

Bayesian Combinatorial Auctions: Expanding Single Buyer Mechanisms to Many Buyers *

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Abstract

We present a general framework for approximately reducing the mechanism design problem for multiple agents to single agent sub problems in the context of Bayesian combinatorial auctions. Our framework can be applied to any setting which roughly satisfies the following assumptions: (i) agents' types are distributed independently (not necessarily identically), (ii) objective function is additively separable over the agents, and (iii) there are no inter agent constraints except for the supply constraints (i.e., that the total allocation of each item should not exceed the supply). Our framework is general in the sense that it makes no direct assumption about agents' valuations, type distributions, or single agent constraints (e.g., budget, incentive compatibility, etc).

We present two generic multi agent mechanisms which use single agent mechanisms as black boxes. If an α -approximate single agent mechanism is available for each agent and assuming no agent ever demands more than $\frac{1}{k}$ of all units of each item, our generic multi agent mechanisms are $\gamma_k \alpha$ -approximation of the optimal multi agent mechanism, where γ_k is a constant which is at least $1 - \frac{1}{\sqrt{k+3}}$.

As a byproduct of our construction, we present a generalization of prophet inequalities where both gambler and prophet are allowed to pick k numbers each to receive a reward equal to their sum. Finally, we use our framework to obtain multi agent mechanisms with improved approximation factor for several settings from the literature.

1 Introduction

The main challenge of stochastic optimization arises from the fact that all instances in the support of the distribution are relevant for the objective and this support is exponentially big in the size of problem. This paper aims to address this challenge by providing a general decomposition technique for assignment problems on independently distributed inputs where the objective is additively separable over the inputs. The main challenge faced by such a decomposition approach is that the feasibility constraint of an assignment problem introduces correlation in the outcome of the optimal solution. In mechanism design problems, such constraints are typically the supply constraints. For

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example, when agents are independent, a revenue maximizing seller with unlimited supply can decompose the problem over the agents and optimize for each agent independently. However, in the presence of supply constraints, a direct decomposition is not possible. Our decomposition technique can be roughly described as the following: (i) compute a mechanism that satisfies the supply constraints only in expectation (*ex ante*); the optimization problem corresponding to such a mechanism can be fully decomposed over the set of agents; (ii) convert the mechanism from the previous step to another mechanism that satisfies the supply constraint at every instance while only losing a small fraction of the objective.

We restrict our discussion to Bayesian combinatorial auctions. We are interested in mechanisms that allocate a set of heterogeneous items of limited supply to a set of agents in order to maximize the expected value of a certain objective function which is additively separable over the agents (e.g., welfare, revenue, etc). The agents' types are assumed to be distributed independently according to publicly known priors. We defer the formal statement of our assumptions to §2.

The following are the main challenges in designing mechanisms for multiple agents.

- (I) The decisions made for each agent should be coordinated with the decisions made for other agents.
- (II) The decisions made for each agent should be (approximately) optimal.

Making coordinated optimal decisions for multiple agents is challenging as it requires optimizing over the joint type space of all agents, the size of which grows exponentially in the number of agents. The second challenge is usually due to incentive compatibility (IC) constraints, specially in multi dimensional settings where these constraints cannot be encoded compactly. In this paper, we mostly address the first challenge by providing a framework for approximately decomposing the mechanism design problem for multiple agents to sub problems dealing with each agent individually.

1.1 Results

We present a general framework for approximately reducing the mechanism design problem for multiple agents to single agent sub problems. Our framework can be summarized as follows. We start by relaxing the supply constraints to hold only in expectation, i.e., we consider mechanisms for which the *ex ante expected* number of allocated units of each item does not exceed the supply of that item. Note that “*ex ante*” means the expectation is over randomness of input (i.e., randomness of agents' types). We show that an optimal mechanism for the relaxed problem can be obtained by independently running n single agent mechanisms, where each single agent mechanism is an optimal single agent mechanism subject to an *ex ante* probabilistic supply constraint. In particular, we show that if an α -approximate single agent mechanism is available for each agent, then running the single agent mechanisms simultaneously and independently yields an α -approximate mechanism for the relaxed multi agent problem. We then present two methods for converting a mechanism for the relaxed problem to a mechanism for the original problem while losing only a small constant factor in the approximation. Each method results in a generic multi agent mechanism that uses single agent mechanisms as black boxes ¹. Each of our two generic multi agent mechanisms requires its own set of technical assumptions about the single agent mechanisms in order to be applicable. In fact, most of the single agent mechanisms presented in this paper satisfy the requirements of only one of the two generic multi agent mechanisms. In the first mechanism, agents are served sequentially by running, for each agent, the corresponding single agent mechanism. However, some of the items

¹Note that the single agent mechanisms do not need to be identical, which allows us to accommodate different classes of agents in the same multi agent mechanism.

are precluded at random from the early agents in order to ensure that late agents get the same chance of being offered those items. We ensure that the ex ante probability of preclusion for each item is equalized over all agents regardless of the order in which the agents are served (i.e., the order in which the agents are served does not affect their chance of being able to buy an item). In the second mechanism, we run all of the single agent mechanisms simultaneously and then modify the outcomes by deallocating some units of the overallocated items at random while adjusting the payments respectively. We ensure that the ex ante probability of deallocation is equalized among all units of each item and therefore simultaneously minimized for all agents.

We introduce a toy problem, *the magician's problem*, along with a near-optimal solution for it in §4. The solution of this problem serves as the main ingredient of our multi agent mechanisms. As a byproduct, we present an improved prophet inequality in a setting when gambler and prophet are each allowed to select up to k numbers (for a given constant k) and receive a reward equal to the sum of the selected numbers. We present a randomized strategy for gambler that guarantees in expectation at least $1 - \frac{1}{\sqrt{k+3}}$ fraction of the reward of a prophet. A generalization of the magician's problem along with a near optimal solution is presented in §7.

In §6, we use our framework to obtain multi agent mechanisms with improved approximation factor for several settings from the literature. For each setting, we only present a single agent mechanism that satisfies the requirements of one of our generic multi agent mechanisms. Table 1 lists several settings for which we obtain multi agent mechanisms with improved approximation factor compared to previous best known approximations in the same class. Note that the approximation factors shown in the table correspond to the resulting multi agent mechanisms and are equal to the approximation factor of the corresponding single agent mechanisms multiplied by γ_k , where k is the minimum number of units available of each item, and γ_k is a constant which is at least $1 - \frac{1}{\sqrt{k+3}}$. Observe that γ_k is at least $\frac{1}{2}$ (i.e., for $k = 1$) and approaches 1 as $k \rightarrow \infty$.

Setting	Approx	Space of Mechanisms	Ref
single item(multi unit), unit demand, budget constraint, revenue maximization	γ_k	item pricing with budget randomization	§6.1
multi item(heterogeneous), unit demand, product distribution, revenue maximization	$\frac{1}{2}\gamma_k$	deterministic	§6.2
multi item(heterogeneous), additive valuations, product distribution, budget constraint, revenue maximization	$(1 - \frac{1}{e})\gamma_k$	item pricing with budget randomization	§6.3
multi item(heterogeneous), additive valuations, correlated distribution with polynomial number of types, budget constraint, matroid constrains, revenue or welfare maximization	γ_k	randomized (BIC)	§6.4

Table 1: Summary of mechanisms obtained using the current framework.

1.2 Related Work

The literature on Bayesian mechanism design can be categorized based on (i) the objective (e.g., maximizing welfare, maximizing revenue, etc), (ii) the dimension of the type space of each agent (e.g., multi dimensional versus single dimensional), (iii) the cardinality of the type space of each agent (e.g., continuous, discrete with polynomially many types, or discrete with exponentially many types).

