Structure of Graphs of $G(n,p)$ as $p$ increases

$p(n) = 0 \cdots \frac{1}{n^2} \cdots \frac{1}{n^3} \cdots \frac{1}{n \log n} \cdots \frac{1}{n} \cdots \frac{1}{2} \frac{3}{4} 1$

1) Forest of trees (no cycles) while $p = o\left(\frac{1}{n}\right)$ [i.e. $\frac{p(n)}{\frac{1}{n}} \rightarrow 0$ as $n \rightarrow \infty$]

2) Cycles will appear as soon as $p = \Theta\left(\frac{1}{n}\right)$ [i.e. $\frac{p(n)}{\frac{1}{n}} \rightarrow \text{const}$]
   - Each component is a tree or unicyclic
   - No component exists of size larger than $\log n$

3) At $p = \frac{1}{n}$, an abrupt phase transition occurs in which a giant component appears. This phase transition occurs as a double jump whereby:
   - At $p < \frac{1}{n}$: largest component has size $\approx \log n$
   - At $p = \frac{1}{n}$: largest component has size $\approx n^{2/3}$
   - At $p > \frac{1}{n}$: largest component has size $\approx \text{const} \cdot n$

4) As $p$ increases, larger components get swallowed up into the giant component. At $p = \frac{1}{n} \frac{\log n}{n}$: There is only the giant component plus isolated vertices.

5) At $p = \frac{\log n}{n}$: Connected graph (no isolated vertices left)

6) At $p = \text{const}$: graph has diameter 2.

More on phase transition at $p = \frac{1}{n}$:

- This is a "sharp" transition, implying the existence of a "threshold" at $p = \frac{1}{n}$.
- We say that $p(n)$ is a threshold for a property if
  - For all $p_1(n)$ such that $\frac{p_1(n)}{p(n)} \rightarrow 0$, almost no graph has the property, and
  - For all $p_2(n)$ such that $\frac{p_2(n)}{p(n)} \rightarrow 0$, almost every graph has the property.
Thresholds occur for all properties of $G(n,p)$ that we list.
Thresholds occur for other structures besides $G(n,p)$.

Review from Preu Lecture: Graph has diameter at most 2 when $p = \text{const}$.

- Define an unordered pair of vertices $(u,v)$ to be "bad" if no vertex $w$ exists that is adjacent to both $u$ and $v$.
- Let $\lambda$ = # of unordered pairs in graph that are bad.
- Let $\lambda_i$ = indicator variable for ith pair ($1 \leq i \leq \lambda$);
  $\lambda_i = 1$ if bad pair, $\lambda_i = 0$ if good pair.
- Then $\lambda = \sum \lambda_i$ and expected value $E(\lambda) = \sum E(\lambda_i)$
  But all $E(\lambda_i)$ are the same, so $E(\lambda) = \binom{n}{2} E(\lambda_1)$
- For a given $u$ and $v$, what is the probability that there does not exist
  a vertex $w$ adjacent to both $u$ and $v$?
  - Prob. that a given vertex is not adjacent to both is $(1 - p^2)$
  - There are $(n-2)$ possible vertices, so the probability that no $w$
    exists is $(1 - p^2)^{n-2}$
- So $E(\lambda) = \binom{n}{2} (1 - p^2)^{n-2} = \frac{n(n-1)}{2} c^{n-2}$ where $c$ is a constant less than 1.
  As $n \to \infty$, $E(\lambda) \to 0$. Therefore, for a random graph, the expected number
  of bad pairs is zero. Thus, the graph has diameter at most 2.
- What if $E(\lambda)$ had been 1? We cannot conclude that all (or many) graphs do
  have bad pairs, because they need not be uniformly distributed.
- This proof relies on $p = \text{const}$ because $(1 - p^2)^{n-2} \to 0$ only if $p$ is a constant.
  For example, $(1 - \frac{d}{n})^n \to e^{-d} \neq 0.$
Disappearance of isolated vertices.

Let $x$ be the number of isolated vertices.

\[ E(x) = n(1-p)^{(n-1)} \]

\[
(1-p)^n = e^{\ln(1-p)^n} \\
= e^{(n \ln(1-p))} \\
[ \ln(1-p) = -p - \frac{p^2}{2} - \frac{p^3}{3} - \ldots ] \\
= e^{-n(p + \frac{p^2}{2} + \frac{p^3}{3} + \ldots)} \\
= e^{-np} * e^{-n(\frac{p^2}{2} + \frac{p^3}{3} + \ldots)} \\
= e^{-np} * e^{-n(\frac{p^2}{2} + \frac{p}{3} + \frac{p^2}{4} + \ldots)} \\
[ p = \frac{c \ln n}{n} ] \\
= e^{-c \ln n} * e^{-n(\frac{c^2}{2} + \frac{c}{3n} + \ldots)} \\
\]

\[ \lim_{n \to \infty} e^{-c \ln n} \]

\[ E(x) = n(n^{-c}) = n^{1-c} \]

$c < 1, \ E(x) \to \infty$  
$c > 1, \ E(x) \to 0$

*note: $p = \frac{c \ln n}{n}$

\[
(1 - \frac{c \ln n}{n})^n = e^{(c \ln n)} = n^{-c} ?? \\
(1 - \frac{d}{n})^n = e^{-d} \\
We \ have \ only \ proved \ this \ for \ constant \ d \ so \ this \ is \ not \ correct.