SEARCHING, SORTING, AND ASYMPTOTIC COMPLEXITY

Constant time

$0 \ 1 \ 2 \ 3 \ \ldots$

$n \times n$ ops

$2n + 2$ ops

$n + 2$ ops

$n$ ops

Constant time

size $n$

Pic: Natalie Diebold
/* Sort b[h..k]. Precondition: b[h..t] and b[t+1..k] are sorted. */

public static merge(int[] b, int h, int t, int k) {
    Copy b[h..t] into another array c;
    Copy values from c and b[t+1..k] in ascending order into b[h..]
}

We leave you to write this method. It is not difficult. Just have to move values from c and b[t+1..k] into b in the right order, from smallest to largest.
Runs in time O(k+1–h)
/** Sort b[h..k] */

public static mergesort(int[] b, int h, int k) {
    if (size b[h..k] < 2) {
        return;
    }
    int t = (h+k)/2;
    mergesort(b, h, t);
    mergesort(b, t+1, k);
    merge(b, h, t, k);
}

merge is $O(k+1-h)$

This is $O(n \log n)$ for an initial array segment of size $n$

But space is $O(n)$ also!
/** Sort b[h..k] */
public static mergesort(
    int[] b, int h, int k)
{
    if (size b[h..k] < 2)
        return;
    int t = (h+k)/2;
    mergesort(b, h, t);
    mergesort(b, t+1, k);
    merge(b, h, t, k);
}
QuickSort versus MergeSort

/** Sort b[h..k] */
public static void QS
  (int[] b, int h, int k) {
  if (k – h < 1) return;
  int j= partition(b, h, k);
  QS(b, h, j-1);
  QS(b, j+1, k);
}

/** Sort b[h..k] */
public static void MS
  (int[] b, int h, int k) {
  if (k – h < 1) return;
  MS(b, h, (h+k)/2);
  MS(b, (h+k)/2 + 1, k);
  merge(b, h, (h+k)/2, k);
}

One processes the array then recurses.
One recurses then processes the array.
Textbook: Chapter 4

Homework:

Recall our discussion of linked lists and A2.

What is the **worst** case complexity for appending an items on a linked list? For testing to see if the list contains X? What would be the **best** case complexity for these operations?

If we were going to talk about complexity (speed) for operating on a list, which makes more sense: worst-case, average-case, or best-case complexity? Why?
What Makes a Good Algorithm?

Suppose you have two possible algorithms or ADT implementations that do the same thing; which is better?

What do we mean by better?
- Faster?
- Less space?
- Easier to code?
- Easier to maintain?
- Required for homework?

How do we measure time and space of an algorithm?
Basic Step: One “constant time” operation

Basic step:
- Input/output of scalar value
- Access value of scalar variable, array element, or object field
- assign to variable, array element, or object field
- do one arithmetic or logical operation
- method call (not counting arg evaluation and execution of method body)

• If-statement: number of basic steps on branch that is executed

• Loop: (number of basic steps in loop body) * (number of iterations) – also bookkeeping

• Method: number of basic steps in method body (include steps needed to prepare stack-frame)
/** return true iff v is in b */
static boolean find(int[] b, int v) {
    for (int i = 0; i < b.length; i++) {
        if (b[i] == v) return true;
    }
    return false;
}

Let \( n = b\.\text{length} \)

<table>
<thead>
<tr>
<th>basic step</th>
<th># times executed</th>
</tr>
</thead>
<tbody>
<tr>
<td>i = 0;</td>
<td>1</td>
</tr>
<tr>
<td>i &lt; b.\text{length}</td>
<td>( n+1 )</td>
</tr>
<tr>
<td>i++</td>
<td>( n )</td>
</tr>
<tr>
<td>b[i] == v</td>
<td>( n )</td>
</tr>
<tr>
<td>return true</td>
<td>0</td>
</tr>
<tr>
<td>return false</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>( 3n + 3 )</td>
</tr>
</tbody>
</table>

