Unless stated otherwise we make the following market structure assumptions: frictionless markets, competitive markets, and no arbitrage opportunities.

1. Suppose you own an American put option with expiration date \( T = 1 \) and \( S(0) = K \) (i.e., the stock price at time 0 is the strike price). Further, suppose \( S(t) < K, \forall 0 < t \leq T \).
   Is it possible to have \( t_* = 0 \), where \( t_* \) is the optimal time to exercise (i.e., the payoff at time \( t = T \) is maximum at \( t = t_* \), \( \forall 0 < t \leq T \))? Explain.

2. Suppose you own an American put option, \( P(0 : T, K) \). Let \( S(T) = 0 \) and \( S(t) > 0, \forall t \leq T \). Does \( t_* = T \) (necessarily)? Explain.

3. Verify (i.e., do the algebra) the formula in the middle of page 5 (Lecture 1 handout):
   \[ c(0) = \left[ \pi (SU - K) + (1 - \pi)0 \right]/R, \text{ where } \pi = [R - D]/[U - D]. \]

4. Verify (i.e., derive) the binomial solution at the bottom of p.5 (Lecture 1 handout) for the case where there are three time periods, \( T = 0, 1, 2 \).

5. Consider the following European call option problem:
   - Strike price = 100
   - Interest rate = .1
   - Dividend rate = .025
   - Volatility = .3
   - Time to maturity = 1

Using the MATLAB financial toolbox and the Black-Scholes functions, and the function \texttt{mesh}, construct surfaces covering the regions \( S = 1 : 10 : 251, \ t = [.5 : .5 : 12]/12 \) where \( t \) is time and \( S \) is the price of the underlying. There should be seven surfaces in total, one for option price and one for each of the “greeks”.
   (a) Explain why each of the graphs makes “financial sense”.

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(b) Compute the implied volatilities of 10 of the data points above and using the \texttt{norm} function, verify the correctness of the implied volatility matrix.

6. Let $D$ be the present value of a dividend paid by a stock share during the lifetime of corresponding call and put options. Prove the following put-call parity equation:
\begin{equation}
c + D + Ke^{-rT} = p + S
\end{equation}
where $K$ is the strike price, $r$ is the riskless interest rate, and the options begin at time 0 and expire at time $T$.

7. It is possible to establish put-call inequalities for the American option case. Assume $K$ is the strike price, $r$ is the riskless interest rate, the options begin at time 0, expire at time $T$, and there are no dividends.

(a) Prove, using put-call parity for European options, that
\begin{equation}
C - P < S - Ke^{-rT}.
\end{equation}

(b) Prove that
\begin{equation}
C - P > S - K.
\end{equation}

Hint: Consider two portfolios: Portfolio $A$ consists of a European call and $K$ dollars, Portfolio $B$ consists of an American put and 1 stock share.

8. It follows from question 7. that
\begin{equation}
c - S + Ke^{-rT} < P < c - S + K.
\end{equation}
Therefore, given the price of the European call option it is possible to bound, from above and below, the American put option price.

Display the the upper and lower bounds, using the MATLAB function \texttt{blsprice}, to determine the European call price, on the following data: $S = 100, r = .1, T = .25, \sigma = .5, D = 0$ and $K = [95 : .1 : 105]$. Finally, use the MATLAB function \texttt{binprice} to illustrate (1) for this data.

9. Consider the example discussed in class (on Wednesday, Feb 4.).

(a) Suppose that the total wealth constraint continues to be $17,000, and that we would like delta, gamma, and vega to be small but shorting is not allowed. Formulate this problem as a minimization problem with nonnegativity constraints and a single linear equality constraint.
(b) Suppose that we have a total wealth constraint ($17,000) and we wish to hedge against delta (only). Shorting is permissible. Show that there are an infinite number of feasible portfolios (based on the given 4 options); obtain an expression for all solutions. What happens if shorting is not allowed?

(c) If $r_i$ is the expected rate of return (per dollar invested in option $i$). Formulate an optimization problem that yields a delta-neutral investment strategy that maximizes expected profit subject to a wealth constraint of ($17,000). Shorting is not allowed.