1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 5.1

2. The main message of this lecture:

**Recursion applies to counting as well.**

**Definition 24.1.** A recurrence relation $R$ for a sequence \( \{a_n\} \) is an equation (understood broadly) that expresses $a_n$ in terms of some of the previous terms $a_0, a_1, \ldots, a_{n-1}$. A solution of a given recurrence relation $R$ is a sequence \( \{a_n\} \) of terms satisfying $R$.

**Example 24.2.** The size of a certain fish population in Cayuga lake can increase 10% a year due to natural growth. The harvesting rate is 1000 individuals per year. If the initial population size is 8000 individuals find the population size after 5 years.

Solution: $a_n$ = the population size after $n$ years.  
$a_0$ = 8000 - the initial condition  
$a_n = 1.1 \cdot a_{n-1} - 1000$ - the recurrence relation proper.

Note that there is no principal difference between an initial condition and a recurrence relation in the narrow sense. In particular, the problem above can be formally presented in the standard unified form 24.1:

\[
a_n = \begin{cases} 
8000, & \text{if } n = 0 \\
1.1 \cdot a_{n-1} - 1000, & \text{if } n \geq 1
\end{cases}
\]

Here $n_0 = 1$. The problem 24.2 has a unique solution:

\[
a_0 = 8000 \\
a_1 = 1.1 \cdot a_0 - 1000 = 8800 - 1000 = 7800 \\
a_2 = 1.1 \cdot a_1 - 1000 = 8580 - 1000 = 7580 \\
a_3 = 1.1 \cdot a_2 - 1000 = 7338 \\
a_4 = 1.1 \cdot a_3 - 1000 = 7072 \\
a_5 = 1.1 \cdot a_4 - 1000 = 6779.2
\]

Can you explain the population size being a rational which is not an integer? Well, this is a difference between a real biological system (where the size of a fish population is always a nonnegative integer) and its mathematical model where this number is not necessarily integer.

**Example 24.3.** Some more familiar examples.

<table>
<thead>
<tr>
<th>Recurrence relation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n = a_{n-1} + d$</td>
<td>$a_0, a_0 + d, a_0 + 2d, \ldots, a_0 + (n-1) \cdot d, \ldots$</td>
</tr>
<tr>
<td></td>
<td>arithmetic progression</td>
</tr>
<tr>
<td>$a_n = a_{n-1} \cdot q$</td>
<td>$a_0, a_0 \cdot q, a_0 \cdot q^2, \ldots, a_0 \cdot q^{n-1}, \ldots$</td>
</tr>
<tr>
<td></td>
<td>geometric progression</td>
</tr>
<tr>
<td>$a_0 = 0, a_1 = 1, a_n = a_{n-2} + a_{n-1}$</td>
<td>$0, 1, 1, 2, 3, 5, 8, 13, \ldots$</td>
</tr>
<tr>
<td></td>
<td>Fibonacci numbers</td>
</tr>
</tbody>
</table>
Example 24.4. (Compound interest) The initial deposit is $10000 at a bank yielding 5% per year with interest compounded annually. How much will be the amount after $n$ years?

Recurrent equation is $S_0 = 10000$, $S_n = 1.05 \cdot S_{n-1}$. Solution sequence: $S_0 = 10000$, $S_1 = 1.05 \cdot S_0 = 10500$, $S_2 = 1.05 \cdot S_1 = 10525 \ldots$ So, everyone can become well off provided he/she lives long enough ...

Example 24.5. Find a recurrence relation for the number $b_n$ of bit strings of length $n$ that do not have two consecutive 0’s: $b_0 = 1$ (only one null string), $b_1 = 2$ (two bit strings of length 1, both fit). Let now $n \geq 2$. We present $b_n = X + Y$, where $X$ is the number of strings ending with 1 and $Y$ the number of strings ending with 0. Note that $X = b_{n-1}$, since each such string ending with 1 is $x1$ where $x$ is a string without two consecutive 0’s. Moreover, $Y = b_{n-2}$, since each such string ending with 0 is $y1$, where $y$ is a string without two consecutive 0’s. The resulting equation is $b_0 = 1$, $b_1 = 2$, $b_n = b_{n-2} + b_{n-1}$ for $n \geq 2$. Solution: 1, 2, 3, 5, 8, 13, 21, ...

Example 24.6. Suppose a codeword is a string of decimal digits, a valid codeword is a codeword with even number of 0’s. Let $C_n$ be the number of ways to parenthesize the product of $n$ terms $x_0 \cdot x_1 \cdot x_2 \ldots \cdot x_n$. For example, there is only one way to parenthesize a ”product” $x_0$, thus $C_0 = 1$. There is also only one way to parenthesize $x_0 \cdot x_1$, therefore, $C_1 = 1$. For $C_2$ consider a product $x_0 \cdot x_1 \cdot x_2$. There are two different ways to do it: $(x_0 \cdot x_1) \cdot x_2$ or $x_0 \cdot (x_1 \cdot x_2)$, thus $C_2 = 2$. For $n = 3$ we already have five possibilities: $x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$, $x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)$, $(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$, $(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$, $((x_0 \cdot x_1) \cdot x_2) \cdot x_3$.

Example 24.7. Messages are transmitted through a communication channel using two signals: one requires 1 microsecond, the other 2 microseconds. Find the total number of messages that can be sent in $n$ microseconds (no blanks are permitted). Note that $a_0 = 1$, $a_1 = 1$. Let $n \geq 2$. Then each message $x$ of length $n$ falls into one of two disjoint classes:

1) $x = y\sigma$, where $y$ is a message of length $n-1$, $\sigma$ a short signal.
2) $x = z\beta$, where $z$ is a message of length $n-2$, $\beta$ a long signal.

As before, $a_n = a_{n-1} + a_{n-2}$, therefore, $a_n = f_{n+1}$.

Example 24.8. Find a recurrence equation for the number $C_n$ of ways to parenthesize the product of $n+1$ terms $x_0 \cdot x_1 \cdot x_2 \ldots \cdot x_n$. For example, there is only one way to parenthesize a ”product” $x_0$, thus $C_0 = 1$. There is also only one way to parenthesize $x_0 \cdot x_1$, therefore, $C_1 = 1$. For $C_2$ consider a product $x_0 \cdot x_1 \cdot x_2$. There are two different ways to do it: $(x_0 \cdot x_1) \cdot x_2$ or $x_0 \cdot (x_1 \cdot x_2)$, thus $C_2 = 2$. For $n = 3$ we already have five possibilities: $x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$, $x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)$, $(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$, $(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$, $((x_0 \cdot x_1) \cdot x_2) \cdot x_3$.

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Here is a general argument: for a product $x_0 \cdot x_1 \cdot x_2 \ldots \cdot x_n$ first of all pick one of $n$ multiplications as the outermost one. Each such pick breaks the problem of size $n$ into two independent problems of the combined size $n-1$. By the Sum Rule and the Product Rule,

$$C_n = C_0 \cdot C_{n-1} + C_1 \cdot C_{n-2} + \ldots + C_{n-1} \cdot C_0 = \sum_{k=0}^{n-1} C_k C_{n-k-1}.$$ 

The numbers $C_n$ are called **Catalan numbers**; it can be shown that $C_n = C(2n,n)/(n+1)$.

**Homework assignments.** (due Friday 03/30).

24A:Rosen5.1-10; 24B:Rosen5.1-22; 24C:Rosen5.1-30