Induction

Overview

• Recursion
  – a strategy for writing programs that compute in a “divide-and-conquer” fashion
  – solve a large problem by breaking it up into smaller problems of same kind
• Induction
  – a mathematical strategy for proving statements about integers (more generally, about sets that can be ordered in some fairly general ways)
• Understanding induction is useful for figuring out how to write recursive code.

Defining Functions

• It is often useful to write a given function in different ways.
  – (eg) Let $S : \text{int} \rightarrow \text{int}$ be a function where $S(n)$ is the sum of the natural numbers from 0 to n.
    $S(0) = 0, S(3) = 0 + 1 + 2 + 3 = 6$
  – One definition: iterative form
    • $S(n) = 0 + 1 + ... + n$
  – Another definition: closed-form
    • $S(n) = \frac{n(n+1)}{2}$

Equality of function definitions

• How would you prove the two definitions of $S(n)$ are equal?
  – In this case, we can use fact that terms of series form an arithmetic progression.
• Unfortunately, this is not a very general proof strategy, and it fails for more complex (and more interesting) functions.
Sum of Squares Functions

• Here is a more complex example.
  – (eg) Let \( SQ: \mathbb{N} \to \mathbb{N} \) be a function where \( SQ(n) \) is the sum of the squares of natural numbers from 0 to \( n \).
  \[ SQ(0) = 0, \quad SQ(3) = 0^2 + 1^2 + 2^2 + 3^2 = 14 \]

• One definition:
  – \( SQ(n) = 0^2 + 1^2 + \ldots + n^2 \)

• Is there a closed-form expression for \( SQ(n) \)?

Closed-form expression for \( SQ(n) \)

• Sum of natural numbers up to \( n \) was \( n(n+1)/2 \)
  which is a quadratic in \( n \).

• Inspired guess: perhaps sum of squares on natural numbers up to \( n \) is a cubic in \( n \).

• So conjecture: \( SQ(n) = a.n^3 + b.n^2 + c.n + d \) where \( a, b, c, d \) are unknown coefficients.

• How can we find the values of the four unknowns?
  – Use any 4 values of \( n \) to generate 4 linear equations, and solve.

Finding coefficients

\( SQ(n) = 0^2 + 1^2 + \ldots + n^2 = a.n^3 + b.n^2 + c.n + d \)

• Let us use \( n=0,1,2,3 \).

- \( SQ(0) = 0 = a.0 + b.0 + c.0 + d \)
- \( SQ(1) = 1 = a.1 + b.1 + c.1 + d \)
- \( SQ(2) = 5 = a.8 + b.4 + c.2 + d \)
- \( SQ(3) = 14 = a.27 + b.9 + c.3 + d \)

• Solve these 4 equations to get
  \( a = 1/3, \quad b = 1/2, \quad c = 1/6, \quad d = 0 \)

• This suggests
  \( SQ(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \)
  \( = \frac{n(n+1)(2n+1)}{6} \)

• Question: How do we know this closed-form solution is true for all values of \( n \)?
  – Remember, we only used \( n = 0,1,2,3 \) to determine these co-efficients. We do not know that the closed-form expression is valid for other values of \( n \).
• One approach:
  – Try a few values of $n$ to see if they work.
  – Try $n = 5$. $\text{SQ}(n) = 0 + 1 + 4 + 9 + 16 + 25 = 55$
  – Closed-form expression: $5 \times 6 \times 11 / 6 = 55$
  – Works!
  – Try some more values….

• Problem: we can never prove validity of closed-form solution for all values of $n$ this way since there are an infinite number of values of $n$.

To solve this problem, let us express $\text{SQ}(n)$ in another way.

\[
\text{SQ}(n) = \sum_{i=0}^{n-1} i^2 + n^2
\]

This leads to the following recursive definition of $\text{SQ}$:

\[
\begin{align*}
\text{SQ}(0) &= 0 \\
\text{SQ}(n) &= \text{SQ}(n-1) + n^2 \quad | \quad n > 0
\end{align*}
\]

Vertical bar $|$ means “whenever”

To get a feel for this definition, let us look at

\[
\begin{align*}
\text{SQ}(4) &= \text{SQ}(3) + 4^2 = \text{SQ}(2) + 3^2 + 4^2 = \text{SQ}(1) + 2^2 + 3^2 + 4^2 \\
&= \text{SQ}(0) + 1^2 + 2^2 + 3^2 + 4^2 = 0 + 1^2 + 2^2 + 3^2 + 4^2
\end{align*}
\]

