1 Generating Functions

Suppose we are given a sequence of numbers $a_0, a_1, a_2, ...$ We associate a single entity, a generating function $g(x)$, to represent this infinite sequence:

$$a_0, a_1, a_2, ... \iff g(x) = \sum_{i=0}^{\infty} a_i x^i$$

For instance, the generating function for $a_0 = a_1 = ... = 1$ is

$$1, 1, 1, ... \iff \sum_{i=1}^{\infty} x^i = \frac{1}{1-x}$$

Suppose we differentiate the generating function:

$$\frac{d}{dx} g(x) = \sum_{i=0}^{\infty} i a_i x^{i-1}$$

Multiply both sides by $x$:

$$\frac{d}{dx} g(x) = \sum_{i=0}^{\infty} i a_i x^i \iff 0, a_1, 2a_2, 3a_3, 4a_4, ...$$

Let $g(x) = (1-x)^{-1}$ as above. Then since $\frac{d}{dx} (1-x)^{-1} = (1-x)^{-2}$ we know

$$x \frac{d}{dx} (1-x)^{-1} = x(1-x)^{-2} = \frac{x}{(1-x)^2} \iff \text{Generating function for } 0, 1, 2, 3, 4, ...$$

2 Powers

Let $g(x) = \sum_{i=0}^{\infty} g_i x^i$ where $g_i$ are the probabilities that an integer valued random variable has value $i$. 
Suppose we have two independent random variables $x_1$, $x_2$ that we want to add together such that

$$\text{Prob}(x_1 + x_2) = i$$

$$\implies \text{Prob}(x_1 = 0) \text{Prob}(x_2 = i) + \text{Prob}(x_1 = 1) \text{Prob}(x_2 = i - 1) + ... = g_0g_i + g_1g_{i-1} + g_2g_{i-2} + ... + g_ig_0$$

You might recognize the above sum as something called a convolution.

Now we try to find $g^2(x)$:

$$g^2(x) = \left(\sum_{i=0}^{\infty} g_i x^i\right)\left(\sum_{j=0}^{\infty} g_j x^j\right) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} g_i g_j x^{i+j}$$

Let $k = i + j$. Then

$$= \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} g_{k-j} g_j \right] x^k$$

Note in brackets the coefficients of $x^k$ are grouped as powers of x.

$$\implies \sum_{j=0}^{k} g_{k-j} g_j \iff g_0g_0, \underbrace{g_1g_0 + g_0g_1, g_2g_0 + g_1g_1 + g_0g_2, ...}_{k=2}$$

3 Examples

Example 1:

Suppose we are given three object types A, B and C:

A can be selected 0 or 1 times
B can be selected 0, 1, or 2 times
C can be selected 0, 1, 2, or 3 times

How many ways can you select five objects?

We calculate the generating functions for A, B and C as:

A $1 + x$
B $1 + x + x^2$
To find the total number of ways to select A, B and C, we multiply their respective generating functions together:

$$(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)$$

$$= 1 + 2x + \overbrace{5x^2}^{\text{check}} + 6x^3 + 5x^4 + \overbrace{3x^5}^{\text{check}} + x^6$$

We check the term $5x^2$. There are five ways to pick two objects as follows:

$CC, BB, CB, CA, BA$

We also check the term $3x^5$. There are three ways to pick five objects as follows:

$CCCBB, CCCBA, CCBBA$

**Example 2:**

Suppose we try to make change with pennies, nickels and dimes.

<table>
<thead>
<tr>
<th>Coins</th>
<th>Generating Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pennies</td>
<td>$1 + x + x^2 + x^3 + x^4 + ...$</td>
</tr>
<tr>
<td>Nickel</td>
<td>$1 + x^5 + x^{10} + x^{15} + ...$</td>
</tr>
<tr>
<td>Dimes</td>
<td>$1 + x^{10} + x^{20} + ...$</td>
</tr>
</tbody>
</table>

To find the change possibilities, we multiply the coins’ generating functions together:

$$(1 + x + x^2 + ...)(1 + x^5 + x^{10} + ...)(1 + x^{10} + x^{20} + ...)$$

$$= 1 + x^2 + x^3 + x^4 + 2x^5 + ... + \overbrace{9x^{23}}^{\text{check}} + ...$$

We want to check the term $9x^{23}$. There are nine ways to make change for 23 cents as follows:

<table>
<thead>
<tr>
<th>Coins</th>
<th>Change Possibilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pennies</td>
<td>$3 3 8 13 3 8 13 18 23$</td>
</tr>
<tr>
<td>Nickel</td>
<td>$0 2 1 0 4 3 2 1 0$</td>
</tr>
<tr>
<td>Dimes</td>
<td>$2 1 1 1 0 0 0 0 0$</td>
</tr>
</tbody>
</table>
4 Branching Trees

Why are branching trees important?

*Hereditary* - Given a family tree, will that family’s bloodline die out?

*Epidemics* - We can model how a disease spreads using a branching tree. Will a given epidemic spread throughout the world or die out (relatively) harmlessly?

*Search in a random graph* - When performing a breadth-first search from a node $n$, what are the expected number of children reachable from $n$? Will that search eventually die out? What is the expected graph component size?

Let $p(x)$ be the probability that $x$ takes on a given value. Let $f(x)$ be the generating function for probability of $i$ children.

Then in the $j^{th}$ generation, the number of children is found to be:

$$f_j(x) = \begin{cases} f(x) & j = 1 \\ f_{j-1}(f(x)) & j > 1 \end{cases}$$

Show by induction:

Coefficient of $x^r$ in $f_{j-1}(x)$ is the probability of $r$ children in $(j-1)^{st}$ gen-
Each of these $r$ children contribute $f(x)$ to $j^{th}$ generation.

For a sum of $r$ sets of descendants, the generating function is $f^r(x)$. Why? Any sum of $r$ random variables has a generating function $f^r(x)$ because if $x \iff g(x)$ then $x + \ldots + x \iff g^r(x)$ by convolution argument.

We know the total contribution of $r$ children is $f^r(x)$.

Substitute in $f_{j-1} = a_0 + a_1 x + a_2 x + \ldots$

Substitute in $f^r(x)$ for $x^r$:

$$= f_{j-1}(f(x))$$

Assuming coefficient of $x^r$ in $f_{j-1}$ give correct probability for $r$ children in $(j - 1)^{th}$ generation, then substituting $g(x)$ for $x$ gives generation for $j^{th}$ generation.