For all these questions, explain how you got the answer that you got.

1. Rosen Section 7.3: 4

**Solution** Let $S_1$ and $S_2$ be the events that Ann picked a ball from the first (resp., second) box; let $O$ be the event that she selected an orange ball. We are given that $\Pr(S_1) = \Pr(S_2) = \frac{1}{2}$, $\Pr(O \mid S_1) = \frac{3}{7}$, and $\Pr(O \mid S_2) = \frac{5}{11}$. We want to know $\Pr(S_2 \mid O)$. By Bayes’ Theorem,

$$\Pr(S_2 \mid O) = \frac{\Pr(O \mid S_2) \Pr(S_2)}{\Pr(O \mid S_1) \Pr(S_1) + \Pr(O \mid S_2) \Pr(S_2)} = \frac{(5/11)(1/2)}{(3/7)(1/2) + (5/11)(1/2)} = \frac{5/11}{3/7 + 5/11}.$$

2. Rosen Section 7.3, 6:

**Solution** Let $P$ be the event that a soccer player tests positive for steroids; Let $N$ be the event that a soccer player tests negative for steroids; let $T$ be the event that a soccer player takes steroids. We are given that $\Pr(T) = .05$, $\Pr(P \mid T) = .98$, and $\Pr(P \mid \overline{T}) = .12$. We want to know $\Pr(T \mid P)$. By Bayes’ Theorem,

$$\Pr(T \mid P) = \frac{\Pr(P \mid T) \Pr(T)}{\Pr(P \mid T) \Pr(T) + \Pr(P \mid \overline{T}) \Pr(\overline{T})} = \frac{.98(.05)}{.98(.05) + .12(.95)}.$$

(This is about $5/17$, so although it’s higher than the prior probability of .05, it nowhere near the accuracy of the test.)

3. Rosen, Section 7.3, 16:

**Solution** Let $L$ be the event that Ramesh is late; let $B$ be the event that Ramesh takes a bus; let $D$ be the event that Ramesh drives; and let $BI$ be the event that Ramesh bikes. We are given that $\Pr(L \mid B) = .2$, $\Pr(L \mid D) = .5$, and $\Pr(L \mid BI) = .05$. We want to compute $\Pr(D \mid L)$.

By Bayes’ Theorem,

$$\Pr(D \mid L) = \frac{\Pr(L \mid D) \Pr(D)}{\Pr(L \mid D) \Pr(D) + \Pr(L \mid B) \Pr(B) + \Pr(L \mid BI) \Pr(BI)} = \frac{.5 \Pr(D)}{.5 \Pr(D) + .2 \Pr(B) + .05 \Pr(BI)}.$$

(a) If $\Pr(D) = \Pr(BI) = \Pr(B) = 1/3$, then

$$\Pr(D \mid L) = \frac{.5(1/3)}{.5(1/3) + .2(1/3) + .05(1/3)} = \frac{50}{75} = \frac{2}{3}.$$

(b) If $\Pr(D) = .3$, $\Pr(BI) = .6$, $\Pr(B) = .1$, then

$$\Pr(D \mid L) = \frac{.5 \cdot .3}{.5 \cdot .3 + .2 \cdot .1 + .05 \cdot .6} = \frac{15}{38}.$$

4. Rosen Section 7.4, 6:

**Solution** The word “chosen” suggests that there are no repeats; in this case, there are $C(50, 6)$ possible choices, so the expected winnings are $10,000,000/C(50, 6)$. This is roughly $1/2$.

5. Rosen Section 7.4, 8:
Solution The expected outcome when one fair die is called is 3.5. Let $X_1$, $X_2$, and $X_3$ represent the outcomes when each of the die is tossed. By the linearity of expectation, $E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3) = 3(3.5) = 10.5$

6. Rosen Section 7.4, 12

Solution (a) We roll the die $n$ times exactly if we get don’t get a six the first $n-1$ times and then get a 6 the $n$th time. That happens with probability $(5/6)^{n-1}(1/6)$.

(b) The expected number of times we roll the die is $\sum_{n=1}^{\infty} n(5/6)^{n-1}(1/6)$. (It can be shown that $\sum_{n=1}^{\infty} n(5/6)^{n-1}(1/6) = 6$, but you didn’t need to do that for this exercise. The intuition should be clear though: since the probability of getting 6 is 1/6, on average, you need to toss a die 6 times to get a 6.)

7. Rosen Section 7.4, 16

Solution This is most easily proved by induction on $n$. Let $P(n)$ be the estatement we want to prove: $\Pr(E_1 \cap \cdots \cap E_n) = \Pr(E_1) \Pr(E_2 \mid E_1 \cap E_2) \cdots \Pr(E_n \mid E_1 \cap \cdots \cap E_{n-1})$. The base case just says that $\Pr(E_1) = \Pr(E_1)$; this is obviously true. Suppose $P(n)$ holds. We prove $P(n+1)$. By the definition of conditioning, $\Pr(E_{n+1} \mid E_1 \cap \cdots \cap E_n) = \Pr(E_{n+1} \cap E_1 \cap \cdots \cap E_n) / \Pr(E_1 \cap \cdots \cap E_n)$. Thus, $\Pr(E_1 \cap \cdots \cap E_n \cap E_{n+1}) = \Pr(E_{n+1} \mid E_1 \cap \cdots \cap E_n) \cdot \Pr(E_1 \cap \cdots \cap E_n)$. By $P(n)$, we can replace $\Pr(E_1 \cap \cdots \cap E_n)$ by $\Pr(E_1) \Pr(E_2 \mid E_1 \cap E_2) \cdots \Pr(E_n \mid E_1 \cap \cdots \cap E_{n-1})$. $P(n+1)$ immediately follows. This completes the proof.

(This is called the chain rule. It turns out to be really useful in practice to get efficient calculations of probability.)

8. Rosen Section 7.4, 24

Solution Remember that a random variable is not random and not a variable. It is a function from the sample space to the reals. Formally, $I_A(s) = 1$ if $s \in A$ and $I_A(s) = 0$ if $s \notin A$. It is almost immediate from the definition that $I_A$ is 1 with probability $\Pr(A)$ and 0 with probability $1 - \Pr(A)$. Thus, $E(I_A) = 1 \cdot \Pr(A) + 0 \cdot (1 - \Pr(A)) = \Pr(A)$.

9. Rosen Section 7.4, 28

Solution Let $X$ represent how many times 6 appears when a die is tossed 10 times. Let $X_k = 1$ if 6 appers on the $k$ toss. Thus, $X = X_1 + \cdots + X_{10}$. Since $X_1, \ldots, X_{10}$ are independent, by Theorem 7 in the Rosen, $\text{Var}(X) = \sum_{k=1}^{10} \text{Var}(X_k)$. For $k = 1, \ldots, n$, since $X_k$ is 1 with probability 1/6 and 0 with probability 5/6, $E(X_k) = 1/6$. Moreover $X_k^2 = X_k$, so $\text{Var}(X_k) = E(X_k^2) - [E(X_k)]^2 = 1/6 - (1/6)^2 = 5/36$. Thus, $\text{Var}(X) = 50/36$. 

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