CS485
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\[ P_0, P_1, P_2, \ldots \]
\[ f(x) = \sum_{i=0}^{\infty} P_i x^i \]

iterated:
\[ f_1(x) = f(x) \]
\[ f_{j+1}(x) = f_j(f(x)) \]

\[ P_0 + P_1 + P_2 + \cdots = 1 \]

\[ f_1(x) \]
\[ x \]
\[ P_0 = 1 \]

Root of equation \( f(x) = x \).
\( x = 1 \) is always a root.

\[ f(x) = P_0 + P_1 x + P_2 x^2 + \cdots \]
\[ f(1) = P_0 + P_1 + P_2 + \cdots = 1 \]

\[ m = f'(1) = P_0 + P_1 + 2P_2 + 3P_3 + \cdots \]

\[ \int_{-\infty}^{\infty} e^{ax^2} \, dx = \sqrt{\frac{\pi}{a}} \]

\[ q^x q^x q^x \]

\[ q < x < 1 \]
\[ x > f_1(x) > f_2(x) > \cdots \]
\[ 0 < x < q \]
\[ x < f_1(x) < f_2(x) < \cdots < q \]
\[ x \in (0, 1) \]
\[ \lim_{x \to 0} f_j(x) = q \]
if \( m < 1 \) then process dies out with probability 1
if \( m > 1 \) then process dies out with probability \( q \), where \( q \) is
unique root of \( f(x) = x \) in interval \((0, 1)\)

Expected Size of Component is finite

Example: Let \( x \) be a random variable
Let \( p_i \) be probability that \( |X| = i \)
\[
  p_i = \frac{6}{\pi} \cdot \frac{1}{i^2} \cdot \sum_{i=0}^{\infty} p_i = 1
\]
\[
  E(|X|) = \sum_{i=0}^{\infty} i p_i = \frac{6}{\pi} \sum_{i=0}^{\infty} \frac{1}{i} = \infty \quad \text{(doesn't exist)}
\]

Lemma: If the slope \( m = f'(1) \neq 1 \), then the expected size of an extinct
family is finite.

What happens if \( m = 1 \), \( p_i < 1 \), I don't know the answer.

Let \( Z_i \) be the random variable denoting the size of the \( i \)th generation.
The expected value of \( Z_i \) over extinct families, it is likely to be small
than expected value of \( Z_i \) over all cases.

\[
  \text{Prob}[Z_i = k \text{ and extinction}] = \text{Prob}[Z_i = k / \text{extinction}] \cdot \text{Prob}[\text{extinction}]
\]
\[
  = \text{Prob}[\text{extinction} / Z_i = k] \cdot \text{Prob}[Z_i = k]
\]

Bayes Rule:
\[
  \text{Prob}[Z_i = k / \text{extinction}] = \frac{\text{Prob}[\text{extinction} / Z_i = k] \cdot \text{Prob}[Z_i = k]}{\text{Prob}[\text{extinction}]}
\]
\[
  = \frac{q^k p_b}{q} = q^{k-1} p_b.
\]
Expected size of $z$, given extinction

$E(z/\text{extinction}) = \sum_{b=0}^{\infty} k b^b p_k = f'(q)$

$E(z_i/\text{extinction}) = [f'(q)]^i$

$E(\text{tree/\text{extinction}}) = \sum_{i=0}^{\infty} [f'(q)]^i = \frac{1}{1-f'(q)}$

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**Diagram:**

- $F_r = F_{r+1} = F_r - 1 + x_1$
- $P_0 + P_1 = P_0 + P_0 P_1 + P_0 P_3 + P_0 P_3 P_3$

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**Expected value of sum of $n$ random variables with identical distribution**

$E(x_1 + x_2 + \ldots + x_n) = E(x_1) + E(x_2) + \ldots + E(x_n)$

Did not use independence

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What if the number $n$ of random variables is itself a random variable

$E(\frac{1}{n} x_1) = E(n)(x_1)$ provided you have independence