Generating Function

Branching process:

Let $P_i$ be the probability of $i$ children.

Let $g(x) = \sum P_i X^i$ be the corresponding generating function.

Define $j$th iteration of $g(x)$

$$g_1(x) = g(x)$$
$$g_2(x) = g(g(x))$$
$$\ldots$$
$$g_j(x) = g_{j-1}(g(x))$$

Two observations:

$g^2(x)$ is the generating function for the sum of two independent random
variables \(x_1+x_2\) where \(x_1\) and \(x_2\) have probability distribution \(P_i\).

\[
g^2(x) = P_0^2 + (P_0P_1 + P_1P_0)x + (P_1P_2 + P_1P_1 + P_2P_0)x^2 + \ldots
\]

For \(x_1+x_2\) to have value 0 both \(x_1\) and \(x_2\) must have value zero.

For \(x_1+x_2\) to have value 1 exactly one of \(x_1, x_2\) must have value 1 and then other have value 0.

In generating \(g'(x)\) is the generating function for \(x_1+x_2+\ldots+x_r\).

\(g_j(x)\) is generating function for number of children in jth generation of branching process.
Proof

By induction on j,

By Induction Hypothesis,

\[ g_{j+1}(x) = b_0 + b_1x + b_2x^2 + \ldots + b_ix^i + \ldots \]

where coefficient of \( x^i \) is probability of \( i \) children in \( j-1 \) level.

If \( i \) children in \( j-1 \) generation, these will contribute in total \( g_i(x) \)

\[ g_i(x) = b_0 + b_1g(x) + b_2g^2(x) + \ldots + b_ig^i(x) = g_{j-1}(g(x)) \]

Generating function for sequence defined by recurrence relationship.
e.g. Fibonacci sequence.

\[ F_0 = 1, \quad F_1 = 1, \quad F_i = F_{i-1} + F_{i-2} \quad (i \geq 2) \]

**How do we get generating function for Fibonacci sequence?**

\[ f_i x^i = f_{i-1} x^i + f_{i-2} x^i \quad (\text{\textbullet} x_i \text{ on both sides}) \]

\[ \sum_{i=2}^{\infty} f_i x^i = \sum_{i=2}^{\infty} f_{i-1} x^i + \sum_{i=2}^{\infty} f_{i-2} x^i \]

Let \( f(x) = \sum_{i=0}^{\infty} f_i x^i \)

\[ f(x) - f_1 x = x f(x) + x^2 f(x) \]

\[ f(x) - x f(x) - x^2 f(x) = x \quad (\text{by rearranging}) \]

\[ f(x) = x / (1 - x - x^2) \]

**Asymptotic Behavior**
\[ f(x) = \left(\frac{\sqrt{5}}{5}\right) / (1 - \varnothing_1 x) + \left(\frac{\sqrt{5}}{5}\right) / (1 - \varnothing_2 x) \]

where \( \varnothing_1 = \frac{(1+\sqrt{5})}{2}, \varnothing_2 = \frac{(1-\sqrt{5})}{2} \)

\[ f(x) = \left(\frac{\sqrt{5}}{5}\right) \left[ 1 + \varnothing_1 x + (\varnothing_1 x)^2 + \ldots - (1 + \varnothing_2 x + (\varnothing_2 x)^2 + \ldots) \right] \]

\[ f_n = \left(\frac{\sqrt{5}}{5}\right) (\varnothing_1^n - \varnothing_2^n) \quad |\varnothing_2| < 1 \]

\[ f_n = \lfloor \left(\frac{\sqrt{5}}{5}\right) \rfloor \varnothing_1^n \]

\[ \lfloor f_n \rfloor + \left(\frac{\sqrt{5}}{5}\right) \varnothing_2^n = \left(\frac{\sqrt{5}}{5}\right) \varnothing_1^n \]

Where \( \lfloor \cdot \rfloor \) sign is round down sign.

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**Mean**
Let $z$ be an integer valued random variable. Let $p_i$ be probability that $z=i$

$$E(z) = \sum_{i=0}^{\infty} i p_i$$

Let $p(x) = \sum_{i=0}^{\infty} p_i x^i$

$$p'(x) = \sum_{i=0}^{\infty} i p_i x^{i-1}$$
$$xp'(x) = \sum_{i=0}^{\infty} i p_i x^i$$

$$p'(1) = \sum_{i=0}^{\infty} i p_i \quad \leftarrow \text{mean}$$

Exponential Generating function

$$a_0 a_1 a_2 \ldots \leftrightarrow g(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$$