Notation for recursive functions

- **Base case**
  \[
  \text{SQ}(0) = 0
  \]
  \[
  \text{SQ}(n) = \text{SQ}(n-1) + n^2 \quad | \quad n > 0
  \]

- **Recursive case**

Can we show that these two definitions of $\text{SQ}(n)$ are equal?

\[
\begin{align*}
\text{SQ}_r(0) &= 0 \\
\text{SQ}_r(n) &= \text{SQ}_r(n-1) + n^2 \quad | \quad n > 0
\end{align*}
\]

\[
\text{SQ}_c(n) = \frac{n(n+1)(2n+1)}{6}
\]

$r$: recursive

$c$: closed-form
Dominoes

- Assume equally spaced dominoes, and assume that spacing between dominoes is less than domino length.
- How would you argue that all dominoes would fall?
- Dumb argument:
  - Domino 0 falls because we push it over.
  - Domino 1 falls because domino 0 falls, domino 0 is longer than inter-domino spacing, so it knocks over domino 1.
  - Domino 2 falls because domino 1 falls, domino 1 is longer than inter-domino spacing, so it knocks over domino 2.
  - …
- Is there a more compact argument we can make?

Better argument

- Argument:
  - Domino 0 falls because we push it over (base case).
  - Assume that domino k falls over (inductive hypothesis).
  - Because domino k’s length is larger than inter-domino spacing, it will knock over domino k+1 (inductive step).
  - Because we could have picked any domino to be the kth one, we conclude that all dominoes will fall over (conclusion).
- This is an inductive argument.
- This is called weak induction. There is also strong induction (see later).
- Not only is it more compact, but it works even for an infinite number of dominoes!

Weak induction over integers

- We want to prove that some property P holds for all integers n ≥ 0.
- Inductive argument:
  - P(0): (base case) show that property P is true for 0
  - P(k): (inductive hypothesis) assume that P(k) is true for a particular integer k.
  - P(k) => P(k+1): (inductive step) show that if property P is true for integer k, it is true for integer k+1
  - P(n): (conclusion) Because we could have picked any k, this means P(n) holds for all integers n ≥ 0.

SQi(n) = SQc(n) for all n?

Define P(n) as SQi(n) = SQc(n)

Prove P(0).
Assume P(k) for particular k.
Prove P(k+1) assuming P(k).
Let \( P(n) \) be the proposition that \( SQ_r(n) = SQ_c(n) \).

Proof by induction:

**Base case**

- \( SQ_r(0) = 0 = SQ_c(0) \)

**Inductive hypothesis**

Assume \( SQ_r(k) = SQ_c(k) \)

**Inductive step**

Prove that \( P(k) \Rightarrow P(k+1) \):

\[
SQ_r(k+1) = SQ_r(k) + (k+1)^2
\]

= \( SQ_c(k) + (k+1)^2 \) (inductive hypothesis)

= \( k(k+1)(2k+1)/6 + (k+1)^2 \) (definition of \( SQ_c \))

= \( (k+1)(k+2)(2k+3)/6 \) (algebra)

= \( SQ_c(k+1) \) (definition of \( SQ_c \))

Therefore, \( SQ_r(n) = SQ_c(n) \) for all integers \( n \). Conclusion

Another example of weak induction

Prove that the sum of the first \( n \) integers is \( n(n+1)/2 \).

Let \( S(i) = 0+1+2+\ldots+i \)

Show that \( S(n) = n(n+1)/2 \).

- **Base case**: \( n=0 \)
  - \( S(0) = 0 \)
- **Inductive hypothesis**: Assume \( S(k) = k(k+1)/2 \) for a particular \( k \).
- **Inductive step**:
  - \( S(k+1) = 0+1+2+\ldots+k+(k+1) = S(k) + (k+1) \)
  - \( = k(k+1)/2 + (k+1) \)
  - \( = (k+1)(k+2)/2 \) (algebra)
  - Therefore, if result is true for \( k \), it is true for \( k+1 \).
- **Conclusion**: result follows for all integers.
- **Note**: we did not use arithmetic progressions theory.

Note on base case

- In some problems, we are interested in showing some proposition is true for integers greater than or equal to some lower bound (say \( b \)).
- Intuition: we knock over domino \( b \), and dominoes in front get knocked over. Not interested in dominoes \( 0,1,\ldots,(b-1) \).
- In general, base case in induction does not have to be 0.
- If base case is some integer \( b \), induction proves proposition for \( n = b, b+1, b+2, \ldots \).
- Does not say anything about \( n = 0, 1, \ldots, b-1 \).
Weak induction: non-zero base case

- We want to prove that some property $P$ holds for all integers $n \geq b$
- Inductive argument:
  - $P(b)$: show that property $P$ is true for integer $b$
  - $P(k)$: assume that $P(k)$ is true for a particular integer $k$.
  - $P(k) \Rightarrow P(k+1)$: show that if property $P$ is true for integer $k$, it is true for integer $k+1$
  - $P(n)$: Because we could have picked any $k$, this means $P(n)$ holds for all integers $n \geq b$.

