1. Derivation of the Black-Scholes partial differential equation

As before, let

\[ dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \]

Suppose we consider a derivative security whose value is:

\[ V(t) = V(t, S(t)) \]

Remark: This could be a call or put option (American or European).

By Ito’s lemma, we can write

\[ dV(t) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 \]

Suppose we form a portfolio over \([t, t+dt]\) consisting of the stock and derivative, with portfolio value

\[ P(t) = V(t) - NS(t) \quad \text{where} \]

\[ N = \frac{\partial V}{\partial S} \]

Then, \( dP(t) = dV(t) - NdS(t) \).

Plugging in the value for \( dV(t) \), we obtain that

\[ dP(t) = \frac{\partial V}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 \]

But \( dS^2 = \sigma^2 S^2 dt \). Substitution yields

\[ dP(t) = \frac{\partial V}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 dt \]

Note that this term is deterministic. To avoid the existence of arbitrage, it must be that

\[ dP(t) = rP(t)dt \]

Making the substitutions, we get that
\[ rV(t)dt - r\frac{\partial V}{\partial S}Sdt = \frac{\partial V}{\partial t} dt + (1/2)\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt. \]

The dt terms cancel, to yield the Black-Scholes p.d.e.

Remarks:
1. The term involving \( \frac{\partial V}{\partial t} \) is going forward in time, if one defined \((T-t)\) as the argument, then \(t\) is replaced by \((T-t)\) and a minus sign appears in front of this term.

2. This p.d.e holds for any derivative. To make it well-formed, we need to specify certain boundary conditions. For example, for the European call, \( V(T,S(T)) = \max[S(T)-K,0] \), for the European put, \( V(T,S(T)) = \max[K-S(T),0] \).

3. This derivation is heuristic because it is done for differentials (keeping \(N\) fixed). It can be done more correctly using stochastic integrals and self-financing trading strategies.

2. **Risk Management**

   To control the risk of an option portfolio, three considerations apply.

1. **Delta Neutral Portfolios**
   According to the derivation of the Black-Scholes formula, if trading took place continuously, then a delta neutral portfolio would be risk free over \([t, t+dt]\). But, trading can only be done discretely.

2. **Gamma Neutral Portfolios**
   If trading takes place continuously, then \(dS\) and \(dS^2\) are of the same magnitude, hence need to hedge movements in \(dS^2\). This is a gamma neutral position. Sometimes called jump or gap risk.

3. **Vega Neutral Portfolios**
   The volatility is assumed constant and known. If this is not true, then the model is misspecified. Hence, a vega neutral portfolio is trying to eliminate model error.