9. Random Simulations

Topics:

The class random
Estimating probabilities
Estimating averages
More occasions to practice iteration
The random Module

Contains functions that can be used in the design of random simulations.

We will practice with these:

   random.randint(a,b)
   random.uniform(a,b)
   random.normalvariate(mu,sigma)

And as a fringe benefit, more practice with for-loops
Generating Random Integers

If $a$ and $b$ are initialized integers with $a < b$ then

$$i = \text{random.randint}(a,b)$$

assigns to $i$ a “random” integer that satisfies

$$a \leq i \leq b$$

That is, we randomly select an element from the set \{a, a+1, ..., b\} and assign it to $n$. 
What Does “Random” Mean?

import random
for k in range(1000000):
    i = random.randint(1, 6)
print i

The output would “look like” you rolled a dice one million times and recorded the outcomes.

No discernible pattern.

Roughly equal numbers of 1’s, 2’s, 3’s, 4’s, 5’s, and 6’s.
Renaming Imported Functions

```python
import random
for k in range(1000000):
    i = random.randint(1,6)
print i
```

```python
from random import randint as randi
for k in range(1000000):
    i = randi(1,6)
print i
```

Handy when the names are long or you just want to name things your way.
Random Simulation

We can use `randint` to simulate genuinely random events, e.g.,

Flip a coin one million times and record the number of heads and tails.
from random import randint as randi

N = 1000000
Heads = 0
Tails = 0
for k in range(N):
    i = randi(1,2)
    if i==1:
        Heads = Heads+1
    else:
        Tails = Tails+1
print N, Heads, Tails

The “count” variables Heads and Tails are initialized
randi returns 1 or 2
Convention: “1” is heads
Convention: “2” is tails
A Handy Short Cut

Incrementing a variable is such a common calculation that Python supports a shortcut.

These are equivalent:

\[ x += 1 \]
\[ x = x + 1 \]

\[ x += c \] is equivalent to \[ x = x + c \]
from random import randint as randi

N = 1000000
Heads = 0
Tails = 0
for k in range(N):
i = randi(1,2)
    if i==1:
        Heads+=1
    else:
        Tails+=1
print  N, Heads, Tails

The “count” variables Heads and Tails are initialized

randi returns 1 or 2

Convention: “1” is heads

Convention: “2” is tails
Different runs produce different results. This is consistent with what would happen if we physically tossed a coin one million times.
Estimating Probabilities

You roll a dice. What is the probability that the outcome is “5”?

Of course, we know the answer is 1/6. But let’s “discover” this through simulation.
from random import randint as randi
N = 6000000
count = 0
for k in range(N):
    i = randi(1,6)
    if i==5:
        count+=1
prob = float(count)/float(N)
print N, count, prob
from random import randint as randi
N = 6000000
count = 0
for k in range(N):
    i = randi(1,6)
    if i==5:
        count+=1
prob = float(count)/float(N)
print prob

N is the number of "experiments".
i is the outcome of an experiment
prob is the probability the outcome is 5
from random import randint as randi
N = 6000000
count = 0
for k in range(N):
    i = randi(1,6)
    if i==5:
        count+=1
prob = float(count)/float(N)
print prob

Output: .166837
Discovery Through Simulation

Roll three dice.

What is the probability that the three outcomes are all different?

If you know a little math, you can do this without the computer. Let’s assume that we don’t know that math.
Solution

N = 1000000

count = 0

for k in range(1, N+1):
    d1 = randi(1,6)
    d2 = randi(1,6)
    d3 = randi(1,6)
    if d1 != d2 and d2 != d3 and d3 != d1:
        count += 1
    if k % 100000 == 0:
        print k, float(count)/float(k)

Note the 3 calls to randi

Prints snapshots of the probability estimates every 100,000 trials
### Sample Output

