1 Recurrence Relations: Bottom Up

For each of the characteristic equations:

1. Find the corresponding recurrence equation.
2. How many boundary conditions are necessary for a complete solution to \( f(n) \)?
3. Show that the roots of the characteristic equations raised to the \( n^{th} \) power are solutions of the recurrence equation. I.e \( f(n) = r^n \), where \( r \) is a root of the corresponding characteristic equation, satisfies the recurrence equation.

The characteristic equations:

1.1 \( x - 2 = 0 \)
   1. \( f(n) = 2f(n - 1) \)
   2. Need 1 boundary condition.
   3. The roots are: 2.

\[
2^n = 2(2)^{n-1}
\]

1.2 \( x^2 + 2x - 15 = 0 \)
   1. \( f(n) = -2f(n - 1) + 15f(n - 2) \)
   2. Need 2 boundary conditions.
   3. The roots are: 3 and -5.

\[
(-5)^n = -2(-5)^{n-1} + 15(-5)^{n-2} \\
25 = 10 + 15 \tag{1}
\]

And:

\[
(3)^n = -2(3)^{n-1} + 15(3)^{n-2} \\
9 = -6 + 15 \tag{2}
\]
1.3 \[8x^3 - 8x^2 + 4x - 1 = 0\]

1. \[8f(n) = 8f(n-1) - 4f(n-2) + f(n-3)\]

2. Need 3 boundary conditions.

3. The roots are: \(\frac{1}{2}, \frac{1 + \sqrt{3}}{4}, \frac{1 - \sqrt{3}}{4}\).

\[
8 \left(\frac{1}{2}\right)^n = 8 \left(\frac{1}{2}\right)^{-1} - 4 \left(\frac{1}{2}\right)^{-2} + \left(\frac{1}{2}\right)^{-3} \\
\frac{8}{2^3} = \frac{8}{2^2} - \frac{4}{2} + 1 \\
1 = 2 - 2 + 1
\]

\[
8 \left(\frac{1 + \sqrt{3}}{4}\right)^n = 8 \left(\frac{1 + \sqrt{3}}{4}\right)^{-1} - 4 \left(\frac{1 + \sqrt{3}}{4}\right)^{-2} + \left(\frac{1 + \sqrt{3}}{4}\right)^{-3} \\
8 \left(\frac{1 + \sqrt{3}}{4}\right)^2 = \left(\frac{1 + \sqrt{3}}{4}\right)^2 - \sqrt{3} \\
\frac{1 + 3\sqrt{3} - 9 - 3\sqrt{3}}{8} = \frac{1 + 2\sqrt{3} - 3}{2} - \sqrt{3} \\
\frac{8}{8} = \frac{2 + 2\sqrt{3}}{2} - \sqrt{3} = 1
\]

In a very similar way the root \(\left(\frac{1 - \sqrt{3}}{4}\right)^n\) can be shown to satisfy the recurrence.

2 Recurrence Relation: Top Down

For each of the recurrence equations:

1. Find the characteristic equation and its roots.

2. The linearly independent set of basis functions.

3. The final closed form solution of \(f(n)\)

4. Show for \(n = 3\) and \(n = 4\), the closed form solution and the recurrence equations agree.

The recurrence relations:

2.1 First Eq

Given, \(f(n) = 5f(n-1) - 6f(n-2)\) with boundary conditions \(f(0) = 0\) and \(f(1) = 1\).

1. The characteristic eq is: \(x^2 - 5x + 6 = 0\), with roots \(r_1 = 3\) and \(r_2 = 2\).

2. The linearly independent basis functions are \(3^n\) and \(2^n\).
3. The closed form solution must be of the form \( f(n) = a3^n + b2^n \), where \( a \) and \( b \) are determined by the boundary conditions:

\[
\begin{align*}
  f(0) &= a(3)^0 + b(2)^0 = a + b = 0 \\
  f(1) &= a(3)^1 + b(2)^1 = 3a + 2b = 1 
\end{align*}
\]

Hence, \( a = 1 \) and \( b = -1 \) \( \implies \) \( f(n) = 3^n - 2^n \)

4. By recurrence:

\[
\begin{align*}
  f(2) &= 5, \\
  f(3) &= 5 \cdot 5 - 6 = 19, \\
  f(4) &= 5 \cdot 19 - 6 \cdot 5 = 65 
\end{align*}
\]

By formula:

\[
\begin{align*}
  f(3) &= 3^3 - 2^3 = 19 \text{ and } f(4) = 3^4 - 2^4 = 65 
\end{align*}
\]

### 2.2 Second Eq

Given, \( f(n) = 6f(n-1) - 12f(n-2) + 8f(n-3) \) with boundary conditions \( f(0) = 1, f(1) = 6, \) and \( f(2) = 32 \).

1. The characteristic eq is: \( x^3 - 6x^2 + 12x - 8 = 0 \), with roots \( r_1 = 2 \) and \( r_2 = 2 \) and \( r_3 = 2 \).