Single dimensional types. Most of the related work from the computer science literature in this category is focused on approximating the VCG mechanism for welfare maximization and/or

approximating the Myerson's mechanism Myerson [1981] for revenue maximization (e.g., Bulow and Roberts [1989], Babaioff et al. [2006], Blumrosen and Holenstein [2008], Hartline and Roughgarden [2009], Dhangwatnotai et al. [2010], Chakraborty et al. [2010], Yan [2011]). Many of them consider mechanisms that have simple implementation and are computationally efficient. For welfare maximization in single dimensional settings, Hartline and Lucier [2010] give a black box reduction from mechanism design to algorithmic design. For revenue maximization with budget constrained agents with i.i.d distribution having monotone hazard rates, Pai and Vohra [2008] compute an optimal BIC mechanism by discretizing agents' type spaces and solving a linear program whose size is polynomial in the number of discrete types of each agent. However, we should point that the main goal of that paper is to identify qualitative behavior of an optimal mechanism rather than computing one. In §6.1, for a similar setting except with possibly non identical (asymmetric) agents, we compute a γ_k -approximate item pricing mechanism by solving a small convex program whose size is polynomial in the number of agents only.

Multidimensional types (continuous or exponentially large support). Most of the related work in this category either consider type distributions with compact representation (e.g., product distributions) or assume the availability of an oracle that can sample from the distribution and/or answer certain types of queries about the distribution. In either case, the size of the support of the distribution is often exponential or infinite (e.g., continuous) in the size of the input. For welfare maximization, Hartline et al. [2011] provide a quite general black box reduction from mechanism design to algorithm design which subsumes the earlier work of Hartline and Lucier [2010]. For revenue maximization, Chawla et al. [2010] have proposed several sequential posted pricing mechanisms for various settings with different types of matroid feasibility constraints. These mechanisms have simple implementation and approximate the revenue of the optimal mechanism. For unit-demand agents with regular and independently distributed valuations, Chawla et al. [2010] have presented a sequential posted pricing mechanism that obtains in expectation at least $\frac{1}{6.75}$ -fraction of the revenue of the optimal deterministic mechanism. For the same problem, we present a $\frac{1}{2}$ -approximate single agent mechanism in §6 which yields an improved $\frac{1}{2}\gamma_k$ -approximate sequential posted pricing multi agent mechanism (recall that k is the minimum number of units available of each item, and γ_k is a constant which is at least $1 - \frac{1}{\sqrt{k+3}}$). For a single agent, Cai and Daskalakis [2011] provide an algorithm for computing a $1 - \epsilon$ approximation of the optimal deterministic mechanism in quasipolynomial time. Their algorithm runs in polynomial time when all distributions have monotone hazard rate, but their running time is too high to be practical (e.g., more than $O(m^{100})$ for $\epsilon = \frac{1}{2}$). Daskalakis and Weinberg [2012] provide an algorithm for computing a $1 - \epsilon$ approximation of the optimal BIC mechanism when either (a) agents are symmetric and there is a constant number of items, or (b) there is a constant number of agents and distributions are symmetric with respect to items. For budget constrained agents, several approximation mechanisms have been proposed by Chawla et al. [2011] for maximizing welfare/revenue in various single and/or multi dimensional settings.

Multidimensional types (polynomial size support). Most of the related work in this category assume the type distribution is explicitly specified as part of the input (i.e., by enumerating the probabilities of all types), hence the support of the distribution being of polynomial size with respect to the input size. For combinatorial auctions with additive/correlated valuations with budget and demand constraints, Bhattacharya et al. [2010] proposed a $\frac{1}{4}$ -approximate BIC mechanism for revenue maximization (similarly for welfare maximization) in the form of an all-pay auction. In §6.4, we present an improved γ_k -approximate BIC mechanism for the same problem assuming k is

the minimum number of units available of each item. Recall that $\gamma_k \geq 1 - \frac{1}{\sqrt{k+3}}$ which is at least $\frac{1}{2}$ and approaches 1 as $k \rightarrow \infty$. For the same setting, Bhattacharya et al. [2010] also proposed sequential posted pricing mechanisms with $O(1)$ approximation factors. We improve on that in §6.3 by presenting a $(1 - \frac{1}{e})\gamma_k$ -approximate sequential posted pricing mechanism. There are several recent papers which present polynomial time computable optimal mechanisms in the presence of extra allocation constraints and when each agent has additive and possibly correlated valuations. Cai et al. [2012a] presented an optimal mechanism when the only allocation constraints are the supply constraints. Alaei et al. [2012] showed how to compute an optimal mechanism when the space of feasible allocations is captured by a polymatroid. Recently, Cai et al. [2012b] presented an optimal mechanism when the space of feasible allocations is captured by an arbitrary but polynomial time solvable polytope². In a follow up work, Cai et al. [2013] generalized the previous result when the polytope of feasible allocations is only approximately solvable in polynomial time. Note that the last four results inevitably rely on the ellipsoid method to compute an optimal mechanism as they need to solve an optimization problem with exponentially many inter agent linear constraints. On the other hand, the γ_k -approximate mechanism presented in this paper needs to solve an optimization problem with only m interagent linear constraints (i.e., convex program (\overline{OPT})), and yet it is close to optimal for relatively large k (recall that γ_k approaches 1 as $k \rightarrow \infty$).

Prophet Inequalities. Krengel and Sucheston [1977, 1978] considered the following problem. A sequence of non-negative random numbers, drawn independently from known distributions, is presented to a gambler one by one. The gambler may choose to stop the sequence at any point to receive a reward equal to the last observed number. On the other hand, a prophet has complete knowledge of the entire sequence in advance and chooses the highest number. The original *prophet inequality*, proved by Krengel and Sucheston [1977, 1978], states that the expected reward of a gambler who uses an optimal stopping rule is at least half of the expected reward of a prophet. Samuel-Cahn [1984] showed that the same guarantee can be obtained using a simple threshold based stopping rule which is not necessarily optimal. Several other variants of this problem have also been studied in which the prophet, the gambler, or both are allowed to make multiple choices (See Hill and Kertz [1992] for a survey). In this paper we consider a variant where, for some constant k , gambler and prophet are each allowed to select up to k numbers and receive a reward equal to the sum of the selected numbers. For this variant, Hajiaghayi et al. [2007] obtained a sequence of k stopping rules for gambler that guarantees in expectation at least $1 - O(\frac{\sqrt{\ln k}}{\sqrt{k}})$ fraction of the reward of a prophet. We improve the previous result by presenting a new randomized strategy for gambler that guarantees in expectation at least $1 - \frac{1}{\sqrt{k+3}}$ fraction of the reward of a prophet. Recently, Kleinberg and Weinberg [2012] proved a variant of prophet inequalities with arbitrary matroid constraints which is more general than the variant considered here.³

2 Preliminaries

We start by defining the model and the common notation. We present our framework for combinatorial auctions, but it can be readily applied to other Bayesian mechanism design problems.

²I.e., there is a polynomial time algorithm for optimizing any linear objective over that polytope.

³In their model, each element of the sequence of random numbers is associated with an element of the ground set of one or more matroids such that a set of random draws can be feasibly selected only if it corresponds to an independent set of each matroid. Our variant can be considered as a special case with a k -uniform matroid.

