On Random Sampling Auctions for Digital Goods

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Outline

1. Introduction
2. Basic Lowerbound on RSOP revenue
3. An upperbound on RSOP revenue
Problem Definition

- Originally proposed by Goldberg & Hartline.
- We have a single type of good with unlimited supply.
- There are $n$ bidders with bids $v_1 \geq \cdots \geq v_n$.
- We want a revenue-maximizing incentive compatible auction.
- We have no prior information on distributions.
- Benchmark is the optimal uniform price auction:
  $$\max_{\lambda \geq 2} \lambda \cdot v_\lambda$$
Random Sampling Optimal Price Auction

The mechanism:
- Partition the bids to two groups $A$ and $B$ uniformly at random.
- Compute the optimal uniform price in each group and offer it to the other group.
Random Sampling Optimal Price Auction

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- RSOP is incentive compatible.
Random Sampling Optimal Price Auction

- The mechanism:
  - Partition the bids to two groups $A$ and $B$ uniformly at random.
  - Compute the optimal uniform price in each group and offer it to the other group.

- RSOP is incentive compatible.

Conjecture

The revenue of RSOP is at least $\frac{1}{4} \text{OPT}$. i.e. RSOP is 4-competitive.
RSOP Example

- Suppose the bids are \( \{7, 6, 5, 1\} \).
- After random partitioning of the bids, \( A = \{6, 1\} \) and \( B = \{7, 5\} \).
- We offer 6 to \( B \) and 5 to \( A \).
- We get a revenue of 11 while OPT is 15.

**Conjecture**

*The worst case performance of RSOP is when bids are \( \{1, \frac{1}{2}\} \).*
Previous/Present Results

- Goldberg & Hartline (2001): $\frac{OPT}{RSOP} < 7600$
- Feige et al (2005): $\frac{OPT}{RSOP} < 15$
- Our result (2008): $\frac{OPT}{RSOP} < 4.68$
Previous/Present Results

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- Our result (2008): \( \frac{OPT}{RSOP} < 4.68 \)

Theorem

The competitive ratio of RSOP is (\( \lambda \) is the index of the winning bid in OPT) (e.g. in \{7, 6, 5, 1\}, \( \lambda = 3 \)):

\[
\begin{align*}
< 4.68 & \quad \lambda < 6 \\
< 4 & \quad \lambda > 6 \\
< 3.3 & \quad \lambda \to \infty
\end{align*}
\] (1)
We have an infinite number of bids (i.e. \( n = \infty \)), by adding 0’s.
Assumptions

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- \( OPT = 1 \), by scaling all the bids.
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- $OPT = 1$, by scaling all the bids.
- $v_1$ is always in $B$ and we only consider the revenue obtained from set $B$. 
A lowerbound on RSOP revenue when $\lambda > 10$

- A dynamic programming method for computing the lower bound given the $\lambda$. 

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A lowerbound on RSOP revenue when $\lambda > 10$

- A dynamic programming method for computing the lower bound given the $\lambda$.
- A second method which is independent of $\lambda$ but assumes it is large (i.e. $> 5000$) and uses Chernoff bound.
Random Partition

**Example**

\[ A = \{v_2, v_3, v_4\} \]
\[ B = \{v_1, v_5, v_7\} \]

**Definition**

\[ S_i = \#\{v_j | v_j \in A, j \leq i\} \]
Random Partition

**Example**

\[ A = \{v_2, v_3, v_4\} \]
\[ B = \{v_1, v_5, v_7\} \]

**Definition**

\[ S_i = \#\{v_j|v_j \in A, j \leq i\} \]
**Random Partition**

**Observation**

\[
\lim_{i \to \infty} \frac{S_i}{i} \to \frac{1}{2}
\]

or

\[
\lim_{i \to \infty} \Pr \left[ \frac{S_i}{i} < \frac{1}{2} - \epsilon \right] \to 0
\]
Worst Profit Ratio

Observations

∀ j : \( \frac{S_j}{j} < \alpha \)
Worst Profit Ratio

Observations

∀j : \( \frac{S_j}{j} < \alpha \)

\( \frac{\text{Prof}(B)}{\text{Prof}(A)} \geq \frac{1 - \alpha}{\alpha} \)
Worst Profit Ratio

