Vectors

- A vector represents 2 things: length and direction.
- The same vector (A direction if length = 0).
- Special vector 0 has zero length, no direction.
- A vector represents an offset from one location in $\mathbb{R}^n$ to another.
- A vector only represents a position if you first choose an origin.

Vector space operations

- Addition $w = u + v$ — "how far have I gone if I first walk $u$, then $v$?"
  - Commutative $(u + v = v + u)$ — from parallelogram
  - Negation $u + (-u) = 0$
  - Reflection $u - v = u + (-v)$ changes length preserves direction
- Scaling (scalar multiplication) $\alpha u$
  - Distributive $\alpha(u + v) = \alpha u + \alpha v$

Coordinates & bases

- Scalar — always in $\mathbb{R}^n$ for us.
- Def: 2 vectors $u, v$ are linearly dependent if $\alpha u + \beta v = 0$ for some $\alpha, \beta$.
- While diagram magnified by $\gamma$ — not going to get back to zero.
  - Can get back to zero — $0 = u$
For 2 vecs, linearly dependent $\iff$ parallel.

Def: 3 vecs $\mathbf{y}, \mathbf{z}, \mathbf{w}$ are linearly dependent if $a \mathbf{x} + b \mathbf{y} = 0$ for some $\neq 0$.

Any 3 vecs in the plane are dependent.

For 3 vecs linearly dependent $\iff$ coplanar.

Useful property: For 2 lin. indep. vecs $\mathbf{x}$ in $\mathbb{R}^2$, you can write any vector as

A linear combination of $\mathbf{x}$:

$\mathbf{w} = a \mathbf{x} + b \mathbf{x}$

(Remember this is the same stuff as just above.)

Same property holds for 3 vecs in $\mathbb{R}^3$.

Def: span of vectors $\mathbf{x}$, $\mathbf{y}$ is $\text{span}\{\mathbf{x}, \mathbf{y}\} = \{a \mathbf{x} + b \mathbf{y} | a, b \in \mathbb{R}\}$.

That is, all the vectors you can get to using $\mathbf{x}$ and $\mathbf{y}$.

Property above is, for lin indep $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, span $\{\mathbf{x}, \mathbf{y}\} = \mathbb{R}^2$

Trivial generalization to $\mathbb{R}^3$.

Q: How do we represent vectors in code?

Can’t really store a vector in memory – only numbers.

So pick some basis vecs and store the coefficients.

I’ll call these vectors the “canonical basis” and name them $e_1, e_2, \ldots$ (and $e_3$).

We do not store $e_1, e_2, e_3$ – we just agree upon them and store the coefficients.

Note I drew those vectors perpendicular ->

Def: orthogonal $\iff$ perpendicular

Ortho-ortho = orthogonal and length = 1

The canonical basis will always be orthonormal.

This is a cartesian coordinate system.

One nice property: can use Pythagorean $\sqrt{\sum m^2}$ to compute lengths:

$\|\mathbf{x}, \mathbf{y}\| = \sqrt{x^2 + y^2}$

$\|\mathbf{x}, \mathbf{y}, \mathbf{z}\| = \sqrt{x^2 + y^2 + z^2}$
Coordinates & bases cont.

\[ (x, y, z) = v \]

Use Pythagorean theorem:

\[ ||v||^2 = x^2 + y^2 \]

\[ ||v|| = \sqrt{||v||^2} = \sqrt{x^2 + y^2} \]

Notation:

\[ (x, y) = \begin{bmatrix} x \\ y \end{bmatrix} = [x \ y]^T \]

often \( u = (u_x, u_y) = u_x e_x + u_y e_y \)

Vector Products

Dot product of two vectors: (also inner prod. or scalar prod.)

\[ a = u \cdot v = <u, v> \]

\[ \uparrow \uparrow \uparrow \]

\[ \text{scalar vectors} \]

has some properties:

\[ u \cdot v = v \cdot u \]  
\[ \text{commutative} \]

\[ u \cdot (v + w) = u \cdot v + u \cdot w \]  
\[ \text{distributes over addition} \]

\[ (aq) \cdot v = a \cdot (qv) = q(a \cdot v) \]  
\[ \text{scalars factor out} \]

Interpretations:

\[ u \cdot v = ||u|| \cdot ||v|| \cdot \cos \theta \]

Special case:

\[ u \cdot u = ||u||^2 \]

Special case if \( u \perp v \):

\[ u \cdot v = 0 \quad (\cos \theta = 0) \]

\[ u \cdot v = ||u|| (u \rightarrow v) = ||v|| (v \rightarrow u) \]

Special case if \( ||u|| = 1 \):

\[ u \cdot a = u \rightarrow a \]

\[ a \rightarrow b \text{ is a projected onto } b \]
Vector products cont.

So far dot prod. is in terms of abstract vectors (no coordinates).

How to compute a dot product? We store $u_1, u_2, v_1, v_2$

$$u \cdot v = (u_1, e_1 + u_2, e_2) \cdot (v_1, e_1 + v_2, e_2)$$
$$= u_1 v_1 e_1^2 + u_1 v_2 e_1^2 + u_2 v_1 e_1^2 + u_2 v_2 e_1^2$$
$$= u_1 v_1 + u_2 v_2$$

In $\mathbb{R}^3$, $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$.