Model. We consider the problem of allocating m indivisible heterogeneous items to n agents where we have k_j units of item j , for each $j \in [m]$. All the relevant private information⁴ of each agent $i \in [n]$ is represented by her type $t_i \in T_i$ where T_i is the type space of agent i . $\mathbf{T} = T_1 \times \dots \times T_n$ denotes the space type profiles. We assume the agents' type profile $\mathbf{t} = (t_1, \dots, t_n) \in \mathbf{T}$ is distributed according to a publicly known prior \mathbf{D} . We use X_{ij} and P_i to denote the random variables respectively corresponding to the allocation of item j to agent i and the payment of agent i . We also use the following vector notation: $X = (X_1, \dots, X_n)$ where $X_i = (X_{i1}, \dots, X_{im})$ for each i , and $P = (P_1, \dots, P_n)$. We define \mathbf{X} to be the space of jointly feasible allocations, i.e.,

$$\mathbf{X} = \left\{ \mathbf{x} \in \mathbb{N}_0^{n \times m} \mid \sum_{i \in [n]} x_{ij} \leq k_j, \forall j \in [m] \right\}.$$

A mechanism $M : \mathbf{T} \rightarrow \Delta(\mathbf{X} \times \mathbb{R}^n)$ is a mapping from type profiles to distributions over allocations/payments. We are interested in computing a mechanism that (approximately) maximizes⁵ the expected value of a given objective function $\text{OBJ}(\mathbf{t}, X, P)$. We are only interested in mechanisms which are within a given space \mathbf{M} of feasible mechanisms. Formally, our goal is to compute a mechanism $M \in \mathbf{M}$ that (approximately) maximizes $\mathbf{E}_{\mathbf{t} \sim \mathbf{D}}[\mathbf{E}_{(X,P) \sim M(\mathbf{t})}[\text{OBJ}(\mathbf{t}, X, P)]]$.

Assumptions. We make the following assumptions.

- (A1) **Independence.** The agents' types must be distributed independently, i.e., $\mathbf{D} = \mathcal{D}_1 \times \dots \times \mathcal{D}_n$ where \mathcal{D}_i is the distribution of types for agent i . Note that if agent i has multidimensional types, \mathcal{D}_i itself does not need to be a product distribution.
- (A2) **Additive separability of objective.** The objective function must be additively separable over the set of agents, i.e., it must be of the form $\text{OBJ}(\mathbf{t}, X, P) = \sum_i \text{OBJ}_i(t_i, X_i, P_i)$.
- (A3) **Single unit demands.** No agent should ever need more than *one unit* of each item, i.e., $X_{ij} \in \{0, 1\}$. This assumption is not necessary, but it simplifies the exposition; it can be avoided as explained in §8.
- (A4) **Convexity.** \mathbf{M} must be a convex space. In other words, for any two mechanisms $M, M' \in \mathbf{M}$, their convex combinations must also be in \mathbf{M} . A convex combination of M and M' is a mechanism M'' which simply runs M with probability β and runs M' with probability $1 - \beta$, for some $\beta \in [0, 1]$. In particular, if \mathbf{M} is restricted to deterministic mechanisms, it is not convex; however if \mathbf{M} includes mechanisms that randomize over deterministic mechanisms, then it is convex⁶.
- (A5) **Decomposability.** The set of constraints defining \mathbf{M} must be decomposable to supply constraints (i.e., $\sum_i X_{ij} \leq k_j$, for all $\mathbf{t} \in \mathbf{T}$, $(X, P) \in M(\mathbf{t})$, and $j \in [m]$) and single agent constraints (e.g., incentive compatibility, single agent allocation constraints, budget, etc). We define this assumption formally as follows. For any mechanism M , let $[[M]]_i$ be the single agent mechanism perceived by agent i , by simulating⁷ the other agents according to their

⁴This may include the agent's valuations, budget, etc.

⁵All of the results can be applied to minimization problems by simply maximizing the negation of the objective function.

⁶As an example of a randomized non-convex space of mechanisms, consider the space of mechanisms where the expected payment of every type must be either less than \$2 or more than \$4.

⁷The single agent mechanism induced on agent i can be obtained by simulating all agents other than i by drawing a random \mathbf{t}_{-i} from \mathbf{D}_{-i} and running M on agent i and the $n - 1$ simulated agents with types \mathbf{t}_{-i} ; note that this is a single agent mechanism because the simulated agents are just part of the mechanism.

respective distributions \mathbf{D}_{-i} . Define $M_i = \{[[M]]_i | M \in \mathbf{M}\}$ to be the space of feasible single agent mechanisms for agent i . The decomposability assumption requires that for any arbitrary mechanism M the following holds: if M satisfies the supply constraints and also $[[M]]_i \in M_i$ (for all agents i), then it must be that $M \in \mathbf{M}$.

We clarify the last assumption by giving an example. Suppose \mathbf{M} is the space of all agent specific item pricing mechanisms, then \mathbf{M} satisfies the last assumption. On the other hand, if \mathbf{M} is the space of mechanisms that offer the same set of prices to every agent, it does not satisfy the decomposability assumption because of the implicit inter agent constraint requiring the same set of prices being offered to every agent.

Throughout the rest of this chapter, we often omit the range of sums whenever the range is clear from the context (e.g., \sum_i means $\sum_{i \in [n]}$, and \sum_j means $\sum_{j \in [m]}$).

Multi agent problem. Formally, the multi agent problem is to find a mechanism M which is a solution to the following program:

$$\begin{aligned} & \text{maximize} && \sum_i \mathbf{E}_{\mathbf{t} \sim \mathbf{D}} [\mathbf{E}_{(X,P) \sim M(\mathbf{t})} [\text{OBJ}_i(t_i, X_i, P_i)]] && (OPT) \\ & \text{subject to} && \sum_i X_{ij} \leq k_j && \forall j \in [m], \forall \mathbf{t} \in \mathbf{T}, \forall (X, P) \in M(\mathbf{t}), \\ & && [[M]]_i \in M_i && \forall i \in [n]. \end{aligned}$$

Observe that we could optimize the mechanism for each agent independently in the absence of the first set of constraints. This observation is the key to our multi to single agent decomposition, which allows us to approximately decompose/reduce the multi agent problem to several single agent problems. A mechanism M is an α -approximation of the optimal mechanism if it is a feasible mechanism for the above program and obtains in expectation at least α -fraction of the optimal value of the program.

Ex ante allocation rules.

Definition 1 (ex ante allocation rule). *For a multi agent mechanism M , the ex ante allocation rule is a vector $x \in [0, 1]^{n \times m}$ in which x_{ij} is the probability of allocating a unit of item j to agent i , where the probability is over all type profiles, i.e.,*

$$x_{ij} = \mathbf{E}_{\mathbf{t} \sim \mathbf{D}} [\mathbf{E}_{(X,P) \sim M(\mathbf{t})} [X_{ij}]].$$

Note that for any feasible mechanism M and for every item j , the ex ante allocation rule satisfies $\sum_i x_{ij} \leq k_j$ by linearity of expectation.

Single agent problem. The single agent problem, for each $i \in [n]$, is to compute both an optimal mechanism and its expected objective value for agent i (i.e., in the absence of all other agents), subject to a given upper bound $\bar{x}_i \in [0, 1]^m$ on the ex ante allocation rule; in other words, the single agent mechanism may allocate a unit of item j to agent i with a probability of at most \bar{x}_{ij} where the probability is over all types of the agent. Formally, the single agent problem is to compute an optimal solution for the following program and to compute its optimal value, for any given \bar{x}_i :

$$\begin{aligned} & \text{maximize} && \mathbf{E}_{t_i \sim \mathcal{D}_i} [\mathbf{E}_{(X_i, P_i) \sim M_i(t_i)} [\text{OBJ}_i(t_i, X_i, P_i)]] && (OPT_i) \\ & \text{subject to} && \mathbf{E}_{t_i \sim \mathcal{D}_i} [\mathbf{E}_{(X_i, P_i) \sim M_i(t_i)} [X_{ij}]] \leq \bar{x}_{ij} && \forall j \in [m], \\ & && M_i \in M_i. \end{aligned}$$

We denote an optimal single agent mechanism for agent i , subject to a given \bar{x}_i (i.e., a solution to the above program), by $M_i^{\text{OPT}}(\bar{x}_i)$, and denote its expected objective value (i.e., the optimal value of the above program as a function of \bar{x}_i) by $R_i^{\text{OPT}}(\bar{x}_i)$. Later, we prove that $R_i^{\text{OPT}}(\bar{x}_i)$, which we refer to as the *optimal benchmark* for agent i , is a concave function of \bar{x}_i . As we will see later, the optimal single agent mechanism/optimal benchmark are usually hard to compute, therefore we often resort to approximation as formally defined next.

