On Random Sampling Auctions for Digital Goods

Saeed Alaei    Azarakhsh Malekian    Aravind Srinivasan
Outline

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2 Introduction

3 Basic Lowerbound on RSOP revenue

4 An upperbound on RSOP revenue
Truthful Auctions (Brief Review)

What is a **truthful** auction?

Any auction where disclosing the private information is a weakly dominant strategy for bidders (e.g. second price auction).
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Why truthful auctions?

- Simpler to analyze (efficiency, revenue, ...)
- Simpler for bidders (no strategic behavior)
- Smaller space of possible mechanisms for auction designer to look at
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Theorem (Revelation Principle)
Any non-truthful mechanism that has a Nash Equilibrium can be converted to a truthful mechanism.
Vickrey-Clarke-Groves (VCG)

Abstract Model

- A set of outcomes $A = \{a_1, \cdots, a_m\}$.
- A set of bidders $N$, each having valuation $v_i(a)$ for each $a \in A$.
- The utility of bidder $i$ is $u_i = v_i(a) - p_i$ where $p_i$ is payment.
- The utility of the auctioneer is $u_0 = \sum_{i \in N} p_i$.
- The social welfare is $U_N(a) = \sum_{i \in N} u_i(a) = \sum_{i \in N} v_i(a)$. 

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Random Sampling Auctions . . .
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Definition (VCG Mechanism)

1. Ask bidders to submit their private valuations \( v_i \).
2. Choose the outcome \( a^* = \arg \max_a U_N(a) \).
3. Set the payment of each bidder \( i \) to \( p_i = (\max_a U_N \setminus \{i\}(a)) - (U_N(a^*) - v_i(a^*)) \).
Truthful Mechanisms

Abstract Model

- Each bidder has a multidimensional type vector $t_i$.
- Let $x_{t-i}(b_i)$ be the allocation function of mechanism for $i$.
- Let $u_i(x_{t-i}(b_i), t_i)$ be the utility of advertiser $i$ if she submits $b_i$ while her true type is $t_i$.

Theorem (Characterization of Truthful Mechanisms)

An allocation mechanism $x$ is truthful if the following payment is well-defined:

$$p_i = \int_0^{t_i} \nabla_{t_i} u_i(x_{t-i}(b_i), b_i) \cdot db_i$$
Truthful Mechanisms for Single Parameter setting

Abstract Model

- Each bidder has a single parameter type $v_i$, the value for the item.
- Let $x_{v_i}(b)$ be the allocation function of mechanism for $i$.
- Let $v_i x_{v_i}(b)$ be the utility of advertiser $i$ if she submits $b$ while her true type is $v_i$.

Theorem (Characterization of Truthful Mechanisms)

An allocation mechanism $x$ is truthful if it is monotone (increasing) and its payment is:

$$p_i = \int_0^{v_i} b \frac{\partial}{\partial b} x_{v_i}(b) db = v_i x_{v_i}(v_i) - \int_0^{v_i} x_{v_i}(b) db$$
Myerson Optimal Auction

Model

We have a single item to sell. Bidders have unit demand and pure private valuations and bidder $i$’s type, $v_i$, is drawn independently from the distribution $F_i(v)$. We are looking for an auction that maximizes revenue in expectation.

Theorem (Optimal Bayesian Auction)

For each bidder, compute the virtual valuation, $\phi_i(v_i)$. Give the item to the bidder $i$ with the highest positive virtual valuation and charge her equal to $\phi_i^{-1}(\phi_j(v_j))$ where $\phi_j(v_j)$ is the second highest virtual valuation or $\phi_i^{-1}(0)$ if all others are negative.

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$ (1)
Digital Goods Auction, Problem Definition

- Originally proposed by Goldberg & Hartline.
- We have a single type of good with unlimited supply.
- There are $n$ bidders with bids $v_1 \geq \cdots \geq v_n$.
- We want a revenue-maximizing incentive compatible auction.
- We have no prior information on distributions.
- Benchmark is the optimal uniform price auction:
  $$\max_{\lambda \geq 2} \lambda \cdot v_\lambda$$
Random Sampling Optimal Price Auction

- The mechanism:
  - Partition the bids to two groups $A$ and $B$ uniformly at random.
  - Compute the optimal uniform price in each group and offer it to the other group.