2. The linearly independent basis functions are \( 2^n, n2^n \) and \( n^22^n \).

3. The closed form solution must be of the form \( f(n) = a2^n + bn2^n + cn^22^n \), where \( a, b, \) and \( c \) are determined by the boundary conditions:

\[
\begin{align*}
  f(0) &= a(2)^0 + b(0)(2^0) + c(0)^22^0 = a = 1 \\
  f(1) &= a(2)^1 + b(1)(2^1) + c(1)^22^1 = 2a + 2b + 2c = 6 \\
  f(2) &= a(2)^2 + b(2)(2^2) + c(2)^22^2 = 4a + 8b + 16c = 32 
\end{align*}
\]

Hence, \( a = 1, b = \frac{1}{2}, \) and \( c = \frac{3}{2} \) \( \implies \) \( f(n) = 2^n + \frac{1}{2}n2^n + \frac{3}{2}n^22^n \)

4. By recurrence:

\[
\begin{align*}
  f(3) &= 6 \cdot 32 - 12 \cdot 6 + 8 \cdot 1 = 128, \\
  f(4) &= 6 \cdot 128 - 12 \cdot 32 + 8 \cdot 6 = 432 
\end{align*}
\]

By formula: \( f(3) = 2^3 + \frac{3}{2}2^3 + \frac{27}{2}2^3 = 128 \) and \( f(4) = 2^4 + \frac{4}{2}2^4 + \frac{48}{2}2^4 = 432 \)

### 3 Bijective Proof

Here we provide the outline of a proof. You should rewrite all the provided steps and fill in the missing steps marked by ???. For definitions of 1-1(injective), onto(surjective), and isomorphism(bijection) please reference wikipedia.

**Theorem 1.** Given sets \( S \) and \( T \), and a mapping \( f : S \to T \) that is onto and a mapping \( g : T \to S \) is onto, prove whether or not \( S \) and \( T \) are isomorphic.

This is not a proof, but to guess whether or not \( S \) and \( T \) are isomorphic, we can consider their cardinality. If \( f \) is an onto mapping from \( S \) to \( T \):

\[
|S| \geq |T|
\]
And similarly, if \( g \) is an onto mapping from \( T \) to \( S \):

\[ |S| \leq |T| \]

Which implies:

\[ |S| = |T| \]

Hence, if \( S \) and \( T \) are finite sets they have the same cardinality and would be isomorphic. But, we need to construct a rigorous proof to handle finite and infinite sets.

**Proof.** The proof idea: To prove \( S \) and \( T \) are isomorphic, we will use mappings \( f \) and \( g \) to construct a 1−1 mapping from \( S \) to \( T \) and a 1−1 mapping from \( T \) to \( S \). From class, if we can show both 1−1 mappings exist, this implies there is a bijection between \( S \) and \( T \). Hence, \( S \) and \( T \) are isomorphic.

The proof:

We now construct a 1−1 mapping from \( T \) to \( S \) using \( f \):

Since, \( f \) maps \( S \) onto \( T \), for every \( t \in T \) there exists a non-empty set \( s_t \) such that for all \( s \in s_t \) \( f(s) = t \). I.e, \( s_t \) is all elements of \( S \) that map to \( t \in T \). Note that because \( s_t \) is non-empty, we can pick one element from every \( s_t \). Now, for every \( t \in T \), we create the set \( s'_t \), by picking exactly one element from \( s_t \).

Note, that because \( f \) was a mapping, no element of \( S \) mapped to. We use If we consider the mapping from \( f' \), \( T \) to \( S \), where \( f'(t) = s \in s'_t \), we have a 1-1 mapping from \( T \) to \( S \). By construction,

We now construct a 1−1 mapping from \( S \) to \( T \) using \( g \):

In exactly the same manner, but this time using \( g \), sets \( t'_s \) of cardinality 1 for every \( t \in T \), can be constructed. The sets \( t'_s \) can then be used for a mapping \( g'(s) = t \in t'_s \). This \( g' \) mapping is a 1-1 mapping from \( S \) to \( T \).

Hence, from class, as there exists a 1-1 mapping from \( S \) to \( T \) and from \( T \) to \( S \), there exists a bijection from \( S \) to \( T \). Hence, \( S \) and \( T \) are isomorphic.

4 **Mapping Examples**

Consider the set up in the previous problem where \( f \) is an onto mapping from \( S \) to \( T \). In the previous problem we show \( S \) and \( T \) are isomorphic.

1. Give an example of two sets \( S \) and \( T \) and mapping \( f \), where the fact \( S \) and \( T \) are isomorphic implies that \( f \) is also 1−1.

   **Example 2.** We have that \( S \) and \( T \) are isomorphic, if we add the constraint that \( S \) and \( T \) have finite cardinality, then any onto mapping from \( S \) to \( T \) must also be 1-1. The easiest way to see this is to draw a picture and realize that there are exactly the same number of elements in \( S \) and \( T \) (follows from finite cardinality). Hence, any onto mapping must have mapped each \( s \in S \) to a unique element of \( T \), which is 1-1.

2. Give an example of two sets \( S \) and \( T \) and mapping \( f \), where the fact \( S \) and \( T \) are isomorphic does not implies that \( f \) is also 1−1.

   **Example 3.** The reason we had to do a careful proof on the previous problem, was the case in which \( S \) and \( T \) are infinite. If \( S \) was the set of natural numbers and \( T \) was the set of real numbers both have cardinality \( |S| = |T| = \infty \), but they are not isomorphic. In fact there does not exist an onto mapping from the natural numbers to the real numbers.

Now, if we consider \( S \) and \( T \) to be the natural numbers, \( S \) and \( T \) are indeed isomorphic, they are the same set! We can construct the mapping where \( f(s \in S) = \lfloor \frac{s}{2} \rfloor \). The mapping is indeed onto, but 3 and 4 map to 1 and 7 and 8 map to 3. So it is not 1-1, it is 2-1.