Approximation for single agent problem. For $i \in [n]$, we say that a single agent mechanism M_i together with a benchmark function R_i provide an α -approximation of the optimal single agent mechanism for agent i if for every $\bar{x}_i \in [0, 1]^m$, $M_i(\bar{x}_i)$ is a feasible mechanism for agent i whose ex ante allocation rule does not exceed \bar{x}_i and such that the expected objective value of $M_i(\bar{x}_i)$ is at least $\alpha R_i(\bar{x}_i)$ and R_i is a concave function that is an upper bound on R_i^{OPT} everywhere.

Example. To make the exposition more concrete, consider the following single agent problem as an example. Suppose there is only one type of item (i.e., $m = 1$) and the objective is to maximize the expected revenue⁸. Suppose agent i 's valuation is drawn from a regular distribution whose CDF is given by F_i . The optimal single agent mechanism for i , subject to $\bar{x}_i \in [0, 1]$, is a deterministic mechanism which offers the item at some fixed price, while ensuring that the probability of sale (i.e., the probability of agent i 's valuation being above the offered price) is no more than \bar{x}_i . In particular, the optimal benchmark $R_i^{\text{OPT}}(\bar{x}_i)$ is the optimal value of the following convex program as a function of \bar{x}_i :

$$\begin{aligned} \text{maximize} \quad & x_i F_i^{-1}(1 - x_i) \\ \text{subject to} \quad & x_i \leq \bar{x}_i, \\ & x_i \in [0, 1]. \end{aligned}$$

Furthermore, the optimal single agent mechanism offers the item at the price $F_i^{-1}(1 - x_i)$ where x_i is the optimal assignment for the above convex program. Note that $x_i F_i^{-1}(1 - x_i)$ is always concave in x_i for a regular distribution, so the above program is a concave program.

3 Decomposition via Ex ante Allocation Rule

In this section we present general methods for approximately decomposing/reducing a multi agent problem to several single agent problems. Recall that a single agent problem is to compute an (approximately) optimal single agent mechanism $M_i(\bar{x}_i)$ subject to an upper bound \bar{x}_i on the ex ante allocation rule, and to compute a corresponding concave benchmark $R_i(\bar{x}_i)$. We present two methods for constructing an approximately optimal multi agent mechanism, using M_i and R_i as black boxes. The two methods we present have different requirements and yield multi agent mechanisms with different properties, therefore the choice of method depends on the specific problem at hand. In some instances only one of the two methods is applicable.

Multi agent benchmark. We start by showing that the expected objective value of the optimal multi agent mechanism is upper bounded by the optimal value of the following convex program in which R_i is a concave benchmark for each agent i :

⁸The optimal multi agent mechanism for this setting is given by Myerson [1981]; however we consider this setting to keep the example simple and intuitive.

$$\begin{aligned}
& \text{maximize} && \sum_i R_i(\bar{x}_i) && (\overline{OPT}) \\
& \text{subject to} && \sum_i \bar{x}_{ij} \leq k_j, && \forall j \in [m] \\
& && \bar{x}_{ij} \in [0, 1], &&
\end{aligned}$$

Theorem 1. *The optimal value of the convex program (\overline{OPT}) is an upper bound on the expected objective value of the optimal multi agent mechanism.*

Proof. Let M^* be an optimal multi agent mechanism. Let x^* denote the ex ante allocation rule corresponding to M^* , i.e., $x_{ij}^* = \mathbf{E}_{\mathbf{t} \sim \mathbf{D}}[\mathbf{E}_{(X,P) \sim M^*(\mathbf{t})}[X_{ij}]]$. Observe that x^* is a feasible assignment for the convex program yielding an objective value of $\sum_i R_i(x_i^*)$ which is upper bounded by the optimal value of the convex program. So to prove the theorem it is enough to show that the contribution of each agent i to the expected objective value of M^* is upper bounded by $R_i(x_i^*)$. Consider $M_i^* = [[M^*]]_i$, i.e., the single agent mechanism induced by M^* on agent i . M_i^* can be obtained by simply running M^* on agent i and simulating the other $n - 1$ agents with random types $\mathbf{t}_{-i} \sim \mathbf{D}_{-i}$. M_i^* is a feasible single agent mechanism whose ex ante allocation rule is exactly x_i^* and obtains the same expected objective value as M^* from agent i . Therefore, the expected objective value of the optimal single agent mechanism, subject to an upper bound of x_i^* on the ex ante allocation rule, could only be higher. Notice that Assumption A1 is necessary to ensure that the simulated agents have correct type distribution; in particular, it ensures that agent i cannot tell if the other agents are being simulated or not. \square

Recall that R_i are required to be concave by definition. However this requirement is not arbitrary. We show that the optimal benchmarks R_i^{OPT} are always concave regardless of the problem.

Theorem 2. *The optimal benchmarks R_i^{OPT} are always concave.*

Proof. We prove this for an arbitrary agent i . To show that R_i^{OPT} is concave, it is enough to show that for any $\bar{x}_i, \bar{x}'_i \in [0, 1]^m$ and any $\beta \in [0, 1]$, the following inequality holds:

$$R_i^{\text{OPT}}(\beta \bar{x}_i + (1 - \beta) \bar{x}'_i) \geq \beta R_i^{\text{OPT}}(\bar{x}_i) + (1 - \beta) R_i^{\text{OPT}}(\bar{x}'_i).$$

Let M_i^{OPT} denote an optimal single agent mechanism for agent i . Consider the single agent mechanism M'' that works as follows: M'' runs $M_i^{\text{OPT}}(\bar{x}_i)$ with probability β and runs $M_i^{\text{OPT}}(\bar{x}'_i)$ with probability $1 - \beta$. Note that M_i is a convex space (this follows from Assumptions A4 and A5), therefore $M'' \in M_i$. Observe that by linearity of expectation, the ex ante allocation rule of M'' is bounded above by $\beta \bar{x}_i + (1 - \beta) \bar{x}'_i$ and the expected objective value of M'' is exactly $\beta R_i^{\text{OPT}}(\bar{x}_i) + (1 - \beta) R_i^{\text{OPT}}(\bar{x}'_i)$. So the expected objective value of the optimal single agent mechanism, subject to $\beta \bar{x}_i + (1 - \beta) \bar{x}'_i$, may only be higher. That implies $R_i^{\text{OPT}}(\beta \bar{x}_i + (1 - \beta) \bar{x}'_i) \geq \beta R_i^{\text{OPT}}(\bar{x}_i) + (1 - \beta) R_i^{\text{OPT}}(\bar{x}'_i)$ which proves the claim. \square

Constructing multi agent mechanisms. Theorem 1 suggests that by computing an optimal assignment of \bar{x} for the convex program (\overline{OPT}) and running the single agent mechanism $M_i(\bar{x}_i)$ for each agent i , one might obtain a reasonable multi agent mechanism; however such a multi agent mechanism would only satisfy the supply constraints in expectation. In other words, there is a good chance that some items are over allocated. We present two generic multi agent mechanisms for combining single agent mechanisms and resolving allocation conflicts in such a way that would

ensure the supply constraints are met at every instance, not just in expectation. In both approaches we first solve the convex program (\overline{OPT}) to compute an optimal assignment \bar{x} . The high level description of each mechanism is explained below.

