On Random Sampling Auctions for Digital Goods

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Outline

Problem Definition

- Originally proposed by Goldberg & Hartline.
- We have a single type of good with unlimited supply
- There are n bidders with bids $v_1 \geq \cdots \geq v_n$.
- We want a revenue-maximizing incentive compatible auction.
- We have no prior information on distributions.
- Benchmark is the optimal uniform price auction: $\max_{\lambda \geq 2} \lambda \cdot \mathbf{v}_\lambda$

Random Sampling Optimal Price Auction

- The mechanism:
  - Partition the bids to two groups A and B uniformly at random.
  - Compute the optimal unform price in each group and offer it to the other group.

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Conjecture

The revenue of RSOP is at least $\frac{1}{4}OPT$. i.e. RSOP is 4-competitive.

RSOP Example

- Suppose the bids are $\{7, 6, 5, 1\}$.
- After random partitioning of the bids, $A = \{6, 1\}$ and $B = \{7, 5\}$.
- We offer 6 to B and 5 to A.
- we get a revenue of 11 while OPT is 15.

Conjecture

The worst case performance of RSOP is when bids are $\{1, \frac{1}{2}\}$.

Previous/Present Results

- Goldberg & Hartline (2001) : $\frac{OPT}{RSOP} < 7600$
- Feige et al (2005) : $\frac{OPT}{RSOP} < 15$
- Our result (2008) : $\frac{OPT}{RSOP} < 4.68$

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Theorem

The competitive ratio of RSOP is (λ is the index of the winning bid in OPT) (e.g. in $\{7, 6, 5, 1\}$, $\lambda = 3$):

$$\begin{cases} < 4.68 & \lambda < 6 \\ < 4 & \lambda > 6 \\ < 3.3 & \lambda \to \infty \end{cases} \tag{1}$$

Assumptions

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- v_1 is always in B and we only consider the revenue obtained from set B.

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A lowerbound on RSOP revenue when $\lambda > 10$

- A dynamic programming method for computing the lower bound given the λ.
- A second method which is independent of λ but assumes it is large (i.e. > 5000) and uses Chernoff bound.

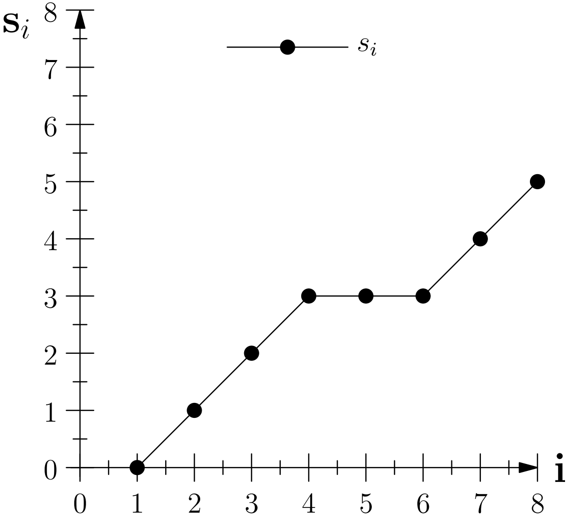
Random Partition

Example

$$A = \{v_2, v_3, v_4\}$$
$$B = \{v_1, v_5, v_7\}$$

Definition

$$S_i = \#\{v_j | v_j \in A, j \leq i\}$$



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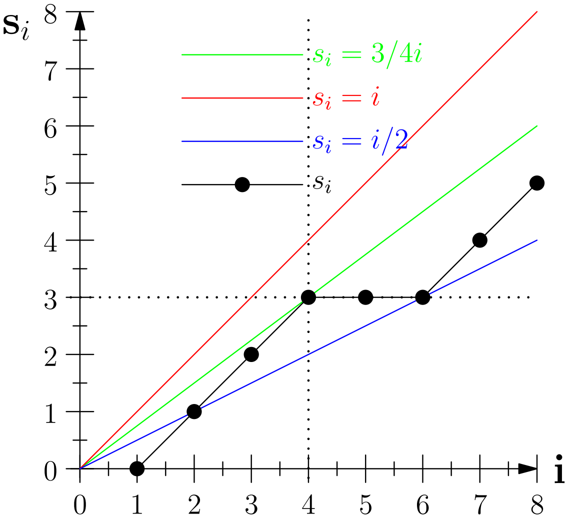