<table>
<thead>
<tr>
<th>$k$</th>
<th>count/k</th>
</tr>
</thead>
<tbody>
<tr>
<td>100000</td>
<td>0.554080</td>
</tr>
<tr>
<td>200000</td>
<td>0.555125</td>
</tr>
<tr>
<td>300000</td>
<td>0.555443</td>
</tr>
<tr>
<td>400000</td>
<td>0.555512</td>
</tr>
<tr>
<td>500000</td>
<td>0.555882</td>
</tr>
<tr>
<td>600000</td>
<td>0.555750</td>
</tr>
<tr>
<td>700000</td>
<td>0.555901</td>
</tr>
<tr>
<td>800000</td>
<td>0.556142</td>
</tr>
<tr>
<td>900000</td>
<td>0.555841</td>
</tr>
<tr>
<td>1000000</td>
<td>0.555521</td>
</tr>
</tbody>
</table>

Note how we say “sample output” because if the script is run again, then we will get different results.

Educated guess: true prob = 5/9
Generating Random Floats

If $a$ and $b$ are initialized floats with $a < b$ then

$$x = \text{random.uniform}(a,b)$$

assigns to $x$ a “random” float that satisfies

$$a \leq x \leq b$$

The actual probability that $x$ is equal to $a$ or $b$ is basically 0.
What Does Random Mean?

Suppose

\[
\text{The probability that } L \leq \text{random.uniform}(a,b) \leq R \text{ is }
\]
\[
\frac{R-L}{b-a}
\]
Illustrate the Uniform Distribution

```
from random import uniform as randu
N = 1000000
a = 0; b = 1000; L = 100; R = 500
count = 0
for k in range(N):
    x = randu(a,b)
    if L<=x<=R:
        count+=1
prob = float(count)/float(N)
fraction = float(R-L)/float(b-a)
print prob, fraction
```

Pick a float in the interval [0,1000]. What is the prob that it is in [100,500]?
Sample Output

Estimated probability: 0.399928

\( \frac{R-L}{b-a} : 0.400000 \)
Estimating Pi Using \texttt{random.uniform}(a,b)

Idea:

Set up a game whose outcome tells us something about pi.

This problem solving strategy is called Monte Carlo. It is widely used in certain areas of science and engineering.
The Game

Throw darts at the 2x2 cyan square that is centered at (0,0).

If the dart lands in the radius-1 disk, then count that as a “hit”.

The text and diagram illustrate a Monte Carlo simulation setup for estimating the area of a circle by randomly throwing darts at a square and counting the hits within the circle.
Facts About the Game

Area of square = 4

Area of disk = $\pi$ since it has radius = 1.

Ratio of hits to throws should approximate $\pi/4$. Thus

$4\cdot \text{hits/throws} = \pi$
Example

1000 throws
776 hits

\[ \pi = 4 \times \frac{776}{1000} = 3.104 \]
When Do We Have a Hit?

The boundary of the disk is given by

\[ x^2 + y^2 = 1 \]

If \((x,y)\) is the coordinate of the dart throw, then it is inside the disk if

\[ x^2 + y^2 \leq 1 \]

is True.
from random import uniform as randu
N = 1000000
Hits = 0
for throws in range(N):
    x = randu(-1,1)
    y = randu(-1,1)
    if x**2 + y**2 <= 1:
        # Inside the unit circle
        Hits += 1
piEstU = 4*float(Hits)/float(N)
Repeatability of Experiments

In science, whenever you make a discovery through experimentation, you must provide enough details for others to repeat the experiment.

We have “discovered” pi through random simulation. How can others repeat our computation?
random.seed

What we have been calling random numbers are actually pseudo-random numbers.

They pass rigorous statistical tests so that we can use them as if they are truly random.

But they are generated by a program and are anything but random.

The seed function can be used to reset the algorithmic process that generates the pseudo random numbers.
from random import uniform as randu
from random import seed
N = 1000000; Hits = 0
seed(0)
for throws in range(N):
    x = randu(-1,1); y = randu(-1,1)
    if x**2 + y**2 <= 1:
        Hits += 1
piEstU = 4*float(Hits)/float(N)

Now we will get the same answer every time
An Example that Uses Both *randi* and *randu*

Repeat:

1. Position a square randomly in the figure window.
2. Choose its side length randomly.
3. Determine its tilt randomly.
4. Color it cyan, magenta, or, yellow randomly.
Sample Output
from random import uniform as randu
from random import randint as randi
from simpleGraphicsE import *

n = 10
MakeWindow(n,bgcolor=BLACK)
for k in range(400):
    # Draw a random colored square
    pass
ShowWindow()
Positioning the square

The figure window is built from MakeWindow(n).