1. **Pre-Rounding.** This mechanism serves the agents sequentially (arbitrary order); for each agent i , it selects a subset $S_i \subseteq [m]$ of available items and runs the single agent mechanism $M_i\langle\bar{x}_i[S_i]\rangle$, where $\bar{x}_i[S_i]$ denotes the vector obtained from \bar{x}_i by setting to zero the entries corresponding to items not in S_i . In particular, this mechanism precludes available items from early agents at random to make them available to late agents. We show that if there are at least k units of each item, this mechanism guarantees that S_i includes item j with probability at least $1 - \frac{1}{\sqrt{k+3}}$, for each agent i and each item j .
2. **Post-Rounding.** This mechanism runs $M_i\langle\bar{x}_i\rangle$ for all agents i simultaneously and independently. It then modifies the outcomes by deallocating the over allocated items at random in such a way that the probability of deallocation observed by all agents are equal, and therefore minimized over all agents. The payments are adjusted respectively. We show that if there are at least k units of each item, this mechanism guarantees that every allocation is preserved with probability $1 - \frac{1}{\sqrt{k+3}}$ from the perspective of the corresponding agent.

We will explain the above mechanisms in detail in §5 and present some technical assumptions that are sufficient to ensure that they retain at least $1 - \frac{1}{\sqrt{k+3}}$ fraction of the expected objective value of each $M_i\langle\bar{x}_i\rangle$.

Main result. The following informal theorem summarizes the main result of this paper. The formal statement of this result can be found in Theorem 7 and Theorem 8.

Theorem 3 (Market Expansion). *If for each agent $i \in [n]$, an α -approximate single agent mechanism M_i and a corresponding concave benchmark R_i are available, then, given some further assumptions (explained later), a multi agent mechanism $M \in \mathbf{M}$ can be constructed in polynomial time by using M_i as building blocks, such that M is a γ_k α -approximation of the optimal multi agent mechanism in \mathbf{M} , where $k = \min_j k_j$ and γ_k is a constant which is at least $1 - \frac{1}{\sqrt{k+3}}$.*

In the next section we present a toy problem called the *the magician's problem* together with a near-optimal solution for it. The solution of this problem is used extensively in both pre-rounding and post-rounding for equalizing the probabilities of preclusion/deallocation over all agents.

4 The Magician's Problem

In this section, we present an abstract online stochastic toy problem and a near-optimal solution for it which provides the main ingredient for combining single agent mechanisms to form multi agent mechanisms. A generalization of this problem and its solution are presented in §7.

Definition 2 (The Magician's Problem). *A magician is presented with a sequence of boxes one by one in an online fashion. There is a prize hidden in one of the boxes. The magician has k magic wands that can be used to open the boxes. If a wand is used on box i , it opens, but with a probability of at most x_i , which is written on the box, the wand breaks. The magician wishes to maximize the probability of obtaining the prize, but unfortunately the sequence of boxes, the written probabilities, and the box in which the prize is hidden are arranged by a villain, and the magician has no prior information about them (not even the number of boxes). However, it is guaranteed that $\sum_i x_i \leq k$,*

and that the villain has to prepare the sequence of boxes in advance (i.e., cannot make any changes once the process has started).

The magician could fail to open a box either because (a) he might choose to skip the box, or (b) he might run out of wands before getting to the box. Note that once the magician fixes his strategy, the best strategy for the villain is to put the prize in the box that has the lowest ex ante probability of being opened based on the magician's strategy. Therefore, in order for the magician to obtain the prize with a probability of at least γ , he has to devise a strategy that guarantees an ex ante probability of at least γ for opening each box. Notice that allowing the prize to be split among multiple boxes does not affect the problem. It is easy to show that the following strategy ensures an ex ante probability of at least $\frac{1}{4}$ for opening each box: for each box randomize independently and use a wand with probability $\frac{1}{2}$. But can we do better? We present an algorithm parameterized by a probability $\gamma \in [0, 1]$ which guarantees a minimum ex ante probability of γ for opening each box while trying to minimize the number of wands broken. In Theorem 4, we show that for $\gamma \leq 1 - \frac{1}{\sqrt{k+3}}$ this algorithm never requires more than k wands.

Definition 3 (γ -Conservative Magician). *The magician adaptively computes a sequence of thresholds $\theta_1, \theta_2, \dots \in \mathbb{N}_0$ and makes a decision about each box as follows: let W_i denote the number of wands broken prior to seeing the i^{th} box; the magician makes a decision about box i by comparing W_i against θ_i ; if $W_i < \theta_i$, it opens the box; if $W_i > \theta_i$, it does not open the box; and if $W_i = \theta_i$, it randomizes and opens the box with some probability (to be defined). The magician chooses the smallest threshold θ_i for which $\Pr[W_i \leq \theta_i] \geq \gamma$ where the probability is computed ex ante (i.e., not conditioned on past broken wands). Note that γ is a parameter that is given. Let $F_{W_i}(\ell) = \Pr[W_i \leq \ell]$ denote the ex ante CDF of random variable W_i , and let S_i be the indicator random variable which is 1 iff the magician chooses to open box i . Formally, the probability with which the magician should open box i condition on W_i is computed as follows:*

$$\Pr[S_i = 1 | W_i] = \begin{cases} 1, & W_i < \theta_i, \\ (\gamma - F_{W_i}(\theta_i - 1)) / (F_{W_i}(\theta_i) - F_{W_i}(\theta_i - 1)), & W_i = \theta_i, \\ 0, & W_i > \theta_i, \end{cases} \quad (S)$$

$$\theta_i = \min\{\ell | F_{W_i}(\ell) \geq \gamma\}. \quad (\theta)$$

Observe that θ_i is in fact computed before seeing box i itself.

The CDF of W_{i+1} can be computed from the CDF of W_i and x_i as follows (assume x_i is the exact probability of breaking a wand for box i , and define $s_i^\ell = \Pr[S_i = 1 | W_i = \ell]$).

$$F_{W_{i+1}}(\ell) = \begin{cases} s_i^\ell x_i F_{W_i}(\ell - 1) + (1 - s_i^\ell x_i) F_{W_i}(\ell), & i \geq 1, \ell \geq 0, \\ 1, & i = 0, \ell \geq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (F_W)$$

Furthermore, if each x_i is just an upper bound on the probability of breaking a wand on box i , then the above definition of F_{W_i} stochastically dominates the actual CDF of W_i , and the magician opens each box with a probability greater than or equal to γ .

In order to prove that a γ -conservative magician does not fail for a given choice of γ , we must show that the thresholds θ_i are no more than $k - 1$. The following theorem states a condition on γ that is sufficient to guarantee that $\theta_i \leq k - 1$ for all i .

Theorem 4 (γ -Conservative Magician). *For any $\gamma \leq 1 - \frac{1}{\sqrt{k+3}}$, a γ -conservative magician with k wands opens each box with an ex ante probability of at least γ . Furthermore, if x_i is the exact*

probability⁹ (not just an upper bound) of breaking a wand on box i for each i , then each box is opened with an ex ante probability exactly γ .

Proof. See §7. The current theorem is a special case of Theorem 16. \square

Definition 4 (γ_k). We define γ_k to be the largest probability such that for any $k' \geq k$ and any instance of the magician's problem with k' wands, the thresholds computed by a γ_k -conservative magician are less than k' . In other words, γ_k is the largest γ which is guaranteed to work for all instances with $k' \geq k$ wands. By Theorem 4, γ_k is at least $1 - \frac{1}{\sqrt{k+3}}$ ¹⁰.

