E(x) \leq \sum_{k=3}^{n} \frac{n^k}{2k} p^k \leq \sum_{k=3}^{n} (np)^k$$

**What if $p$ is asymptotically less than $1/n$? (i.e. $\lim_{n \to \infty} np = 0$)**

Consider: $\sum_{k=0}^{n} a^k = 1 + a + a^2 + ... = \frac{1}{1-a}$, for all $a < 1$

$$\Rightarrow E(x) \leq \sum_{k=3}^{n} (np)^k = 0$$

because $k$ starts at 3 which means it doesn’t include the first term of the series 1.

Therefore almost surely a graph selected at random has no cycle of $p$ is asymptotically less than $1/n$. 
What if np = constant, c?

\[ E(x) = \sum_{k=3}^{\infty} \binom{n}{k} (k-1)! \frac{p^k}{2} = \frac{1}{2} \sum_{k=3}^{\infty} \frac{n(n-1)\ldots(n-k)}{kn^k} (np)^k \]

- If c < 1, it converges.
- If c >= 1, it diverges, but why?

Let us add up the first log n terms:

\[ E(x) \geq \frac{1}{2} \sum_{k=3}^{\log n} \binom{n}{k} (k-1)! \frac{p^k}{n^k} (np)^k \geq \frac{(n-\log n)}{n} \log n \rightarrow \log n \text{ as } n \rightarrow \infty \]

**Other Structures**

\[ N = \{1, 2, \ldots, n\} \]

Flip a coin which has head with probability p and put integer in the set the head occurs. \( N_p = \{1, 2, 5, 9, 13\} \)

Does \( N_p \) contain an arithmetic progression of length \( k \)?

\[ a, a+b, a+2b, a+3b, \ldots, a+(k-1)b \]

Yes, arithmetic progression of length \( k \) abruptly appears when \( p \) reaches \( n^{\frac{2}{k}} \)

Why?
There are \( n^k \) potential numbers of arithmetic progression
Let \( X_k \) be the expected number of arithmetic progression, then

\[ E(X_k) = n^2 p^k \]

If \( p \ll n^{\frac{2}{k}} \)

\[ E(X_k) \ll n^2 \times n^{-2} \ll 1 \]

\[ \therefore \lim_{n \rightarrow \infty} E(X_k) = 0 \]

If \( p \gg n^{\frac{2}{k}} \)

\[ \lim_{n \rightarrow \infty} E(X_k) = \infty \]
Aside: Covariance

Cov(x, y) = E((x - E(x))(y - E(y)))

Var(x + y) = E[((x + y) - E(x + y))^2]
= Var(x) + Var(y) + 2Cov(x, y)

If x and y > 0, then
Cov(x, y) < E(xy)

We now want to establish that
\[ \lim_{n \to \infty} E(X_k) = 0, \quad \text{for } p \gg n \frac{2}{k} \]

Let \( I_i \) be the indicator variable for the \( i^{th} \) arithmetic progression, then

\[ X_k = I_1 + I_2 + \ldots \]
\[ Var(X_k) = \sum_i \sum_j Cov(I_i, I_j) \]