More on induction

- In some problems, it may be tricky to determine how to set up the induction:
  - What are the dominoes?
- This is particularly true in geometric problems that can be attacked using induction.

Tiling problem

- Problem:
  - A chess-board has one square cut out of it.
  - Can the remaining board be tiled using tiles of the shape shown in the picture?
- Not obvious that we can use induction to solve this problem.

Idea

- Consider boards of size $2^n \times 2^n$ for $n = 1, 2, \ldots$.
- Base case: show that tiling is possible for $2 \times 2$ board.
- Inductive hypothesis: assume $2^n \times 2^n$ board can be tiled
- Inductive step: assuming $2^k \times 2^k$ board can be tiled, show that $2^{k+1} \times 2^{k+1}$ board can be tiled.
- Draw conclusion
  - Chess-board (8x8) is a special case of this argument
  - We have proved special case of chess-board by proving generalized problem!
Base case

- For a 2x2 board, it is trivial to tile the board regardless of which one of the four pieces has been cut.

4x4 case

- Divide 4x4 board into four 2x2 sub-boards.
- One of the four sub-boards has the missing piece.
- That sub-board can be tiled since it is a 2x2 board with a missing piece.
- Tile the center squares of the three remaining sub-boards as shown.
- This leaves 3 2x2 boards with a missing piece, which can be tiled.

8x8 case

- Divide board into 4 sub-boards and tile the center squares of the three complete sub-boards.
- The remaining portions of the 4 sub-boards can be tiled by assumption about 4x4 boards.

Inductive proof

- Claim: Any board of size $2^n \times 2^n$ with one missing square can be tiled.
- Proof: by induction.
  - Base case: (n = 1) trivial since board with missing piece is isomorphic to tile.
  - Inductive hypothesis: board of size $2^k \times 2^k$ can be tiled
  - Inductive step: consider board of size $2^{k+1} \times 2^{k+1}$
    - Divide board into four equal sub-boards of size $2^k \times 2^k$
    - One of the sub-boards has the missing piece; by inductive hypothesis, this can be tiled.
    - Tile the central squares of the remaining three sub-boards as discussed before.
    - This leaves three sub-boards with a missing square each, which can be tiled by inductive hypothesis.
  - Conclusion: any board of size $2^n \times 2^n$ with one missing square can be tiled.
When induction fails

• Sometimes, an inductive proof strategy for some proposition may fail.
• This does not necessarily mean that the proposition is wrong.
  – It just means that the inductive strategy you are trying fails.
• A different induction or a different proof strategy altogether may succeed.

Tiling example (contd.)

• Let us try a different inductive strategy which will fail.
• Proposition: any \( n \times n \) board with one missing square can be tiled.
• Problem: a \( 3 \times 3 \) board with one missing square has 8 remaining squares, but our tile has 3 squares. Tiling is impossible.
• Therefore, any attempt to give an inductive proof is proposition must fail.
• This does not say anything about the 8x8 case.

Strong induction

• We want to prove that some property \( P \) holds for all integers.
• Weak induction:
  – \( P(0) \): show that property \( P \) is true for integer 0
  – Assume \( P(k) \) for a particular integer \( k \).
  – \( P(k) \Rightarrow P(k+1) \): show that if property \( P \) is true for integer \( k \), it is true for \( k+1 \)
  – Conclude that \( P(n) \) holds for all integers \( n \).
• Strong induction:
  – \( P(0) \): show that property \( P \) is true for integer 0
  – Assume \( P(0) \) and \( P(1) \) … and \( P(k) \) for particular \( k \).
  – \( P(0) \) and \( P(1) \) and … and \( P(k) \) \( \Rightarrow P(k+1) \): show that if \( P \) is true for integers less than or equal to \( k \), it is true for \( k+1 \)
  – Conclude that \( P(n) \) holds for all integers \( n \).
• For our purpose, both proof techniques are equally powerful.

Editorial comments

• Induction is a powerful technique for proving propositions.
• We used recursive definition of functions as a step towards formulating inductive proofs.
• However, recursion is useful in its own right.
• There are closed-form expressions for sum of cubes of natural numbers, sum of fourth powers etc. (see any book on number theory).