1. Definitions

Let trading take place continuously over $[0, \tau]$.

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $(\mathcal{F}_t : t \in [0, \tau])$ generated by $n$ independent Brownian motions $\{W_1(t), \ldots, W_n(t)\}$.

Let $E(.)$ denote expectation with respect to $\mathbb{Q}$.

All traded assets are default free. Consider zero coupon bonds with time $t$ price $P(t,T)$.

Forward rates are implicitly defined by

$$P(t,T) = \exp\left\{-\int_t^T f(t,u) du \right\}.$$

The forward rate $f(t,T)$ corresponds to the riskless lending/borrowing rate one could contract at time $t$ for period $[T, T+dt]$.

The spot rate is defined by

$$r(t) = f(t,t).$$

A money market account is defined by

$$B(t) = \exp\left\{\int_0^t r(u) du \right\}.$$

2. Arbitrage Pricing Theorems

There are two fundamental theorems of asset pricing.

**Theorem 1:** No arbitrage

Given trading in the zeros, there is no arbitrage if and only if there exists a probability measure $\tilde{\mathbb{Q}}$ such that:

$$\tilde{\mathbb{Q}}(A) = 0 \iff \mathbb{Q}(A) = 0 \quad \text{for all } A \in \mathcal{F} \text{ and}$$
P(t, T)/B(t) is a \( \tilde{Q} \) martingale.

**Theorem 2: Complete markets**

Given trading in the zeros and no arbitrage, then markets are complete if and only if \( \tilde{Q} \) is unique.

By complete we mean that any interest rate derivative can be synthetically constructed using a dynamic trading strategy in the zero coupon bonds and the money market account.

**Lemma (Risk Neutral Valuation):**

Given no arbitrage and complete markets. Let \( x(T) \) be a \( F_t \) measurable random variable which is squared integrable under \( \tilde{Q} \). Then, the time \( t \) cost of generating \( x(T) \) is \( \tilde{E}_t (x(T)/B(T))B(t) \).

This lemma provides a technique for computing prices of interest rate derivatives.

**3. Bond Dynamics**

Assume that forward rates follow the stochastic process:

\[
f(t, T) = f(0, T) + \int_0^T \alpha(v, T)dv + \sum_{i=1}^n \sigma_i(v, T) dW_i(v)
\]

(technical conditions omitted).

Applying Ito’s lemma one can show that (using the definition of \( P(t, T) \) in terms of \( f(t, u) \)) that

\[
P(t, T) = P(0, T) \exp \left\{ \int_0^t [r(v) + b(v, T)]dv - (1/2) \sum_{i=1}^n \int_0^t a_i(v, T)^2 dv \right\}
\]

\[
+ \sum_{i=1}^n \int_0^t a_i(v, T) dW_i(v)
\]

where
Applying the above theorems on asset pricing, assuming the market is arbitrage free and complete implies that 

There exists $\phi_i(t)$ for $i = 1, \ldots, n$ such that 

$$
\alpha(t, T) = -\sum_{i=1}^{n} \sigma_i(t, T)[\phi_i(t) - \int_{t}^{T} \sigma_i(t, v)dv].
$$

This is the contribution of Heath, Jarrow and Morton.

The $\phi_i(t)$ are called the market prices of risk.

Using Girsanov’s theorem, one can show that under $\tilde{Q}$ the dynamics of $P(t,T)$ are:

$$
P(t, T) = P(0, T) \exp\left\{ \int_{0}^{t} r(v)dv - (1/2)\sum_{i=1}^{n} \int_{0}^{t} a_i(v, T)^2dv + \sum_{i=1}^{n} \int_{0}^{t} a_i(v, T)d\tilde{W}_i(v) \right\}
$$

where 

$$
\tilde{W}_i(t) = W_i(t) - \int_{0}^{t} \phi_i(v)dv.
$$

Combined with the lemma in section 2, we can now price interest rate derivatives.

4. Example (Extended Vasicek)

This section considers an important example. It is a one-factor model.

$$
df(t, T) = \alpha(t, T)dt + \sigma e^{-(\lambda/2)(T-t)}dW(t)
$$

No arbitrage and market completeness implies there exists a $\phi(t)$ such that
\[
\alpha(t, T) = -\sigma e^{-((\lambda/2)T-t)} \phi(t) - 2(\sigma/\lambda)^2 e^{-((\lambda/2)(T-t))} (e^{-((\lambda/2)(T-t))} - 1)
\]

Under \( \tilde{Q} \) the forward rate process is:

\[
df(t, T) = -2(\sigma/\lambda)^2 e^{-((\lambda/2)(T-t))} (e^{-((\lambda/2)(T-t))} - 1)dt + \sigma e^{-((\lambda/2)(T-t))} d\tilde{W}(t)
\]

Consider a European call option on the bond \( P(t, T) \) with an exercise price of \( K \) and maturity date \( t^* \) where \( 0 <= t <= t^* <= T \).

Let \( C(t) \) denote the call’s price, then

\[
C(t) = E_t(\max(P(t^*, T) - K, 0) / B(t^*))B(t)
\]

\[
= P(t, T)\Phi(h) - KP(t, t^*)\Phi(h - q)
\]

and

\[
h = [\log(P(t, T) / KP(t, t^*)) + (1/2)q^2] / q
\]

\[
q^2 = (4\sigma^2 / \lambda^2)(e^{-((\lambda/2)T)} - e^{-((\lambda/2)t^*)})^2 (e^{\lambda t^*} - e^{\lambda t}).
\]

Remarks:

1. Other interest rate derivatives can easily be priced.
2. The evolution in \( f(t, T) \) can be approximated via a lattice and numerical procedures akin to the binomial model used.
3. P.D.E. procedures can also be invoked.