By Definition 4 γ_k is non-decreasing in k , and by Theorem 4 it is at least $\frac{1}{2}$ (when $k = 1$) and it approaches 1 as $k \rightarrow \infty$. The next theorem implies that the lower bound of $1 - \frac{1}{\sqrt{k+3}}$ on γ_k is almost tight.

Theorem 5 (Hardness of The Magician's Problem). For any $\epsilon > 0$, there is no strategy for the magician that can guarantee an ex ante probability of $1 - \frac{k^k}{e^k k!} + \epsilon$ for opening each box. Note that $1 - \frac{k^k}{e^k k!} \approx 1 - \frac{1}{\sqrt{2\pi k}}$ by Stirling's approximation.

Proof. See §A. \square

Prophet Inequalities. A sequence of non-negative random numbers, drawn independently from known distributions, is presented to a gambler one by one. The gambler may choose to stop the sequence at any point to receive a reward equal to the last observed number. On the other hand, a prophet has complete knowledge of the entire sequence in advance and chooses the highest number. The original *prophet inequality*, proved by Krengel and Sucheston [1977, 1978], states that the expected reward of a gambler who uses an optimal stopping rule is at least half of the expected reward of a prophet. We are interested in a variant where, for some constant k , gambler and prophet are each allowed to select up to k numbers and receive a reward equal to the sum of the selected numbers. The problem is formally defined as follows.

Definition 5 (k -Choice Sum). A sequence of n non-negative random numbers V_1, \dots, V_n are drawn independently from distributions¹¹ F_1, \dots, F_n one by one in an arbitrary order. A gambler observes the process and may select up to k of the random numbers with the goal of maximizing the sum of the selected ones; a random number may only be selected at the time it is drawn, and it cannot be unselected later. The gambler knows all the distributions in advance, and observes the distribution from which the current number is drawn, but does not know the order in which the future random numbers are drawn. On the other hand, a prophet knows the entire sequence in advance, so he always selects the k highest random numbers. The order in which the random numbers are drawn is fixed in advance (i.e., the order may not change based on decisions of the gambler or based on previous draws).

Next, we present a randomized strategy for gambler by employing a γ_k -conservative magician as a black box. Our proposed strategy uses only a single threshold; however, it may skip some of

⁹The fact that the probability of the event of breaking a wand for the i^{th} box is exactly x_i , conditioned on any sequence of prior events, implies that these events are independent for different boxes.

¹⁰Because for any $k' \geq k$ obviously $1 - \frac{1}{\sqrt{k+3}} \leq 1 - \frac{1}{\sqrt{k'+3}}$.

¹¹Assume each V_i has a bounded expectation.

the numbers in the sequence at random.

Definition 6 (Gambler for k -Choice Sum).

- (I) Find a threshold $\tau \in \mathbb{R}_+$ such that $\sum_i \mathbf{Pr}[V_i \geq \tau] \geq k$ and $\sum_i \mathbf{Pr}[V_i > \tau] \leq k$ (i.e., do a binary search to find τ). Choose $\rho \in [0, 1]$ such that $\sum_i (\rho \mathbf{Pr}[V_i \geq \tau] + (1 - \rho) \mathbf{Pr}[V_i > \tau]) = k$.
- (II) Create an instance of γ_k -conservative magician with k wands.
- (III) Upon seeing V_i (for each $i \in [n]$):
 - (a) Write $x_i = \rho \mathbf{Pr}[V_i \geq \tau] + (1 - \rho) \mathbf{Pr}[V_i > \tau]$ on a box and present it to the magician.
 - (b) If the magician chooses to open the box, then
 - if $V_i > \tau$, select V_i ;
 - otherwise, if $V_i = \tau$, select V_i with probability ρ ;
 - otherwise skip V_i .
 - (c) If V_i was selected in the previous step, break the magician's wand.

Theorem 6 (Prophet Inequality for k -Choice Sum). *For any $\gamma \leq \gamma_k$, the gambler strategy of Definition 6 obtains at least γ fraction of the reward of a prophet in expectation (recall that $\gamma_k \geq 1 - \frac{1}{\sqrt{k+3}}$).*

Proof. To simplify the proof we initially assume that the distributions have no point mass concentrations, so F_i^{-1} is well defined over $[0, 1)$ for every i . Given the previous assumption, we can argue that $\sum_i \mathbf{Pr}[V_i \geq \tau] = \sum_i \mathbf{Pr}[V_i > \tau] = k$, and without loss of generality assume $\rho = 0$.

First, we compute an upper bound on the expected reward of the prophet. Let x_i be the ex ante probability (i.e., before any random number is drawn) that the prophet selects V_i (i.e. the probability that V_i is among the k highest draws). By linearity of expectation we must have $\sum_i x_i \leq k$ because the prophet cannot select more than k random numbers at any instance. Let $u_i(x_i)$ denote the maximum possible contribution of V_i to the expected reward of the prophet if V_i is selected with an ex ante probability of x_i . The optimal value of the following program is an upper bound on the expected reward of the prophet:

$$\begin{aligned}
 & \text{maximize} && \sum_i u_i(x_i) \\
 & \text{subject to} && \sum_i x_i \leq k, & (\tau) \\
 & && x_i \geq 0 \quad \forall i \in [n]. & (\mu_i)
 \end{aligned}$$

It is easy to see that $u_i(x_i)$ is exactly equal to the expected value of V_i conditioned on V_i being above the $1 - x_i$ quantile, multiplied by the probability of V_i being above that quantile. Let f_i denote the PDF of V_i . We can write $u_i(x_i) = \int_{F_i^{-1}(1-x_i)}^{\infty} v f_i(v) dv$. By changing the integration variable and applying the chain rule we get $u_i(x_i) = \int_0^{x_i} F_i^{-1}(1-q) dq$. Observe that the derivative of $u_i(x_i)$ is non-increasing, so u_i is concave; therefore the above program is a convex program.

Define the Lagrangian for the above convex program as

$$L(x, \tau, \mu) = - \sum_i u_i(x_i) + \tau \left(\sum_i x_i - k \right) - \sum_i \mu_i x_i.$$

By the KKT stationarity condition, at the optimal assignment, it must be $\frac{\partial}{\partial x_i} L(q, \tau, \mu) = 0$. On the other hand, $\frac{\partial}{\partial x_i} L(q, \tau, \mu) = -F_i^{-1}(1-x_i) + \tau - \mu_i$. If $x_i > 0$, complementary slackness implies $\mu_i = 0$, and therefore $x_i = 1 - F_i(\tau)$, so $x_i = \Pr[V_i > \tau]$. Furthermore, it is easy to show that the first constraint must be tight, which implies $\sum_i \Pr[V_i > \tau] = k$. Observe that the contribution of each V_i to the objective value of the convex program is exactly $\mathbf{E}[V_i | V_i > \tau] \Pr[V_i > \tau]$. By invoking Theorem 4 we argue that each box is opened with probability γ_k which implies the contribution of each V_i to the expected reward of the gambler is $\mathbf{E}[V_i | V_i > \tau] \Pr[V_i > \tau] \gamma_k$. Therefore the expected reward of the gambler is γ_k fraction of the optimal value of the convex program, which was itself an upper bound on the expected reward of the prophet. That completes the proof under the assumption that the distributions have no point mass concentrations.