RSOP is incentive compatible.

Conjecture

The revenue of RSOP is at least $\frac{1}{4}$ $\text{OPT}$. i.e. RSOP is $4$-competitive.

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**Conjecture**

*The revenue of RSOP is at least $\frac{1}{4} \text{OPT}$. i.e. RSOP is 4-competitive.*
RSOP Example

- Suppose the bids are \{7, 6, 5, 1\}.
- After random partitioning of the bids, \(A = \{6, 1\}\) and \(B = \{7, 5\}\).
- We offer 6 to \(B\) and 5 to \(A\).
- We get a revenue of 11 while OPT is 15.

Conjecture

*The worst case performance of RSOP is when bids are \(\{1, \frac{1}{2}\}\).*
Previous/Present Results

- Goldberg & Hartline (2001): $\frac{OPT}{RSOP} < 7600$
- Feige et al (2005): $\frac{OPT}{RSOP} < 15$
- Our result (2008): $\frac{OPT}{RSOP} < 4.68$
Previous/Present Results

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- Our result (2008): $\frac{\text{OPT}}{\text{RSOP}} < 4.68$

**Theorem**

The competitive ratio of RSOP is (\(\lambda\) is the index of the winning bid in OPT) (e.g. in \(\{7, 6, 5, 1\}\), \(\lambda = 3\)):

\[
\begin{align*}
&< 4.68 & \lambda < 6 \\
&< 4 & \lambda > 6 \\
&< 3.3 & \lambda \to \infty
\end{align*}
\]
Assumptions

- We have an infinite number of bids (i.e. $n = \infty$), by adding 0’s.
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- $OPT = 1$, by scaling all the bids.
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- We have an infinite number of bids (i.e. \( n = \infty \)), by adding 0’s.
- \( OPT = 1 \), by scaling all the bids.
- \( v_1 \) is always in \( B \) and we only consider the revenue obtained from set \( B \).
A lowerbound on RSOP revenue when \( \lambda > 10 \)

- A dynamic programming method for computing the lower bound given the \( \lambda \).
A lowerbound on RSOP revenue when $\lambda > 10$

- A dynamic programming method for computing the lower bound given the $\lambda$.
- A second method which is independent of $\lambda$ but assumes it is large (i.e. $> 5000$) and uses Chernoff bound.
Random Partition

Example

\[ A = \{v_2, v_3, v_4, v_7, v_8\} \]
\[ B = \{v_1, v_5, v_6\} \]

Definition

\[ S_i = \#\{v_j \mid v_j \in A, j \leq i\} \]
Random Partition

Example

\[ A = \{v_2, v_3, v_4, v_7, v_8\} \]
\[ B = \{v_1, v_5, v_6\} \]

Definition

\[ S_i = \#\{v_j \mid v_j \in A, j \leq i\} \]
Random Partition

Observation

\[
\lim_{i \to \infty} \frac{S_i}{i} \rightarrow \frac{1}{2}
\]

or

\[
\lim_{i \to \infty} \Pr \left[ \frac{S_i}{i} < \frac{1}{2} - \epsilon \right] \rightarrow 0
\]
Worst Profit Ratio

Observations

∀j : \( \frac{S_j}{j} < \alpha \)
Worst Profit Ratio

Observations

∀j : \( \frac{S_j}{j} < \alpha \)

\[
\frac{\text{Prof}(B)}{\text{Prof}(A)} \geq \frac{1 - \alpha}{\alpha}
\]
**Basic Lowerbound on RSOP revenue**

