1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 1.8

2. The main message of this lecture:

There are two good reasons to compare functions according to their asymptotic behavior. First: very often it takes big values of the argument for a function to show its real strength. Second: often (but not always!) big arguments correspond to real size problems.

Real numbers (as well as rationals, integers, natural numbers) are linearly ordered: for any two numbers \(a, b\) either \(a < b\) or \(a > b\) or \(a = b\). Unfortunately, this does not hold for functions. Suppose by \(f \leq g\) we understand the condition \(\forall x (f(x) \leq g(x))\), for example, \(\cos x \leq x^2 + 1\) since for all \(x \in \mathbb{R}\) \(\cos x \leq 1\) and \(1 \leq x^2 + 1\). Then too many pairs of functions become incomparable. For example, \(\cos x\) versus \(x^2\). For some of the \(x\)’s, \(e.g.\ x = 0\) \(\cos x\) is greater than the squaring function: \(\cos 0 = 1 > 0 = 0^2\). However, for \(x \geq 1\) the latter clearly dominates: \(1^2 = 1, 2^2 = 4, \ldots, 10^2 = 100, \ldots\), whereas \(\cos x\) stays somewhere between \(-1\) and \(1\). Another typical example: \(x^2\) versus \(x + 100\). For small \(x\)’s the squaring function again loses: \(0^2 = 0, 1^2 = 1, 2^2 = 4, \ldots\) whereas the linear function with a “big” constant \(x + 100\) scores much better points: \(0 + 100 = 100, 1 + 100 = 101, 2 + 100 = 102, \ldots\). However, the rate of growth of \(x^2\) is much higher and this function catches up quickly: \(10^2 = 100 < 10 + 100 = 110,\) but \(11^2 = 121 > 11 + 100 = 111, \ldots\ 20^2 = 400 > 20 + 100 = 120,\ldots 100^2 = 10000 > 100 + 100 = 200.\) According to a common sense judgment, \(x^2\) is much larger than \(x + 100\).

Even more striking difference is provided by yet another canonical example: \(n^2 + 100\) versus \(2^n\). Here is the table of the initial values of those functions.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(n^2 + 100)</th>
<th>(2^n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>101</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>104</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>109</td>
<td>8</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>10</td>
<td>200</td>
<td>1024</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>10100</td>
<td>(&gt; 10^{30})</td>
</tr>
</tbody>
</table>

Note that for \(n = 100\) (a moderate size input), \(n^2 + 100\) constitute a negligible fraction of \(2^n\), something like \(10^{-25}\). The main reason for such a mismatch is, of course, the exponential function \(2^n\) which becomes ridiculously large. In computational complexity, those algorithms which make exponential (of the size of input) number of steps are regarded **non-feasible**.

In what follows \(f, g, h\) may be regarded as functions from \(\mathbb{R}\) to \(\mathbb{R}\) (as well as from to \(\mathbb{N}\) to \(\mathbb{R}\)). Reminder: \(|a|\) stands for the absolute value of \(a\) when the sign of \(a\) is stripped off. Formally speaking, \(|a| = a\), if \(a \geq 0\) and \(|a| = -a\), if \(a < 0\). For example, \(|2| = 2, \ |-2| = -(-2) = 2, \ |0| = 0. Some common inequalities which follow immediately from the definitions: \(a \leq |a|, \ |a \pm b| \leq |a| + |b|\).
Definition 6.1. \( f = O(g) \) (read \( f \) is big-O of \( g \)) if \( \exists C \exists k \forall x (x > k \rightarrow (|f(x)| \leq C \cdot |g(x)|)) \).

The informal meaning of “\( f \) is \( O(g) \)” is that some multiple of \( g \) eventually overruns \( f \). This relation is usually regarded a sort of inequality on functions \( f \preceq g \). We say that \( f \) and \( g \) have the same order (notation \( f = \Theta(g) \)), of both \( f = O(g) \) and \( g = O(f) \).

Examples:

- \( x + 100 \) is \( O(x^2) \). Indeed, put \( C = 1, k = 100 \) and \( x > k \). Then \( x + 100 < x \cdot 100 < x^2 \).
- \( x^2 + 100x \) is \( O(x^2) \) (despite the fact that the former functions is always greater than the latter for positive \( x \)s, moreover, their difference \( (x^2 + 100x) - x^2 = 100x \) grows to +\( \infty \)). Indeed, put \( C = 2, k = 100 \). Then \( x > 100 \rightarrow x^2 + 100x < x^2 + x \cdot x = 2x^2 \).
- \( x^2 \) is \( O(x^2 + 100x) \) (\( C = 1, k = 1 \)), therefore, \( f \) and \( g \) have the same order.
- \( x + 100 \) and \( x^2 \) do not have the same order, on particular, \( x^2 \) is not \( O(x + 100) \). Indeed, let us first negate the definition of “big-O” above: \( \forall C \forall k \exists x (x > k \wedge (|f(x)| > C \cdot |g(x)|)) \), and then prove that the pair \( f(x) = x + 100 \) and \( g(x) = x^2 \) satisfy the negation of the “big-O” definition. Let \( C, k \) are any given reals. Take any \( x > \max (k, 2C, 100) \). Then \( x^2 > 2Cx = C \cdot 2x = C(x + x) > C(x + 100) \).

