1. Consider a program to evaluate

\[ F(a, b) = \sqrt{a^2 + b^2} - |a|. \]

Implementing this formula directly in Matlab (i.e., as \( \text{sqrt}(a^2+b^2)-\text{abs}(a) \)) is prone to overflow (e.g., in the case \( a, b \approx 10^{180} \)), underflow (e.g., in the case \( a, b \approx 10^{-180} \)), and severe cancellation (e.g., in the case \( |a| \gg |b| \)). Write (on paper) a matlab program to evaluate \( F \) that should be more robust against overflow, underflow and cancellation than the direct implementation. It is OK if your program needs some \texttt{if} statements.

2. Let \( U \) be an \( n \times n \) nonsingular upper triangular matrix. (a) Show that \( \| U^{-1} \|_\infty \geq 1/\min_i |U(i, i)| \). This fact leads to a simple but not very reliable condition-number estimator (namely, \( \| U^{-1} \|_\infty \approx 1/\min_i |U(i, i)| \)) for upper triangular matrices. (b) In fact, show that this estimator is not reliable by constructing a \( 2 \times 2 \) upper triangular matrix \( U \) in which \( \| U^{-1} \|_\infty \geq 10^8/\min_i |U(i, i)| \).

3. Let \( A \) be a symmetric positive \textit{semidefinite} matrix.

(a) Show that \( A(1, 1) \) must be nonnegative.

(b) Show that if \( A(1, 1) = 0 \), then the whole first row and column of \( A \) must be all zeros.

These two facts play a role in an efficient algorithm for testing whether a matrix is positive semidefinite.

4. Write a Matlab function \texttt{invlower} that computes \( L^{-1} \) given a lower triangular matrix \( L \) by applying forward substitution to the columns of the identity matrix. Make sure the inner loop is vectorized, and make sure that unnecessary operations on 0’s are omitted.

Then write an m-file called \texttt{mycond} that computes the condition number of a lower triangular matrix by multiplying its norm (the matrix 2-norm, which is computed by \texttt{norm}) in matlab by the norm of the inverse as computed by \texttt{invlower}. Compare this to the builtin \texttt{cond} function. They should nearly identical answers for reasonably well-conditioned matrices, e.g., the matrix returned by \texttt{tril(randn(10,10))}. Which seems to be more accurate for extremely ill-conditioned lower triangular matrices? You can make a lower triangular matrix ill-conditioned by putting a number very close to 0 (say 1e-40) on the main diagonal, or by putting a very big number in an off-diagonal position, or both. You can get some idea of which routine (\texttt{cond} vs \texttt{mycond}) is more accurate.
accurate by checking whether the inequalities of question 1 are satisfied by the results. Hand in listings of all m-files, some sample runs, and a paragraph of conclusions.