Observations

∀j : \( \frac{S_j}{j} < \alpha \)

\[
\frac{\text{Prof}(B)}{\text{Prof}(A)} \geq \frac{1 - \alpha}{\alpha}
\]

\[
\text{Prof}(A) \geq \frac{S_{\lambda}}{\lambda}
\]
Observations

\[ \forall j : \frac{S_j}{S_j} < \alpha \]

\[ \frac{\text{Prof}(B)}{\text{Prof}(A)} \geq \frac{1 - \alpha}{\alpha} \]

\[ \text{Prof}(A) \geq \frac{S_\lambda}{\lambda} \]

\[ Z = \min_i \frac{i - S_i}{S_i} \]

\[ \text{Prof}(B) \geq E[Z \frac{S_\lambda}{\lambda}] \]
**α-Event**

**Definition** ($\mathcal{E}_\alpha$ event)

$$\mathcal{E}_\alpha : \forall j : \frac{S_j}{j} \leq \alpha$$

The diagram illustrates the relationship between $S_i$ and $i$ for different values of $\alpha$. The line $s_i = i$ is shown with a red line, $s_i = i/2$ with a blue line, and $s_i$ with a green line for $\alpha = 3/4$. The graph highlights the behavior of $S_i$ as $i$ increases for different thresholds of $\alpha$. The definition of $\alpha$-Event is visualized through these lines and the corresponding values of $S_i$.
$\alpha$-Events

$E_{[\alpha', \alpha]} = E_{\alpha} - E_{\alpha'}$

$Z | E_{[\alpha', \alpha]} \geq \frac{1 - \alpha}{\alpha}$
### $\alpha$-Events

- $\mathcal{E}_{[\alpha', \alpha]} = \mathcal{E}_\alpha - \mathcal{E}_{\alpha'}$

- $Z|_{\mathcal{E}_{[\alpha', \alpha]}} \geq \frac{1 - \alpha}{\alpha}$

- $Z = \sum_i \Pr[\mathcal{E}_{[\alpha_i, \alpha_{i+1}]}] \frac{1 - \alpha_i}{\alpha_i}$

- $Z = \sum_i (\Pr[\mathcal{E}_{\alpha_{i+1}}] - \Pr[\mathcal{E}_{\alpha_i}]) \frac{1 - \alpha_i}{\alpha_i}$
**α-Events**

\[ \mathcal{E}_{[\alpha', \alpha]} = \mathcal{E}_\alpha - \mathcal{E}_{\alpha'} \]

\[ Z | \mathcal{E}_{[\alpha', \alpha]} \geq \frac{1 - \alpha}{\alpha} \]

\[ Z = \sum_i \Pr[\mathcal{E}_{\alpha_i, \alpha_{i+1}}] \frac{1 - \alpha_i}{\alpha_i} \]

\[ Z = \sum_i (\Pr[\mathcal{E}_{\alpha_{i+1}}] - \Pr[\mathcal{E}_{\alpha_i}]) \frac{1 - \alpha_i}{\alpha_i} \]
Lemma

The worst ratio of profit of set $B$ to profit of set $B$ can be computed using the following:

$$E[Z] = \sum_i Pr[E[\alpha_{i-1}, \alpha_i]] \frac{1 - \alpha_i}{\alpha_i}$$

$$= \sum_i (Pr[E_{\alpha_i}] - Pr[E_{\alpha_{i-1}}]) \frac{1 - \alpha_i}{\alpha_i}$$
The Dynamic Program for computing $P[\mathcal{E}_\alpha]$ 

**Definition**

Let $P_\alpha(k, j)$ be the probability that for any $1 \leq i \leq k$, at most $\alpha$ fraction of the $v_1, \ldots, v_i$ are in $A$ and exactly $j$ of $v_1, \ldots, v_k$ are in $A$. Let $P_\alpha(k) = \sum_{j=0}^{k} P_\alpha(k, j)$, then $Pr[\mathcal{E}_\alpha] = P_\alpha(\infty)$

**Dynamic Program for computing $P_\alpha(k, j)$**

$$P_\alpha(k, j) = \begin{cases} 
0 & j > \alpha k \\
1 & j = k = 0 \\
1/2P_\alpha(k - 1, j) & j = 0, k > 0 \\
1/2P_\alpha(k - 1, j) + 1/2P_\alpha(k - 1, j - 1) & 0 < j < \alpha k 
\end{cases}$$
When \( \lambda \) is large