Cross product of two vectors: (only for 3D!)

$$w = u \times v$$

has some properties:

- $u \times v = -v \times u$ \text{(anti commutative)}
- $u \times (v + w) = u \times v + u \times w$ \text{(distributes over +)}
- $(\alpha v) \times u = \alpha (v \times u)$ \text{(scalar factor out)}
- $u \cdot (u \times v) = (v \times u) \cdot u = 0$ \text{(orthogonal to both)}

interpretations:

$$\|u \times v\| = \|u\| \|v\| \sin \theta$$

$\|u \times v\|$ is the area of the parallelogram spanned by $u, v$.

$u \times v$ is a kind of oriented area.

special case: $u \times u = 0$

arbitrary choice (really a convention about ordering of coordinates):

- $e_1 \times e_2 = e_3 \quad ; \quad e_2 \times e_3 = e_1 \quad ; \quad e_3 \times e_1 = e_2$ \text{ - forward pairs}
- $e_2 \times e_1 = -e_3 \quad , \quad \text{etc.}$ \text{ - backward pairs}

arbitrary geometric convention - right hand rule (for lefthanders):

- $e_1$ - thumb, $e_2$ - forefinger, $e_3$ - palmward.
Vector products cont.

So far $X$ in terms of abstract vectors — what about coordinates?

$$u \times v = (u_1 e_1 + u_2 e_2 + u_3 e_3) \times (v_1 e_1 + v_2 e_2 + v_3 e_3)$$

$$= u_1 v_2 e_1 \times e_2 + u_1 v_3 e_1 \times e_3 + u_2 v_1 e_2 \times e_3 + u_2 v_3 e_2 \times e_1 + u_3 v_1 e_3 \times e_1 + u_3 v_2 e_3 \times e_2$$

$$= (u_1 v_3 - u_3 v_1) e_1 + (u_2 v_3 - u_3 v_2) e_2 + (u_2 v_1 - u_1 v_2) e_3$$

Note: forward pairs are $\Theta$, backward pairs are $\Theta$, and each coordinate depends on the other two coordinates.

To me this is the best mnemonic aid — straight from the definition.

ONB's and Coordinate Frames

An orthonormal basis $\mathbb{B}$ is a set of $n$ mutually orthogonal unit-length vectors ($n=2$ for plane $\mathbb{R}^2$, for $n$-space).

(Usual notation will be $e_1, e_2, e_3$.)

Convention: right-handed $uvw = w$ (same as $e_1, e_2, e_3$).

Dual property: coordinates of $x \times u$ are $u \times v, v \times w, w \times x$:

$$x \times (x_1 u + x_2 v + x_3 w) = x_1 (u \times v) + x_2 (v \times w) + x_3 (w \times u).$$

This, in a nutshell, is why we care about ONBs: it is very easy to transform to and from canonical coordinates.

If we have the coordinates $(x_1, x_2, x_3)$ of $x$ in the ONB $(u, v, w)$ and we have the coordinates $(u_1, u_2, u_3), (v_1, v_2, v_3),$ and $(w_1, w_2, w_3)$ of $u, v, w$ in the canonical basis, then

$$x_1 = x u_1 + x v_1 + x w_1$$

$$x_2 = x u_2 + x v_2 + x w_2$$

$$x_3 = x u_3 + x v_3 + x w_3$$

just the defn $x = x_1 u + x_2 v + x_3 w$ expanded.

(This is the same for a non-orthonormal basis.)
If we have the coordinates \((x, y, z)\) of \(x\) in the canonical basis then
\[
\begin{align*}
x_u &= x \cdot u = x_1 u_1 + x_2 u_2 + x_3 u_3 \\
x_v &= x \cdot v = x_1 v_1 + x_2 v_2 + x_3 v_3 \\
x_w &= x \cdot w = x_1 w_1 + x_2 w_2 + x_3 w_3
\end{align*}
\]

Note this (and the above) is a matrix multiplication — more on this in a couple of lectures...

(This depends on orthonormality.)

This ONB discussion has just been about vectors that represent offsets, not ones that represent points (as we've been using the same origin).

Generally we want the freedom to change the origin away so we associate an origin with the ONB to make a coordinate frame \((x, y, z, w)\).

There are chains to make some computations more convenient. E.g. flight sim: (x-axis)

\[
\begin{align*}
x &= p + u x_u + v x_v + w x_w \\
x_u &= (x-p) \cdot u \\
x_v &= (x-p) \cdot v \\
x_w &= (x-p) \cdot w
\end{align*}
\]

Useful example: constructing ONB from one vector \(u\) (e.g. if we need to compute something that's symmetric about that vector)

We'll align \(w\) with \(u\), but it's supposed to be a unit vector so \(w = u / \|u\|\)

Now if we can get our hands on a vector perpendicular to \(w\), the cross product will finish the job. The cross product solves this too: if \(v\) is not parallel to \(w\), then we can use
\[
\begin{align*}
v &= \frac{v \times w}{\|v \times w\|} \quad \text{that is, compute cross product and then normalize}
\end{align*}
\]

\(v \perp w\). To complete the right-handed ONB:
\[
\begin{align*}
y &= w \times u, \quad \text{note forward pair (convention for RH)}
\end{align*}
\]

Now a new pair is related closely to this construction.