We sometimes simplify counting by counting only important things. Here, it’s the number of array element comparisons \( b[i] == v \). That’s the number of loop iterations: \( n \).
/** b is sorted. Return h satisfying 
  \( b[0..h] \leq v < b[h+1..] \) */

```java
static int bsearch(int[] b, int v) {
    int h = -1;
    int k = b.length;
    while (h + 1 != k) {
        int e = (h + k) / 2;
        if (b[e] <= v) h = e;
        else k = e;
    }
    return h;
}
```

Second solution: 
*Binary Search*

inv:  
\( b[0..h] \leq v < b[k..] \)

Number of iterations (always the same):  
\( \sim \log b.\text{length} \)

Therefore,  
\( \log b.\text{length} \)

array comparisons
What do we want from a definition of “runtime complexity”?

1. Distinguish among cases for large $n$, not small $n$

2. Distinguish among important cases, like
   - $n^2$ basic operations
   - $n$ basic operations
   - $\log n$ basic operations
   - 5 basic operations

3. Don’t distinguish among trivially different cases.
   - 5 or 50 operations
   - $n$, $n+2$, or $4n$ operations
Definition of $O(\ldots)$

**Formal definition:** $f(n)$ is $O(g(n))$ if there exist constants $c$ and $N$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$.

**Graphical view**

Get out far enough –for $n \geq N$– $c \cdot g(n)$ is bigger than $f(n)$. 

\[ c \cdot g(n) \]

\[ f(n) \]
What do we want from a definition of “runtime complexity”?

<table>
<thead>
<tr>
<th>Number of operations executed</th>
<th>size n of problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 ops</td>
<td>0</td>
</tr>
<tr>
<td>2+n ops</td>
<td>1</td>
</tr>
<tr>
<td>n*n ops</td>
<td>2</td>
</tr>
</tbody>
</table>

Formal definition: \( f(n) \) is \( O(g(n)) \) if there exist constants \( c \) and \( N \) such that for all \( n \geq N \), \( f(n) \leq c \cdot g(n) \)

Roughly, \( f(n) \) is \( O(g(n)) \) means that \( f(n) \) grows like \( g(n) \) or slower, to within a constant factor.
Prove that \((n^2 + n)\) is \(O(n^2)\)

Formal definition: \(f(n)\) is \(O(g(n))\) if there exist constants \(c\) and \(N\) such that for all \(n \geq N\), \(f(n) \leq c \cdot g(n)\)

Example: Prove that \((n^2 + n)\) is \(O(n^2)\)

\[
f(n) = \begin{align*}
&= \text{<definition of } f(n)> \\
&= n^2 + n \\
&\leq \text{<for } n \geq 1> \\
&= n^2 + n^2 \\
&= \text{<arith>} \\
&= 2n^2 \\
&= \text{<choose } g(n) = n^2> \\
&= 2g(n)
\end{align*}
\]

Choose \(N = 1\) and \(c = 2\)
Prove that $100n + \log n$ is $O(n)$

Formal definition: $f(n)$ is $O(g(n))$ if there exist constants $c$ and $N$ such that for all $n \geq N$, $f(n) \leq c \cdot g(n)$

$$f(n) = \text{<put in what } f(n) \text{ is>}$$

$$100n + \log n$$

$$\leq \text{<We know } \log n \leq n \text{ for } n \geq 1>$$

$$100n + n$$

$$= \text{<arith>}$$

$$101n$$

$$= \text{<g(n) = n>}$$

$$101g(n)$$

Choose $N = 1$ and $c = 101$
O(...) Examples

Let \( f(n) = 3n^2 + 6n - 7 \)
- \( f(n) \) is \( O(n^2) \)
- \( f(n) \) is \( O(n^3) \)
- \( f(n) \) is \( O(n^4) \)
- \( \ldots \)

\( p(n) = 4n \log n + 34n - 89 \)
- \( p(n) \) is \( O(n \log n) \)
- \( p(n) \) is \( O(n^2) \)

\( h(n) = 20 \cdot 2^n + 40n \)
- \( h(n) \) is \( O(2^n) \)

\( a(n) = 34 \)
- \( a(n) \) is \( O(1) \)

Only the leading term (the term that grows most rapidly) matters

If it’s \( O(n^2) \), it’s also \( O(n^3) \) etc! However, we always use the smallest one
Suppose a computer can execute 1000 operations per second; how large a problem can we solve?