In case of distributions with point mass concentrations (for example discrete distributions), we define

$$F_i^{-1}(x_i) = \min \{v \in \mathbb{R}_+ \mid \Pr[V_i \leq v] \geq x_i\}.$$

The maximum contribution of V_i to the expected reward of the prophet is still given by $u_i(x_i) = \int_0^{x_i} F_i^{-1}(1-q) dq$. Let τ and ρ be defined as in the first step of the gambler strategy (Definition 6) and let $x_i = \rho \Pr[V_i \geq \tau] + (1-\rho) \Pr[V_i > \tau]$ for each i . It is easy to show that $u_i(x_i) = \mathbf{E}[V_i | V_i > \tau] \Pr[V_i > \tau] + \rho \tau \Pr[V_i = \tau]$. On the other hand, it is easy to compute μ_i (for each i) such that x , τ and μ satisfy the KKT conditions. Therefore x is an optimal assignment for the convex program and $\sum_i u_i(x_i)$ is an upper bound on the expected reward of the prophet. Again by invoking Theorem 4 we argue that each box is opened with probability γ_k and the contribution of each V_i to the expected reward of the gambler is $\gamma_k u_i(x_i)$, so the expected rewards of the gambler is at least γ_k fraction of the expected reward of the prophet. \square

5 Generic Multi Agent Mechanisms

In this section, we present a formal description of the two generic multi agent mechanisms outlined toward the end of §3. Throughout the rest of this section we assume, for each agent $i \in [n]$, the availability of a single agent mechanism M_i and a corresponding concave benchmark R_i , which together provide α -approximation of the optimal single agent mechanism for agent i . Define $k = \min_j k_j$. We show that the resulting multi agent mechanism will be a $\gamma_k \alpha$ -approximation of the optimal multi agent mechanism in \mathbf{M} where γ_k is the optimal magician parameter which is at least $1 - \frac{1}{\sqrt{k+3}}$ (see Definition 4).

5.1 Pre-Rounding

This mechanism serves the agents sequentially (arbitrary order); for each agent i , it selects a subset S_i of available items and runs the single agent mechanism $M_i(\bar{x}_i[S_i])$ in which \bar{x} is an optimal assignment for the benchmark convex program (\overline{OPT}) and $\bar{x}_i[S_i]$ denotes the vector obtained from \bar{x}_i by setting to zero the entries corresponding to items not in S_i . In particular, this mechanism precludes some of the items from early agents at random to make them available to late agents. For each item, the mechanism tries to minimize the probability of preclusion simultaneously for all agents by equalizing this probability over all agents. Note that, for any given pair of agent and item, we only care about the ex ante probability of preclusion which is computed over the random types of other agents and the randomness of the mechanism. The mechanism is explained in detail

next.

Definition 7 (γ -Pre-Rounding).

- (I) Solve the convex program (\overline{OPT}) and let \bar{x} be an optimal assignment.
- (II) For each item $j \in [m]$, create an instance of a γ -conservative magician (Definition 3) with k_j wands (this will be referred to as the j^{th} magician). Note that γ is a parameter that is given.
- (III) For each agent $i \in [n]$ do:
 - (a) For each item $j \in [m]$, write \bar{x}_{ij} on a box and present it to the j^{th} magician. Let S_i be the set of items for which the corresponding magicians opened their boxes.
 - (b) Run $M_i(\bar{x}_i[S_i])$ and declare its outcome as the final outcome for agent i .
 - (c) For each item $j \in [m]$, if a unit of item j was allocated to agent i in the previous step, break the wand of the j^{th} magician.

The above mechanism can be used with any $\gamma \in [0, \gamma_k]$. By invoking Theorem 4 and using Definition 4 we can argue that, for each agent i and item j , S_i includes j with probability at least γ .

Further technical assumptions are needed in order for γ -pre-rounding to retain at least a γ fraction of the expected objective value of each $M_i(\bar{x}_i)$. Observe that γ -pre-rounding only guarantees the marginal probability of availability of each item to be at least γ ; however it does not provide any guarantee on the joint probability of availability of multiple items. In fact, this joint probability heavily depends on the behavior of agents served prior to the current agent. For example, if there are only two items and $\gamma = 0.5$, it could be that only one item is available to the last single agent mechanism most of the times, and rarely both items are available at the same time. Therefore there should be almost no complementarity among items with respect to the objective function. For example, if the objective is to maximize welfare and there are two items available, then the expected welfare of a single agent when both items are available to the mechanism should be no more than the sum of the expected welfare when each item is available to the mechanism individually. As another example, for revenue maximization, the optimal single agent revenue from multiple items should be no more than the sum of the optimal single agent revenue from each item individually (i.e., no revenue complementarity for seller). Strictly speaking, it is the benchmark function with respect to which there should be no complementarity among items. That means when there is limited complementarity among items (for example, a constant factor) with respect to the objective function, it might still be possible to find a benchmark function that exhibits no complementarity (for example, by losing a constant factor in the approximation). Next we present a condition for a benchmark function which is equivalent to the lack of complementarity among items with respect to that benchmark.

Definition 8 (Budget Balanced Cross Monotonic Cost Sharing Scheme).

A function $R : [0, 1]^m \rightarrow \mathbb{R}_+$ has a budget balanced cross monotonic cost sharing scheme iff there exists a cost sharing function $\xi : [m] \times [0, 1]^m \rightarrow \mathbb{R}_+$ with the following two properties:

- **Budget Balanced.** $\sum_{j \in S} \xi(j, x[S]) = R(x[S])$ for all $x \in [0, 1]^m$ and $S \subseteq [m]$.
- **Cross Monotonic.** $\xi(j, x[S]) \geq \xi(j, x[S \cup S'])$ for all $x \in [0, 1]^m$, $j \in [m]$, and $S, S' \subseteq [m]$.

Intuitively, such a cost sharing scheme for a benchmark function associates with each item some fraction of the benchmark value. Being budget balanced ensures that the sum of the fractions add up to the total benchmark value exactly. Cross monotonicity ensures that the fraction associated with each item does not decrease when other items become unavailable. In particular, if $R(x[S])$ is submodular in S , then such a cost sharing scheme always exists (for example, in the case of welfare maximization with submodular valuations).¹² We should emphasize that it is enough to show an appropriate cost sharing function exists; however it is never used in the mechanism and its computation is not required.

Theorem 7 (γ -Pre-Rounding). *Suppose for each agent i , M_i is an α -approximate incentive compatible single agent mechanism, and R_i is the corresponding concave benchmark. Also suppose R_i has a budget balanced cross monotonic cost sharing scheme. Then, for any $\gamma \in [0, \gamma_k]$, the γ -pre-rounding mechanism (Definition 7) is a $\gamma\alpha$ -approximation of the optimal mechanism in \mathbf{M} and is dominant strategy incentive compatible (DSIC).*

Proof. See §A. □

Observe that the γ -pre-rounding mechanism assumes no control and no prior information about the order in which agents are served. The order specified in the mechanism is arbitrary and could be replaced with any other ordering which may even be unknown to the mechanism. In particular, this mechanism can be used in an online settings where agents are served in an unknown order.

Corollary 1. *If items are not complementary for the seller's objective for any single agent (i.e., there is a budget balanced cross monotonic cost sharing scheme for each optimal benchmark) and \mathbf{M} includes all feasible BIC mechanisms, then the gap between the optimal DSIC mechanism and the optimal BIC mechanism is at most $1/\gamma_k$, which is at most 2 (for $k = 1$) and vanishes as $k \rightarrow \infty$.*

5.2 Post-Rounding

This mechanism runs $M_i(\bar{x}_i)$ simultaneously and independently for all agents i to compute a tentative allocation/payment for each agent; it then deallocates some of the items at random to ensure that the supply constraints are met at every instance; it ensures that the probabilities of deallocation perceived by the agents¹³ are equalized and therefore simultaneously minimized for all agents.

¹²We conjecture that it holds for revenue maximization when agents' valuations are submodular and \mathbf{M} is restricted to mechanisms which use agent specific item pricing.