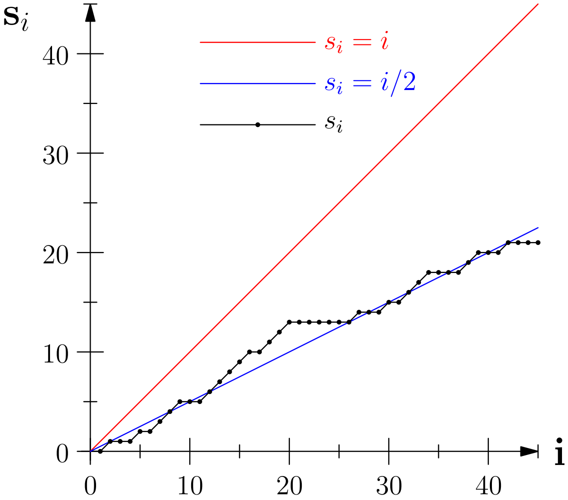
Random Partition

Observation

$$\lim_{i\to\infty} \frac{S_i}{i} \to \frac{1}{2}$$

or

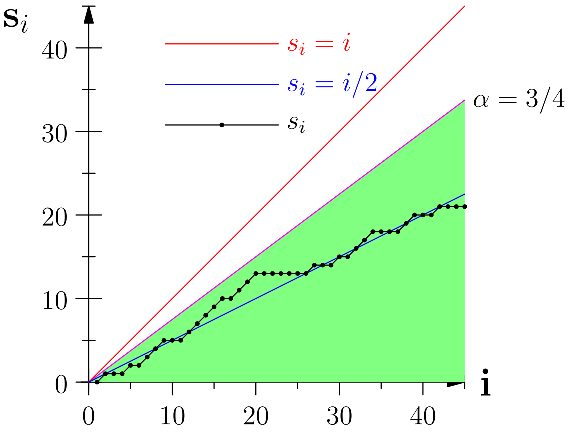
$$\lim_{i\to\infty} Pr\left[\frac{S_i}{i} < \frac{1}{2} - \epsilon\right] \to 0$$



Worst Profit Ratio

Observations

$$\forall j : \quad \frac{S_j}{j} < \alpha$$



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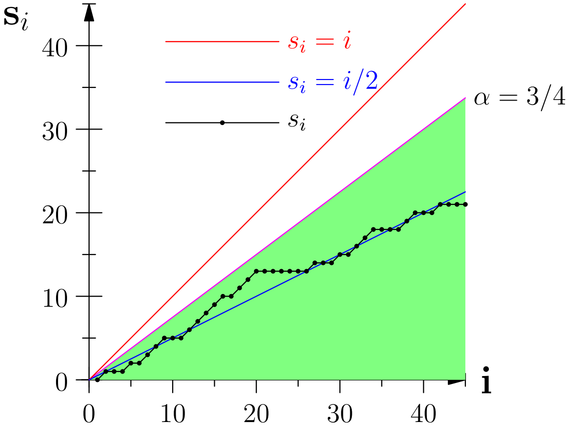
$$\frac{Prof(B)}{Prof(A)} \geq \frac{1-\alpha}{\alpha}$$



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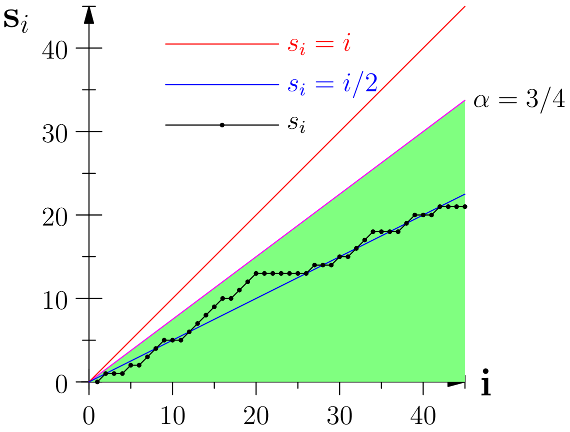

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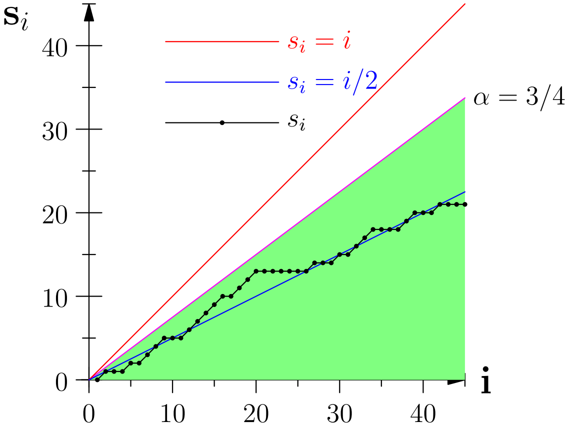
$$Z = \min_i \frac{i - S_i}{S_i}$$