<table>
<thead>
<tr>
<th>alg</th>
<th>1 second</th>
<th>1 minute</th>
<th>1 hour</th>
</tr>
</thead>
<tbody>
<tr>
<td>O(n)</td>
<td>1000</td>
<td>60,000</td>
<td>3,600,000</td>
</tr>
<tr>
<td>O(n log n)</td>
<td>140</td>
<td>4893</td>
<td>200,000</td>
</tr>
<tr>
<td>O(n^2)</td>
<td>31</td>
<td>244</td>
<td>1897</td>
</tr>
<tr>
<td>3n^2</td>
<td>18</td>
<td>144</td>
<td>1096</td>
</tr>
<tr>
<td>O(n^3)</td>
<td>10</td>
<td>39</td>
<td>153</td>
</tr>
<tr>
<td>O(2^n)</td>
<td>9</td>
<td>15</td>
<td>21</td>
</tr>
</tbody>
</table>
## Commonly Seen Time Bounds

<table>
<thead>
<tr>
<th>Time Bound</th>
<th>Characteristic</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(1)$</td>
<td>constant</td>
<td>excellent</td>
</tr>
<tr>
<td>$O(\log n)$</td>
<td>logarithmic</td>
<td>excellent</td>
</tr>
<tr>
<td>$O(n)$</td>
<td>linear</td>
<td>good</td>
</tr>
<tr>
<td>$O(n \log n)$</td>
<td>$n \log n$</td>
<td>pretty good</td>
</tr>
<tr>
<td>$O(n^2)$</td>
<td>quadratic</td>
<td>OK</td>
</tr>
<tr>
<td>$O(n^3)$</td>
<td>cubic</td>
<td>maybe OK</td>
</tr>
<tr>
<td>$O(2^n)$</td>
<td>exponential</td>
<td>too slow</td>
</tr>
</tbody>
</table>
Worst-Case/Expected-Case Bounds

May be difficult to determine time bounds for all imaginable inputs of size $n$

Simplifying assumption #4:
Determine number of steps for either
- worst-case or
- expected-case or average case

- Worst-case
  - Determine how much time is needed for the worst possible input of size $n$

- Expected-case
  - Determine how much time is needed on average for all inputs of size $n$
Use the size of the input rather than the input itself – \( n \)

Count the number of “basic steps” rather than computing exact time

Ignore multiplicative constants and small inputs (order-of, big-O)

Determine number of steps for either
- worst-case
- expected-case

These assumptions allow us to analyze algorithms effectively
Worst-Case Analysis of Searching

**Linear Search**
// return true iff v is in b
static bool find (int[] b, int v) {
    for (int x : b) {
        if (x == v) return true;
    }
    return false;
}

worst-case time: \(O(n)\)

**Binary Search**
// Return \(h\) that satisfies
// \(b[0..h] \leq v < b[h+1..]\)
static bool bsearch(int[] b, int v {  
    int h= -1;  int t= b.length;  
    while ( h != t-1 ) {  
        int e= (h+t)/2;  
        if (b[e] <= v) h= e;  
        else t= e;  
    }
}

Always takes \(\sim (\log n + 1)\) iterations.
Worst-case and expected times: \(O(\log n)\)
Comparison of linear and binary search

![Graph showing comparison of linear vs. binary search]
Comparison of linear and binary search

![Graph comparing linear and binary search]

- X-axis: Number of Items in Array
- Y-axis: Max Number of Comparisons
- Comparison of Linear Search (purple square) and Binary Search (yellow triangle)

The graph illustrates the efficiency of binary search over linear search as the size of the array increases. Binary search requires significantly fewer comparisons than linear search.
Analysis of Matrix Multiplication

Multiply $n$-by-$n$ matrices $A$ and $B$:

Convention, matrix problems measured in terms of $n$, the number of rows, columns

- Input size is really $2n^2$, not $n$
- Worst-case time: $O(n^3)$
- Expected-case time: $O(n^3)$

```plaintext
for (i = 0; i < n; i++)
    for (j = 0; j < n; j++) {
        c[i][j] = 0;
        for (k = 0; k < n; k++)
            c[i][j] += a[i][k]*b[k][j];
    }
```
Remarks

Once you get the hang of this, you can quickly zero in on what is relevant for determining asymptotic complexity

- Example: you can usually ignore everything that is not in the innermost loop. Why?