Example 6.2. Some simple reference functions of \( n \) used in the complexity theory. Here \( f \prec g \) means “\( f = O(g) \) but not \( g = O(f) \)”, \( n! = 1 \cdot 2 \cdot 3 \ldots n \), \( \log \) is the base 2 logarithm.

\[ 1 < \log n \prec \log n \prec n \prec n \log n \prec n \log n \prec n^2 \prec n^3 \prec \ldots \prec 2^n \prec 3^n \prec \ldots \prec n^n \]

Theorem 6.3. If \( f_1 = O(g_1) \) and \( f_2 = O(g_2) \) then \( f_1 + f_2 = O(\max (|g_1|, |g_2|)) \).

Proof. For appropriate \( C_1, k_1, C_2, k_2 \) we have
\[ x > k_1 \Rightarrow |f_1(x)| \leq C_1|g_1(x)| \quad x > k_2 \Rightarrow |f_2(x)| \leq C_2|g_2(x)|. \]

Without loss of generality we assume that \( C_1, C_2 > 0 \). Put \( k = \max (k_1, k_2), C = C_1 + C_2, g = \max (|g_1|, |g_2|) \). Then \( x > k \Rightarrow |f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)| \leq C_1|g_1(x)| + C_2|g_2(x)| \leq (C_1 + C_2)|g(x)| = C|g(x)|. \)

Example: \( x^2 = O(x^2), 100x = O(x^2) \), therefore \( x^2 + 100x = O(x^2) \).

Theorem 6.4. If \( f_1 = O(g_1) \) and \( f_2 = O(g_2) \) then \( f_1 \cdot f_2 = O(g_1 \cdot g_2) \).

Proof. For appropriate \( C_1, k_1, C_2, k_2 \) we have
\[ x > k_1 \Rightarrow |f_1(x)| \leq C_1|g_1(x)| \quad x > k_2 \Rightarrow |f_2(x)| \leq C_2|g_2(x)|. \]

Put \( k = \max (k_1, k_2), C = C_1C_2 \). Then \( x > k \Rightarrow |f_1(x)f_2(x)| = |f_1(x)| \cdot |f_2(x)| \leq C_1|g_1(x)| \cdot C_2|g_2(x)| = C_1C_2|g_1(x) \cdot g_2(x)| = C|g_1(x) \cdot g_2(x)|. \)

Example: Give as good big-O estimate as possible in terms of simple reference functions for \( f(n) = (3n + 1) \log (5n^3 + 1) + 10n^2 \). We may assume that \( n \) is sufficiently large. \( 3n + 1 = O(n), \log (5n^3 + 1) < \log (6n^3) = \log 6 + 3 \log n < 4 \log n = O(\log n) \) (this works for \( n > 6 \)). Therefore, \( (3n + 1) \log (5n^3 + 1) = O(n \log n) \). Since \( 10n^2 = O(n^2) \), and \( n \log n, n^2 \) are dominated, by 6.3, \( f(n) = O(n^2) \). Moreover, since \( n^2 < f(n) \) we have \( n^2 = O(f) \) and thus \( f(n) = \Theta(n^2) \).

Theorem 6.5. \( F(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) has order \( x^n \).

Proof. We show \( f(x) = O(x^n) \) as an exercise. Let \( k = 1 \) and \( x > k \). Then
\[ |a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0| \leq |a_n|x^n + |a_{n-1}|x^{n-1} + \ldots + |a_1|x + |a_0| \leq x^n(|a_n| + \frac{|a_{n-1}|}{x} + \ldots + \frac{|a_1|}{x^{n-1}} + \frac{|a_0|}{x^n}) \leq x^n(|a_n| + |a_{n-1}| + \ldots + |a_1| + |a_0|). \]

Now put \( C = |a_n| + |a_{n-1}| + \ldots + |a_1| + |a_0| \) and get the desired \( |f(x)| \leq C \cdot x^n \) whenever \( x > k \).

Homework assignments. (due Friday 02/09).

A. Section 1.8: 2, 8ab, 20ab, 28a

B. Show that \( 2^n = O(n!) \), but \( n! \) is not \( O(2^n) \).