We choose $x$ randomly from the interval $[-n,n]$.

We also choose $y$ randomly from the interval $[-n,n]$.

\[
x = \text{randu}(-n,n) \\
y = \text{randu}(-n,n)
\]
The Size of the square

Let’s make the squares no bigger than $n/3$ on a side.

$$s = \text{randu}(0, n/3.0)$$
The tilt of the square

Pick an integer from 0 to 45 and rotate the square that many degrees.

\[
t = \text{randi}(0,45)
\]
The Color of the square

With probability 1/3, color it cyan
With probability 1/3 color it magenta
With probability 1/3, color it yellow.

```matlab
i = randi(1,3)
if i==1:
    c = CYAN
elif i==2:
    c = MAGENTA
else:
    c = YELLOW
```
x = randu(-n,n)  The center
y = randu(-n,n)
s = randu(0,n/3.0)  The side
t = randi(0,45)  The tilt
i = randi(1,3)
if i==1:
    c = CYAN
elif i==2:
    c = MAGENTA
else:
    c = YELLOW
DrawRect(x,y,s,s,rotate=t,color=c)
Developing For-Loop Solutions

Illustrate the thinking associated with the design of for-loops

Again we illustrate the methodology of stepwise refinement.

An example...
A Game: TriStick

Pick three sticks each having a random length between zero and one.

You win if you can form a triangle whose sides are the sticks. Otherwise you lose.
Win:

Lose:
Problem

Estimate the probability of winning a game of TriStick by simulating a million games and counting the number of wins.
Pseudocode

Initialize running sum variable.
Repeat 1,000,000 times:
  Play a game of TriStick by picking the three sticks.
  If you win increment the running sum
Estimate the probability of winning
# Initialize running sum variable.
wins = 0

for n in range(1000000):
    Play the nth game of TriStick by picking the three sticks.
    If you win
        increment the running sum.

# Estimate the prob of winning
p = float(wins)/1000000
Refine the Loop Body

Play the nth game of TriStick by picking the three sticks.

If you win
  increment the running sum.

```
a = randu(0,1)
b = randu(0,1)
c = randu(0,1)
if a<b+c and b<=a+c and c<=a+b:
  wins +=1
```
Key Problem-Solving Strategy

Progress from pseudocode to Python through a sequence of refinements.

Comments have an essential role during the transitions. They remain all the way to the finished code.
Generating floats from the Normal Distribution
Generating floats from the Normal Distribution

If $\mu$ and $\sigma$ (positive) are floats, then

$$x = \text{random.normalvariate}(\mu, \sigma)$$

assigns to $x$ a “random” float sampled from the normal distribution with mean $\mu$ and standard deviation $\sigma$. 
Normal Distribution
Mean = 0, Standard Deviation = 1
Typical Situation: Test Scores

```
from random import normalvariate as randn
for k in range(450):
    x = randn(70, 7)
    print round(x)
```

This would look like a report of test scores from a class of 450 students.

The mean is approximately 70 and the standard deviation is approximately 7.
Back to Computing Pi

Using `random.uniform`

Using `random.normalvariate`
Back to Computing Pi

Using `random.uniform`

Using `random.normalvariate`
More on Standard Dev

Generate a million random numbers using

```
random.normalvariate(mu, sigma)
```

and confirm that the generated data has Mean mu and std sigma
N = 1000000; sum1 = 0; sum2 = 0
mu = 70; sigma = 7
for k in range(N):
    x = randn(mu,sigma)
    sum1 += x
    sum2 += (x-mu)**2
ApproxMean = float(sum1)/float(N)
ApproxSTD = sqrt(float(sum2)/float(N))

Sample Output: 70.007824 6.998934
Final Reminder

randi, randu, and randn are RENAMED versions of

random.randint
random.uniform
random.normalvariate