Midterm
Friday, July 25, 2003

Answer all below questions. You may consult only the proctor. Submit all
scratch paper and full, clear reasoning; this will help us give you partial credit.

1) (34 pts) You have the three data points

\[(x_1, f(x_1)) = (0, 0),\]

\[(x_2, f(x_2)) = (\frac{\pi}{2}, 1)\]

and

\[(x_3, f(x_3)) = (\pi, 0),\]

where \(f\) is the alleged underlying unknown function. Using the representation

\[p_{n-1}(x) = a_1 + a_2 x + \cdots + a_n x^{n-1},\]

set up, but do not solve, the equations that give the \(a_i\)'s so that \(p_{n-1}\) interpolates
\(f\) at the three data points. Design a 3-point interpolation function for this data
based on the representation

\[p_{n-1}(x) = a_1 + a_2 \left(\frac{x}{\pi}\right) + a_3 \left(\frac{x}{\pi}\right)^2 + \cdots + a_n \left(\frac{x}{\pi}\right)^{n-1}.\]

Be sure to explicitly write down your function. Explain your method’s virtues,
drawbacks.

Bonus: If possible, give interpretations of the unknowns in your method.

2) (33 pts) Clearly delineate how to find \(x\) given that

\[x = b^T C^{-1} b,\]

where \(C = A A^T\), matrices \(A\) and \(C\) are nonsingular, as efficiently as possible.
Assume that pivoting is not necessary. What does the cost of your method boil
down to? Suppose we formed \(C\), formed \(C^{-1}\), then \(x\), and we called this method
2. What would be the savings from your method, given above, over method 2
(Assume that forming \(C^{-1}\) from \(C\) requires \(\frac{8}{3} n^3\))? Note that forming \(C^{-1}\) from
\(C\) is usually unstable and therefore a bad idea, but we consider it for sake of
comparison.

3a) (23 pts) Find \(w_i, x_i\), so that the integration formula
\[ \int_{-1}^{1} x^2 f(x) dx = w_1 f(x_1) + w_2 f(x_2) \]

is exact for all \( f \in \Pi_3 \) (Hint: following the methodology in the text is one way to do it.).

3b) (10 pts) Let

\[ I_1 = \frac{1}{3} \int_a^b (f - g - h) dx \]

and

\[ I_2 = \frac{1}{3} \int_a^b (f + g + h) dx. \]

Clearly show how to estimate \( I_i \) (\( i = 1, 2 \)) to within a tolerance of \( tol \) using Matlab’s \texttt{quad} function. Show why your approximations serve their purpose, clearly stating any assumptions you are using.

Bonus. Discuss two more reasons your method wins over a brute-force method.