
This test lasted 75 minutes. All the questions were weighted equally even though they are not equally difficult. Students were allowed to consult a 8.5-by-11 sheet of paper that they had prepared in advance.

1. Let $x$ be a vector in $\mathbb{R}^n$. (a) Show that

$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty.$$ 

(b) Exhibit two nonzero vectors $x_1, x_2 \in \mathbb{R}^n$ such that the first inequality of part (a) is tight (i.e., is satisfied as an equation) for $x_1$, while the second inequality is tight for $x_2$.

2. Consider the function $f(x) = \cos x - 1$. (a) Show that the obvious way for evaluating this function is prone to catastrophic cancellation for $x$ close to 0. (b) Propose an alternative way to evaluate this function when $x$ is close to 0. [Hint for (b): recall \( \cos 2\theta = \cos^2 \theta - \sin^2 \theta \).]

3. **Threshold pivoting** is a strategy sometimes used in place of partial pivoting within Gaussian elimination applied to an $n \times n$ matrix $A$. In threshold pivoting, any uneliminated entry $A(p,k)$ in the pivot column $k$ may be selected as pivot provided $|A(p,k)| \geq \alpha \max |(A(k : n,k))|$ where $\alpha$ is a parameter between 0 and 1. (For example, $\alpha = 1$ would be partial pivoting.) Assuming threshold pivoting is used, derive an upper bound on $\|L\|_\infty$ in terms of $n$ and $\alpha$, where $L$ is the lower triangular factor resulting from elimination.

4. Consider the problem of evaluating a real-valued differentiable function $f(x)$ of a scalar variable $x$. The condition number of this problem for argument $x_1$ is sometimes defined to be $|f'(x_1) \cdot x_1|/|f(x_1)|$. Explain why. [Hint: consider small relative perturbations to $x_1$. The derivative comes from a Taylor approximation.]

5. Suppose Gaussian elimination with pivoting is performed on a nonsingular $2n \times 2n$ matrix with block structure

$$
\begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
$$

where all the blocks are size $n \times n$. Show that both factors $L$ and $U$ will have a block of zero entries, and determine the number of flops (accurate to the leading term) for computing the $P^T L U$ factorization of this matrix.