**An upperbound on RSOP revenue**

**Worst Profit Ratio**

### Observations

\[ \forall j : \frac{S_j}{j} < \alpha \]

\[ \frac{\text{Prof}(B)}{\text{Prof}(A)} \geq \frac{1 - \alpha}{\alpha} \]

\[ \text{Prof}(A) \geq \frac{S_\lambda}{\lambda} \]

\[
\begin{align*}
S_i & \geq 40 \\
30 & \geq 30 \\
20 & \geq 20 \\
10 & \geq 10 \\
0 & \geq 0
\end{align*}
\]

\[ s_i = i \]

\[ s_i = i/2 \]

\[ \alpha = 3/4 \]

\[ S_i \]

\[ i \]

\[ 0 \quad 10 \quad 20 \quad 30 \quad 40 \]

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Worst Profit Ratio

Observations

\[ \forall j : \frac{S_j}{j} < \alpha \]

\[
\frac{\text{Prof}(B)}{\text{Prof}(A)} \geq \frac{1 - \alpha}{\alpha}
\]

\[
\text{Prof}(A) \geq \frac{S_\lambda}{\lambda}
\]

\[
Z = \min_i \frac{i - S_i}{S_i}
\]

\[
\text{Prof}(B) \geq E[Z \frac{S_\lambda}{\lambda}]
\]
\( \alpha \)-Event

Definition (\( \mathcal{E}_\alpha \) event)

\[ \mathcal{E}_\alpha : \forall j : \frac{S_j}{j} \leq \alpha \]
\( \alpha \)-Events

\[ E_{[\alpha', \alpha]} = E_\alpha - E_{\alpha'} \]

\[ Z | E_{[\alpha', \alpha]} \geq \frac{1 - \alpha}{\alpha} \]
**α-Events**

\[ \mathcal{E}_{[\alpha', \alpha]} = \mathcal{E}_\alpha - \mathcal{E}_{\alpha'} \]

\[ Z | \mathcal{E}_{[\alpha', \alpha]} \geq \frac{1 - \alpha}{\alpha} \]

\[ E[Z] \geq \sum_i Pr[\mathcal{E}_{[\alpha_i, \alpha_{i+1}]}] \frac{1 - \alpha_i}{\alpha_i} \]

\[ E[Z] \geq \sum_i (Pr[\mathcal{E}_{\alpha_{i+1}}] - Pr[\mathcal{E}_{\alpha_i}]) \frac{1 - \alpha_i}{\alpha_i} \]
**α-Events**

\[ E_{[\alpha', \alpha]} = E_{\alpha} - E_{\alpha'} \]

\[ Z | E_{[\alpha', \alpha]} \geq \frac{1 - \alpha}{\alpha} \]

\[ E[Z] \geq \sum_{i} \Pr[E_{[\alpha_i, \alpha_{i+1}]}] \frac{1 - \alpha_i}{\alpha_i} \]

\[ E[Z] \geq \sum_{i} (\Pr[E_{\alpha_{i+1}}] - \Pr[E_{\alpha_i}]) \frac{1 - \alpha_i}{\alpha_i} \]
Lemma

The worst ratio of profit of set B to profit of set A can be computed using the following:

\[
E[Z] = \sum_{i} Pr[\mathcal{E}_{[\alpha_{i-1}, \alpha_i]}] \frac{1 - \alpha_i}{\alpha_i} \\
= \sum_{i} (Pr[\mathcal{E}_{\alpha_i}] - Pr[\mathcal{E}_{\alpha_{i-1}}]) \frac{1 - \alpha_i}{\alpha_i}
\]
The Dynamic Program for computing $P[\mathcal{E}_\alpha]$}

**Definition**

Let $P_\alpha(k, j)$ be the probability that for any $1 \leq i \leq k$, at most $\alpha$ fraction of the $v_1, \ldots, v_i$ are in $A$ and exactly $j$ of $v_1, \ldots, v_k$ are in $A$. Let $P_\alpha(k) = \sum_{j=0}^{k} P_\alpha(k, j)$, then $\Pr[\mathcal{E}_\alpha] = P_\alpha(\infty)$

**Dynamic Program for computing $P_\alpha(k, j)$**

$$
P_\alpha(k, j) = 
\begin{cases} 
0 & j > \alpha k \\
1 & j = k = 0 \\
1/2P_\alpha(k - 1, j) & j = 0, k > 0 \\
1/2P_\alpha(k - 1, j) + 1/2P_\alpha(k - 1, j - 1) & 0 < j < \alpha k
\end{cases}
$$
When $\lambda$ is large

