Equivalence of types

Assume, $A$ and $B$ are types.

When is $\Gamma \vdash t \in A$ equivalent to $\Gamma \vdash t \in B$ (for all $\Gamma$)?

When is $\Gamma; x : A; \Delta[x] \vdash C[x]$ equivalent to $\Gamma; x : B; \Delta[x] \vdash C[x]$?

The answer is: when $\vdash A =_e B$ — but when is it provable?

Ex: $T /\!\!/ (E_1 & E_2)$ and $(T /\!\!/ E_1) \cap (T /\!\!/ E_2)$

Let $T_1 := T /\!\!/ (E_1 & E_2)$ and $T_2 := (T /\!\!/ E_1) \cap (T /\!\!/ E_2)$.

Then $T_1 \subseteq T_2$ is provable. But $T_2 \subseteq T_1$ is not!

On the other hand, we can prove that the membership is the same:

$T_1 /\!\!/ True =_e T_2 /\!\!/ True =_e T /\!\!/ True$

and that the equality is the same:

$x : T; y : T \vdash (x = y \in T_1) \iff (x = y \in T_2)$
Equality Structures: Informal Explanation

\[ \vdash A \subseteq B \text{ is the same as } \vdash \lambda x.x \in (A \rightarrow B). \text{ Let's ask a more general question: when does } \vdash f \in (A \rightarrow B). \]

We know that \( f \) has to respect:

- membership relation
- equality relation
- \( \sim \) equality relation
- ??? relation

If \( f \) has to respect some “functionality relation” \( X \), then \( A \subseteq B \) implies that \( B \) has “more” \( X \) than \( A \) and \( A =_e B \) implies that their \( X \) is the same!

Functionality Structures in \( \cap \) and \( // \)

\[ x : A; y : A; u : E[x,y] \vdash x = y \in B \]
\[ x : (A//E) \vdash x \in B \]
means that \( A//E \) is the smallest type containing all functionality structures of \( A \) and equivalence relation \( E \).

\[ x : C \vdash x \in A \quad x : C \vdash x \in B \]
\[ x : C \vdash x \in (A \cap B) \]
means that \( A \cap B \) contains all functionality structure that both \( A \) and \( B \) contain.

Equality Rules — I

\[ \Gamma \vdash A \subseteq B \quad \Gamma \vdash B \subseteq (A//True) \]
\[ \Gamma \vdash B \subseteq (u,v : A//((u = v \in B))) \quad (\text{Antiquotient}) \]

(Antiquotient) is sufficient to prove

\[ x : ((A//E_1) \cap (A//E_2)) \vdash x \in (A//(E_1 \land E_2)) \]
(provided all quotients are well-formed types)

Equality Rules — II

An equivalent formulation of (Antiquotient):

\[ \Gamma \vdash X \subseteq A \subseteq (X//True) \]
\[ \Gamma \vdash X \subseteq B \subseteq (X//True) \]
\[ \Gamma ; x_1 : X ; x_2 : X \vdash (x_1 = x_2 \in A) \Leftrightarrow (x_1 = x_2 \in B) \]
\[ \Gamma \vdash A =_e B \quad (\text{EqEq}) \]

A simpler (and more general) version (Kopylov & Nogin):

\[ \Gamma ; x_1 : X ; x_2 : X \vdash (x_1 = x_2 \in A) \Rightarrow (x_1 = x_2 \in B) \]
\[ \Gamma ; x : A ; \llbracket x \in X \rrbracket \vdash x \in B \quad (\text{EqMemEq}) \]