Please hand in each problem on separate sheets with your name and netID on each. If a problem requires multiple sheets, please staple the sheets for that problem together.

Reading: Chapters 7.8 - 7.12.

Question 1

You’ve been hired as a consultant to decide whether a newly proposed start-up might actually be able to turn a profit. This ambitious company, which is calling itself Zero Gravity Research (nasdaq: ZGR), plans to build a space station on which experiments can be performed in a zero gravity (or at least a very low gravity) environment. The space station will contain bunch of research equipment and be operated by a team of intrepid scientists. Other companies and labs will then pay ZGR to do particular zero-g experiments for them.

The company founders give the following motivation: There are presumably a good number of companies and labs out there that are interested in low gravity experiments, but most of them can’t justify the immense cost of building a space station to do so. By building a single station that can be used by lots of companies, ZGR hopes to make it affordable to do this sort of research, and make a profit as well. Of course, most experiments require equipment, and so in addition to building the space station, ZGR needs to buy whatever equipment is needed for the jobs it gets paid to do. Equipment is reusable, but they still need to decide what is worth buying and what isn’t.

The company has begun asking around, and has found $k$ companies, each with a single zero-gravity experiment it would like to pay for. Company $i$ is willing to pay $p_i$ for their experiment, which requires a set of research devices $S_i \subseteq \{1, 2, \ldots, m\}$. So in total there are $m$ devices (a mass spectrometer, an x-ray, a gene sequencer, a robotic penguin, etc) that ZGR might want to buy to run these experiments. Device $j$ has a cost $c_j$. The founders of ZGR have already paid for the station and the scientists, and now want to know what equipment should they buy and which research projects should they carry out so as to maximize their profit.

Give a polynomial time algorithm to solve this problem. In particular, your algorithm should return a set of projects $P$ and a set of devices $D$, such that $D$ contains all devices needed for each research project in $P$, and subject to this constraint, the profit $\sum_{i \in P} p_i - \sum_{j \in D} c_j$ is maximized.

Question 2

Consider the task of tiling a chessboard ($8 \times 8$ squares) with dominoes which are the size of two squares. That is, placing dominoes on the board such that every domino covers exactly two squares of the chessboard and every square on the chessboard is covered by exactly one domino. Dominoes may be placed horizontally ($1 \times 2$) or vertically ($2 \times 1$). This is clearly not hard to do, and in fact, it is pretty clear that there are many many ways to do this.
Now suppose you remove two opposite corners from the chessboard. Can this new area still be tiled with dominoes? It turns out that the answer is “no.” After messing around with dominoes for a few minutes you may begin to suspect that it can’t be done. But the easy way to see that it is impossible is to notice that the two opposite corner squares of a chessboard are always of the same color. If we consider the original black and white coloring on the chessboard, then the new area has 32 squares of one color and 30 squares of the other color. But every domino always covers exactly 1 white square and 1 black square. So any tiling will always leave at least 2 squares uncovered.

(a) One might ask whether this sort of problem is the only sort of problem preventing you from tiling a region. More precisely, is it the case that for any \(n \times n\) chess board where \(n\) is even, removing an equal number of white squares and black squares leaves a region that can be tiled with dominoes? Give a proof or a counterexample.

(b) Give an efficient algorithm that, given a region (a subset of squares from an \(n \times n\) chessboard), decides whether that region can be tiled with dominoes.

(c) Suppose the answer for a particular input is “no.” Show how your algorithm can be used to provide a simple proof of this fact. This proof should be a simple one-line statement that you could give to your grandparents and have them understand why tiling would be impossible. In other words, your proof shouldn’t require any knowledge of flows.

Question 3

Upon graduating, you decide to start your own hip and trendy women’s magazine, Maximelle. You plan on having regular sections dedicated to topics such as fitness, fashion, relationships, monster trucks, etc. Each section will have some minimum number of articles. Unfortunately, at the moment, you’re totally broke. You decide that initially, you’ll need your college buddies to write the articles. Most of your friends are pretty excited to be a part of this new project, but most of them have other jobs, so they just don’t have time to write too many articles for you.

In addition, you’ve got some constraints of your own. For starters, no one should ever write more than one article per section. Also, you’ve got your own opinion about how much each of your friends actually knows about each of the topics; you don’t really want Jasmine, who routinely wears purple shoes, green pants, and an orange polka dot top, to be writing your fashion articles. For each of your friends and each topic, you rate that friend’s knowledge on that topic as either expert, decent, or totally clueless. (Presumably you don’t actually tell Samantha that you think she’s clueless when it comes to relationships, but that’s up to you). You don’t want anyone writing an article on a topic that they are clueless about, and furthermore, you’d like at least two articles per section to be written by experts.

More formally, you have a list of \(k\) topics you’d like covered, and for topic \(i\) you’d like to have at least \(a_i\) articles. You have \(m\) friends, and each friend \(j\) is willing to write up to \(b_j\) articles. Finally, for each friend \(j\) and topic \(i\), you have a rating \(r_{ij} \in \{\text{expert, decent, clueless}\}\). You want to decide whether you’ll be able to assign articles for your friends to write such that no friend writes two articles for the same section or more articles than they volunteered for, and each section has enough articles, no clueless authors, and at least two expert authors.

Give a polynomial time algorithm that determine if it is possible to assign articles to friends in such a way that all requirements are satisfied. If it is possible, your algorithm should explain how.