Problem 1: Convolution filters

Here is a definition for the discrete convolution of two sequences of numbers, $f$ and $g$:

$$(f \star g)[j] = \sum_i f[i]g[j - i]$$

That is, we can compute the value of the convolution at any index $j$ by summing products of the elements of $f$ and $g$ with an offset that depends on $j$. In the following let $b$ be the sequence that has $b[0] = 1$ and $b[i] = 0$ for all other $i$.

This sum runs over all $i$, so there are no boundary conditions to worry about. You can think of this as saying that out-of-bounds accesses to the arrays $f$ and $g$ will return zero.

1. Verify that discrete convolution has the following properties:

$$f \star b = f = b \star f$$

This is $b$ is an identity

$$(\alpha f) \star g = \alpha (f \star g)$$

Scalars factor out

$$f \star (g + h) = f \star g + f \star h$$

Distributes over $+$

$$f \star g = g \star f$$

Commutative

$$(f \star g) \star h = f \star (g \star h)$$

Associative

Hints: The general approach is to expand out the definitions on both sides and then manipulate one side to show it’s equal to the other. Also, remember that you can use a change of variable in a sum: if $p : \mathbb{Z} \rightarrow \mathbb{Z}$ is a function that renumbers the terms without losing any or duplicating any (that is, it’s a bijection), then

$$\sum_i X(i) = \sum_i X(p(i))$$

where $X$ can be any expression that depends on $i$.

2. Use these properties to derive a radius 5 (11 by 11) discrete convolution filter that is equivalent to the “unsharp mask” procedure:
CS 465 Homework 2a

- Begin with the image $I_{in}$.
- Blur $I$ with a radius 5 gaussian filter of width $\sigma = 2$ pixels, storing the result in $I_{blur}$.
- Set the final image $I_{out}$ to $(1 + \beta)I_{in} - \beta I_{blur}$.

List the values of the filter itself (round to 2 significant figures), but take advantage of separability and symmetry to avoid having to write down 121 numbers (fewer than 10 should suffice). Explain where you used the properties from part 1 in your derivation.

For reference, the 2D Gaussian filter with width $\sigma$ is defined as $h(s, t) = h(s)h(t)$ where $h(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/(2\sigma^2)}$.

**Problem 2: Reconstruction and resampling**

The equation that defines reconstruction of a continuous function $g$ from a discrete sequence of samples $f$ is

$$g(x) = \sum_i f[i]h(x - i).$$

This can be translated into pseudocode as:

```pseudocode
function g(f, h, x) {
    // Evaluates the reconstruction g of the samples f using the
    // filter h at the point x.
    result = 0;
    for i = 0 to N - 1
        result += f[i] * h(x - i);
    return result;
}
```

where $N$ is the number of elements in the array $f$.

1. This code is inefficient because it loops over the whole array. If we know that the filter radius is $r$ (that is, $h(t) = 0$ for $|t| > r$) What should the loop bounds be to make the minimum number of computations while still computing the correct result? Assume you have the functions round, floor, and ceil available.\(^1\)

2. A reconstruction filter is **interpolating** if $g(x) = f(x)$ for the original sample points (that is, when $x$ is an integer). It is **ripple-free** if $g$ is a constant function whenever $f$ is a constant sequence (when $f[i]$ has the same value for all $i$). Give criteria that one can use to examine a given filter $h$ and determine whether each of these properties holds.

\(^1\)The value round($x$) is the nearest integer to $x$; floor($x$) is the greatest integer that is $\leq x$, and ceil($x$) is the smallest integer that is $\geq x$. 
Problem 3: Math review questions

1. Which of the following functions has an inverse?
   (a) $f : \mathbb{R} \to \mathbb{Z} : x \mapsto \text{round}(x)$
   (b) $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^3 + x^2/100$
   (c) $f : \mathbb{Z} \to \mathbb{Z} : x \mapsto x^3 + x^2/100$

2. Consider the function $f : \mathbb{R} \to \mathbb{R}^2 : f(t) = (\cos t, \sin t)$.
   (a) What is the range of $f$?
   (b) What is the image of the interval $[0, \pi/2]$ under $f$?
   (c) What is the preimage of the set $\{(x, y) \mid |x| < \sqrt{2}\}$?

3. Show that the cross product is not associative by giving a counterexample using the vectors $e_1$, $e_2$, and $e_3$.

4. Shirley Exercise 2.9.2

Problem 4: Curves and surfaces


2. Consider the parametric curve
   
   \[ x = t^3 + at \]
   \[ y = t^2. \]
   
   What does the curve look like as $t$ ranges over $[-1, 1]$ when $a = 0$? When $a > 0$? When $a < 0$? Draw rough sketches of the shape.

3. Consider the parametric surface
   
   \[ x = t^3 + st \]
   \[ y = t^2 \]
   \[ z = s. \]
   
   What does the surface look like as $(s, t)$ ranges over $[-1, 1] \times [-1, 1]$? Sketch the intersections of the surface with the three coordinate planes (the planes $x = 0$, $y = 0$, and $z = 0$).

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2By 2.9 I mean Exercise 9 in Chapter 2.