Claim

As $\lambda$ increases, the correlation between $S_\lambda/\lambda$ and $Z$ decreases so we can separate them.

$$
Prof(b) \geq E \left[ \frac{S_\lambda}{\lambda} Z \right]
\approx E\left[ \frac{S_\lambda}{\lambda} \right] E[Z]
\approx \frac{1}{2} E[Z]
$$

We use a variant of Chernoff bound to bound the error caused by separating the two terms.
The Dynamic Program for $E\left[\frac{S_{\alpha}}{\lambda} Z\right]$

**Definition**

Let $R_{\alpha}(k, j)$ the expected value of lowerbound for profit of set $A$ conditioned and multiplied by the probability that for any $1 \leq i \leq k$, at most $\alpha$ fraction of the $v_1, \ldots, v_i$ are in $A$ and exactly $j$ of $v_1, \ldots, v_k$ are in $A$.

**Dynamic Program for computing $R_{\alpha}(k, j)$**

$$R_{\alpha}(k, j) = \begin{cases} 0 & j = 0 \text{ or } j > \alpha k \\ 1/2 R_{\alpha}(k - 1, j) + 1/2 R_{\alpha}(k - 1, j - 1) & 0 < j \leq \alpha k \\ \frac{j}{\lambda} P_{\alpha}(k - 1, j) & k = \lambda \end{cases}$$
The Dynamic Program for $E\left[\frac{S_\lambda}{\lambda} Z\right]$ (Continued)

Dynamic Program for computing $E\left[\frac{S_\lambda}{\lambda} Z \mid \mathcal{E}_\alpha\right]$

\[
R_\alpha(k) = \sum_{i=0}^{j} R_\alpha(k, j)
\]

\[
R_\alpha(\infty) = E\left[\frac{S_\lambda}{\lambda} \mid \mathcal{E}_\alpha\right] \Pr[\mathcal{E}_\alpha]
\]

\[
E\left[\frac{S_\lambda}{\lambda} Z\right] = \sum_{i} (R_{\alpha_i} - R_{\alpha_{i-1}}) \frac{1 - \alpha_i}{\alpha_i}
\]
An upperbound on the revenue of RSOP with large $\lambda$

**Theorem**

For any given $\lambda$, there is a set of bids with $\lambda$ being the index of the winning price and such that RSOP does not get a revenue of more than $3/8$. 
The equal revenue instances

Definition

An Equal Revenue Instance with $n$ bids consists of the bids $\{1, \frac{1}{2}, \ldots, \frac{1}{n}\}$. 
The equal revenue instances

Definition
An **Equal Revenue Instance** with \( n \) bids consists of the bids \( \{1, \frac{1}{2}, \ldots, \frac{1}{n}\} \).

Observation
In an equal revenue instance, the price offered from each set is the worst price for the other set.
The equal revenue instances, RSOP’

**Definition (RSOP’)**

It is the same as RSOP except that when set $A$ is empty, the price that is offered from $A$ to $B$ is $v_n$ instead of 0. The difference between the revenue of RSOP and RSOP’ is $1/2^n$. 
The equal revenue instances, RSOP’

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Claim

The revenue of RSOP’ on an equal revenue instance with $n+1$ bids is less than that with $n$ bids. The proof is by induction.
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Fact

Revenue of RSOP for equal revenue instances with $n \leq 10$ is at most $1/2.65$. 
### RSOP revenue (basic lowerbound)

<table>
<thead>
<tr>
<th>λ</th>
<th>$E[RSOP]$</th>
<th>Competitive-Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.125148</td>
<td>7.99</td>
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<td>3</td>
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<td>2000</td>
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</table>

Based on dynamic programming up to $n = 5000$ and then Chernoff bound.

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**Revenue**
### RSOP revenue (secondary lowerbound)

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<tr>
<th>$\lambda$</th>
<th>$E[RSOP]$</th>
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Questions?