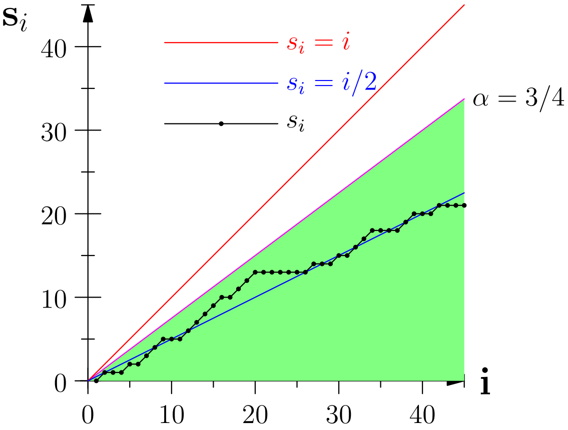
$$Prof(B) \geq E[Z \frac{S_\lambda}{\lambda}]$$



α-Event

Definition ($\mathcal{E}_\alpha$ event)

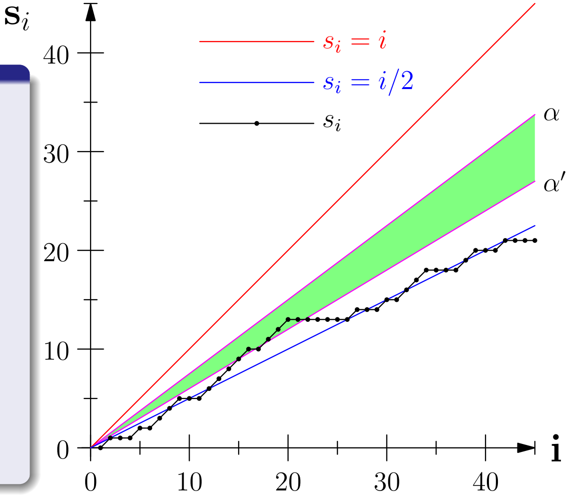
$$\mathcal{E}_\alpha : \quad \forall j : \frac{S_j}{j} \leq \alpha$$



α-Events

$$\mathcal{E}_{[\alpha',\alpha]} = \mathcal{E}_{\alpha} - \mathcal{E}_{\alpha'}$$

$$Z|\mathcal{E}_{[\alpha',\alpha]} \geq \frac{1-\alpha}{\alpha}$$



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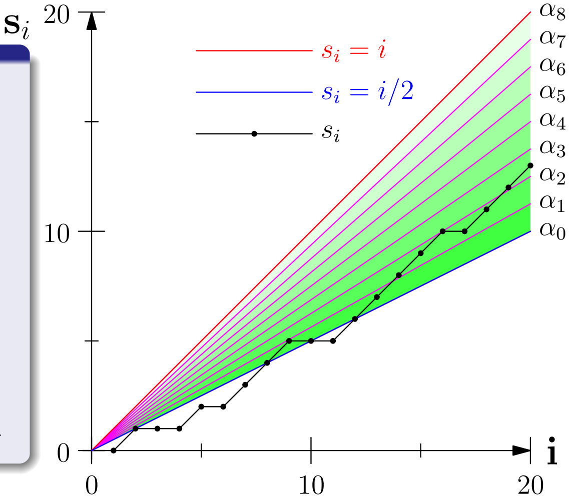

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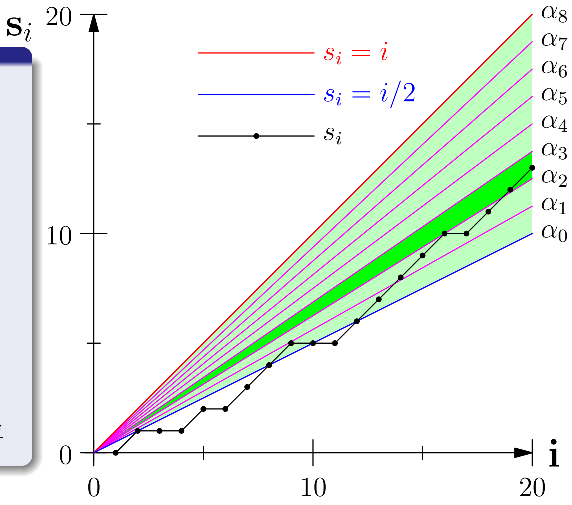

Computing $E[Z]$

Lemma

The worst ratio of profit of set B to profit of set B can be computed using the following:

$$E[Z] = \sum_i Pr[\mathcal{E}_{[\alpha_{i-1},\alpha_i]}]\frac{1-\alpha_i}{\alpha_i}$$
$$= \sum_i (Pr[\mathcal{E}_{\alpha_i}] - Pr[\mathcal{E}_{\alpha_{i-1}}])\frac{1-\alpha_i}{\alpha_i}$$