Claim

As \( \lambda \) increases, the correlation between \( S_\lambda/\lambda \) and \( Z \) decreases so we can separate them.

\[
\text{Prof}(b) \geq E \left[ \frac{S_\lambda}{\lambda} Z \right]
\]

\[
\approx E\left[ \frac{S_\lambda}{\lambda} \right] E[Z]
\]

\[
\approx \frac{1}{2} E[Z]
\]

We use a variant of Chernoff bound to bound the error caused by separating the two terms.
The Dynamic Program for $E\left[\frac{S_{\alpha}}{\lambda} Z\right]$

**Definition**

Let $R_{\alpha}(k,j)$ the expected value of lowerbound for profit of set $A$ conditioned and multiplied by the probability that for any $1 \leq i \leq k$, at most $\alpha$ fraction of the $v_1, \ldots, v_i$ are in $A$ and exactly $j$ of $v_1, \ldots, v_k$ are in $A$.

**Dynamic Program for computing $R_{\alpha}(k,j)$**

\[
R_{\alpha}(k,j) = \begin{cases} 
0 & j = 0 \text{ or } j > \alpha k \\
1/2R_{\alpha}(k-1,j) + 1/2R_{\alpha}(k-1,j-1) & 0 < j \leq \alpha k \\
j \lambda P_{\alpha}(k-1,j) & k = \lambda
\end{cases}
\]
Dynamic Program for computing $E\left[ \frac{S_\lambda}{\lambda} Z \mid \mathcal{E}_\alpha \right]$:

$$R_\alpha(k) = \sum_{i=0}^{j} R_\alpha(k, j)$$

$$R_\alpha(\infty) = E \left[ \frac{S_\lambda}{\lambda} \mid \mathcal{E}_\alpha \right] Pr [\mathcal{E}_\alpha]$$

$$E\left[ \frac{S_\lambda}{\lambda} Z \right] = \sum_{i} \left( R_{\alpha_i} - R_{\alpha_{i-1}} \right) \frac{1 - \alpha_i}{\alpha_i}$$
An upperbound on the revenue of RSOP with large $\lambda$

**Theorem**

For any given $\lambda$, there is a set of bids with $\lambda$ being the index of the winning price and such that RSOP does not get a revenue of more than $3/8$. 
The equal revenue instances

Definition
An Equal Revenue Instance with $n$ bids consists of the bids
\( \{1, \frac{1}{2}, \ldots, \frac{1}{n}\} \).
The equal revenue instances

Definition
An **Equal Revenue Instance** with \( n \) bids consists of the bids \( \{1, \frac{1}{2}, \ldots, \frac{1}{n}\} \).

Observation
In an equal revenue instance, the price offered from each set is the worst price for the other set.
The equal revenue instances, RSOP’

**Definition (RSOP’)**

It is the same as RSOP except that when set $A$ is empty, the price that is offered from $A$ to $B$ is $v_n$ instead of 0. The difference between the revenue of RSOP and RSOP’ is $1/2^n$. 
The equal revenue instances, RSOP’

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The revenue of RSOP’ on an equal revenue instance with $n + 1$ bids is less than that with $n$ bids. The proof is by induction.
The equal revenue instances, RSOP’

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**Fact**

Revenue of RSOP for equal revenue instances with $n \leq 10$ is at most $\frac{1}{2.65}$.
### RSOP revenue (basic lowerbound)

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$E[RSOP]$</th>
<th>Competitive-Ratio</th>
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<tbody>
<tr>
<td>2</td>
<td>0.125148</td>
<td>7.99</td>
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<tr>
<td>3</td>
<td>0.166930</td>
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<td>4</td>
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Based on dynamic programming up to $n = 5000$ and then Chernoff bound.
RSOP revenue (secondary lowerbound)

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<th>$E[RSOP]$</th>
<th>Competitive-Ratio</th>
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<td>0.2669</td>
<td>3.75</td>
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Introduction
Basic Lowerbound on RSOP revenue
An upperbound on RSOP revenue

Questions?

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