One difficulty:

- Determining runtime for recursive programs
  Depends on the depth of recursion
Why bother with runtime analysis?

Computers so fast that we can do whatever we want using simple algorithms and data structures, right?

Not really — data-structure/algorithm improvements can be a very big win

Scenario:
- A runs in $n^2$ msec
- A' runs in $n^2/10$ msec
- B runs in $10n \log n$ msec

Problem of size $n=10^3$
- A: $10^3$ sec $\approx$ 17 minutes
- A': $10^2$ sec $\approx$ 1.7 minutes
- B: $10^2$ sec $\approx$ 1.7 minutes

Problem of size $n=10^6$
- A: $10^9$ sec $\approx$ 30 years
- A': $10^8$ sec $\approx$ 3 years
- B: $2 \cdot 10^5$ sec $\approx$ 2 days

1 day $= 86,400$ sec $\approx 10^5$ sec
1,000 days $\approx$ 3 years
Algorithms for the Human Genome

Human genome
= 3.5 billion nucleotides
~ 1 Gb

@1 base-pair instruction/µsec
- $n^2 \rightarrow 388445$ years
- $n \log n \rightarrow 30.824$ hours
- $n \rightarrow 1$ hour
Limitations of Runtime Analysis

Big-O can hide a very large constant
- Example: selection
- Example: small problems

The specific problem you want to solve may not be the worst case
- Example: Simplex method for linear programming

Your program may not be run often enough to make analysis worthwhile
- Example: one-shot vs. every day
- You may be analyzing and improving the wrong part of the program
- Very common situation
- Should use profiling tools
What you need to know / be able to do

- Know the definition of \( f(n) \) is \( O(g(n)) \)
- Be able to prove that some function \( f(n) \) is \( O(g(n)) \). The simplest way is as done on two slides.
- Know worst-case and average (expected) case \( O(\ldots) \) of basic searching/sorting algorithms: linear/binary search, partition alg of Quicksort, insertion sort, selection sort, quicksort, merge sort.
- Be able to look at an algorithm and figure out its worst case \( O(\ldots) \) based on counting basic steps or things like array-element swaps/
**Goal:** Determine minimum time required to sort $n$ items

**Note:** we want worst-case, not best-case time

- Best-case doesn’t tell us much. E.g. Insertion Sort takes $O(n)$ time on already-sorted input
- Want to know worst-case time for best possible algorithm

- How can we prove anything about the best possible algorithm?

- Want to find characteristics that are common to all sorting algorithms

- Limit attention to comparison-based algorithms and try to count number of comparisons
Comparison Trees

- Comparison-based algorithms make decisions based on comparison of data elements
- Gives a comparison tree
- If algorithm fails to terminate for some input, comparison tree is infinite
- Height of comparison tree represents worst-case number of comparisons for that algorithm
- Can show: Any correct comparison-based algorithm must make at least $n \log n$ comparisons in the worst case
Say we have a correct comparison-based algorithm

Suppose we want to sort the elements in an array $b[]$

Assume the elements of $b[]$ are distinct

Any permutation of the elements is initially possible

When done, $b[]$ is sorted

But the algorithm could not have taken the same path in the comparison tree on different input permutations
How many input permutations are possible? \( n! \sim 2^{n \log n} \)

For a comparison-based sorting algorithm to be correct, it must have at least that many leaves in its comparison tree.

To have at least \( n! \sim 2^{n \log n} \) leaves, it must have height at least \( n \log n \) (since it is only binary branching, the number of nodes at most doubles at every depth)

Therefore its longest path must be of length at least \( n \log n \), and that it its worst-case running time.