The Dynamic Program for computing $P[\mathcal{E}_\alpha]$

Definition

Let $P_\alpha(k,j)$ be the probability that for any $1 \le i \le k$, at most α fraction of the $v_1, \ldots, v_i$ are in A and exactly j of $v_1, \cdots, v_k$ are in A. Let $P_\alpha(k) = \sum_{j=0}^{k} P_\alpha(k,j)$, then $Pr[\mathcal{E}_\alpha] = P_\alpha(\infty)$

Dynamic Program for computing $P_\alpha(k,j)$

$$P_\alpha(k,j) = \begin{cases} 0 & j > \alpha k \\ 1 & j = k = 0 \\ 1/2 P_\alpha(k-1,j) & j = 0, k > 0 \\ 1/2 P_\alpha(k-1,j) + 1/2 P_\alpha(k-1,j-1) & 0 < j < \alpha k \end{cases}$$

When λ is large

Claim

As λ increases, the correlation between S_λ/λ and Z decreases so we can separate them.

$$Prof(b) \geq E\left[\frac{S_\lambda}{\lambda} Z\right]$$
$$\approx E[\frac{S_\lambda}{\lambda}]E[Z]$$
$$\approx \frac{1}{2}E[Z]$$

We use a variant of Chernoff bound to bound the error caused by separating the two terms.

The Dynamic Program for $E[\frac{S_\lambda}{\lambda} Z]$

Definition

Let $R_\alpha(k, j)$ the expected value of lowerbound for profit of set A conditioned and multiplied by the probability that for any $1 \le i \le k$, at most α fraction of the $v_1, \ldots, v_i$ are in A and exactly j of $v_1, \cdots, v_k$ are in A.

Dynamic Program for computing $R_\alpha(k, j)$

$$R_\alpha(k,j) = \begin{cases} 0 & j = 0 \text{ or } j > \alpha k \\ 1/2 R_\alpha(k-1, j) + 1/2 R_\alpha(k-1, j-1) & 0 < j \le \alpha k \\ \frac{j}{\lambda} P_\alpha(k-1, j) & k = \lambda \end{cases}$$

The Dynamic Program for $E[\frac{S_\lambda}{\lambda} Z]$ (Continued)

Dynamic Program for computing $E\left[\frac{S_\lambda}{\lambda} Z \middle| \mathcal{E}_\alpha\right]$

$$R_\alpha(k) = \sum_{i=0}^{j} R_\alpha(k, j)$$

$$R_\alpha(\infty) = E\left[\frac{S_\lambda}{\lambda} \middle| \mathcal{E}_\alpha\right] Pr\left[\mathcal{E}_\alpha\right]$$

$$E[\frac{S_\lambda}{\lambda} Z] = \sum_i (R_{\alpha_i} - R_{\alpha_{i-1}}) \frac{1 - \alpha_i}{\alpha_i}$$

An upperbound on the revenue of RSOP with large λ

Theorem

For any given λ, there is a set of bids with λ being the index of the winning price and such that RSOP does not get a revenue of more than $3/8$.

The equal revenue instances

Definition

An Equal Revenue Instance with n bids consists of the bids $\{1, \frac{1}{2}, \cdots, \frac{1}{n}\}$.

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Observation

In an equal revenue instance, the price offered from each set is the worst price for the other set.

The equal revenue instances, RSOP'

Definition ($RSOP'$)

It is the same as RSOP except that when set A is empty, the price that is offered from A to B is v_n instead of 0. The difference between the revenue of RSOP and RSOP' is $1/2^n$.

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The revenue of RSOP' on an equal revenue instance with $n+1$ bids is less than that with n bids. The proof is by induction.

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Fact

Revenue of RSOP for equal revenue instances with $n \leq 10$ is at most $\frac{1}{2.65}$.

Revenue

RSOP revenue (basic lowerbound)

λ	$E[RSOP]$	Competitive-Ratio
2	0.125148	7.99
3	0.166930	5.99
4	0.192439	5.20
5	0.209222	4.78
6	0.221407	4.52
7	0.230605	4.34
8	0.237862	4.20
9	0.243764	4.10
10	0.248647	4.02
15	0.264398	3.78
20	0.273005	3.66
100	0.296993	3.37
500	0.302792	3.30
1000	0.303560	3.29
1500	0.303818	3.29
2000	0.303949	3.29

Based on dynamic programming up to $n = 5000$ and then Chernoff bound.

RSOP revenue (secondary lowerbound)

λ	$E[RSOP]$	Competitive-Ratio
2	0.2138	4.68
3	0.2178	4.59
4	0.238	4.20
5	0.243	4.11
6	0.2503	3.99
7	0.2545	3.93
8	0.2602	3.84
9	0.2627	3.81
10	0.2669	3.75

Questions?

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