¹³For each agent, this probability is over the random types of other agents and the randomness of the mechanism

The payments are also adjusted respectively. The mechanism is explained in detail next.

Definition 9 (γ -Post-Rounding).

- (I) Solve the convex program (\overline{OPT}) and let \bar{x} denote an optimal assignment.
- (II) Run $M_i\langle\bar{x}_i\rangle$ simultaneously and independently for all agents $i \in [n]$, and let $X'_i \subseteq [m]$ and $P'_i \in \mathbb{R}_+$ denote respectively the allocation (subset of items) and the payment computed by $M_i\langle\bar{x}_i\rangle$ for agent i .
- (III) For each item $j \in [m]$, create an instance of a γ -conservative magician (Definition 3) with k_j wands (this will be referred to as the j^{th} magician). Note that γ is a parameter that is given.
- (IV) For each agent $i \in [n]$ do:
 - (a) For each item $j \in [m]$, write \hat{x}_{ij} on a box and present it to the j^{th} magician, where \hat{x}_{ij} is the exact probability¹⁴ of $M_i\langle\bar{x}_i\rangle$ allocating a unit of item j to agent i ; let S_i be the set of items for which the corresponding magicians opened their boxes.
 - (b) Let $X_i \leftarrow S_i \cap X'_i$ and $P_i \leftarrow \gamma P'_i$. Declare X_i and P_i respectively as the final allocation and the final payment for agent i .
 - (c) For each item $j \in X_i$, break the wand of the j^{th} magician.

Notice that $\sum_i \hat{x}_{ij} \leq \sum_i \bar{x}_{ij} \leq k_j$; so by choosing $\gamma \in [0, \gamma_k]$ and invoking Theorem 4 and using Definition 4 we can argue that, for each agent i and each item j , S_i includes j with probability exactly γ . Consequently, any item that is in X'_i is also in X_i with probability exactly γ .

Further technical assumptions are needed in order for γ -post-rounding to retain at least a γ fraction of the expected objective value of each $M_i\langle\bar{x}_i\rangle$ and also to preserve feasibility and other required properties (such as incentive compatibility). Observe that γ -post-rounding explicitly modifies the outcome of each single agent mechanism which is in contrast to γ -pre-rounding, so we also need to ensure that feasibility and other required properties are preserved. We are interested in a minimally restrictive set of assumptions that guarantees the following.

- **Objective.** To preserve a γ fraction of the expected objective it is enough to ensure that the final outcome for each agent (after post rounding) yields at least a γ fraction of the objective of the original outcome of the corresponding single agent mechanism. Formally, for each agent i with reported type t_i , it is enough to ensure $\mathbf{E}_{X_i, P_i}[\text{OBJ}_i(t_i, X_i, P_i)] \geq \gamma \text{OBJ}_i(t_i, X'_i, P'_i)$. Notice that any function OBJ_i that is linear in P_i and submodular in X_i satisfies this requirement.
- **Incentive compatibility.** To preserve incentive compatibility it is enough to ensure that the expected utility of every type is scaled exactly by γ as a result of post rounding, while ensuring that the expected utility of every misreported type is scaled by a factor of less than or equal to γ . Formally, for each agent i with reported type t_i , it is enough to ensure $\mathbf{E}_{X_i, P_i}[u_i(t_i, X_i, P_i)] = \gamma u_i(t_i, X'_i, P'_i)$, and also $\mathbf{E}_{X_i, P_i}[u_i(\tau, X_i, P_i)] \leq \gamma u_i(\tau, X'_i, P'_i)$, for all $\tau \in T_i$. The magicians are presented with the exact ex ante allocation rule \hat{x} as opposed to \bar{x} to make sure that each box is opened with probability *exactly* γ as opposed to *at least* γ . Notice that any function u_i that is linear in both X_i ¹⁵ and P_i satisfies this requirement. It

¹⁴Note that \bar{x}_{ij} is only an upper bound on the probability of allocation, so $\hat{x}_{ij} \leq \bar{x}_{ij}$

¹⁵It needs to be linear in X_i only for X_i that are in the support of the single agent mechanism.

is worth mentioning that the inequality $\mathbf{E}_{X_i, P_i}[u_i(t_i, X_i, P_i)] \geq \gamma u_i(t_i, X'_i, P'_i)$ holds for any u_i that is submodular in X_i and linear in P_i , so one might be able to obtain an exact equality by somehow reducing the utility of the types for which the inequality is strict ¹⁶.

- **Feasibility and other properties.** To preserve feasibility of allocation it is enough to assume the space of feasible allocations is downward closed. Preserving other properties requires problem specific treatment. For example, γ -post-rounding can easily be modified to preserve ex post individual rationality by setting the payments as $P_i \leftarrow \frac{v_i(t_i, X_i)}{v_i(t_i, X'_i)} P'_i$. Note that the expected payment will still be $\gamma P'_i$.

Next, we present a set of assumptions that guarantee the above properties.

- (A'1) For each $M_i(\bar{x}_i)$, the exact ex ante allocation rule (i.e., \hat{x}) should be available (i.e., efficiently computable). Recall that \bar{x} is only an upper bound on the ex ante allocation rule. This can be computed in general via sampling.
- (A'2) Each function OBJ_i should be of the form $\text{OBJ}_i(t_i, X_i, P_i) = \text{OBJ}_i(t_i, X_i, 0) + c_i P_i$ in which $c_i \in \mathbb{R}_+$ is any constant and $\text{OBJ}_i(t_i, X_i, 0)$ should have a budget balanced cross monotonic cost sharing scheme in X_i .
- (A'3) The resulting mechanism should be in \mathbf{M} . In particular, that implies \mathbf{M} may not be restricted to any form of incentive compatibility stronger than Bayesian incentive compatibility (BIC), because the γ -post-rounding is only BIC.
- (A'4) Agents must have matroid valuations.

Observe that Assumption A'2 obviously holds for revenue maximization (i.e., $\text{OBJ}_i(t_i, X_i, P_i) = P_i$), and also for welfare maximization with quasilinear utilities and submodular valuations (i.e., $\text{OBJ}_i(t_i, X_i, P_i) = v_i(t_i, X_i)$ where $v_i(t_i, X_i)$ is the valuation of agent i with type t_i for allocation X_i ¹⁷). Next, we formally define Assumption A'4.

Definition 10 (Matroid Valuation). *A valuation function $v : 2^m \rightarrow \mathbb{R}_+$ is a matroid valuation iff there exists a matroid $\mathcal{M} = ([m], \mathcal{I})$ and a weight function $w : [m] \rightarrow \mathbb{R}_+$ such that, for any subset $S \subseteq [m]$ of items, $v(S)$ is equal to the weight of a maximum weight independent subset of S , i.e.,*

$$v(S) = \max_{\substack{I \subseteq S \\ I \in \mathcal{I}}} \sum_{j \in I} w(j) \quad \forall S \subseteq [m].$$

Matroid valuations include additive valuations with demand constraints, unit demand valuations, etc.

Theorem 8 (γ -Post-Rounding). *Suppose, for each agent i , M_i is an α -approximate incentive compatible single agent mechanism, and R_i is a corresponding concave benchmark. Also suppose Assumptions A'1 through A'4 hold. Then, for any $\gamma \in [0, \gamma_k]$, the γ -post-rounding mechanism (Definition 9) is a $\gamma\alpha$ -approximation of the optimal mechanism in \mathbf{M} and BIC.*

Proof. See §A. □

¹⁶This can potentially be achieved by discarding some items from X_i at random or increasing P_i . However that requires computing $\mathbf{E}_{\mathbf{t}_{-i}, X_i, P_i}[u_i(t_i, X_i, P_i)]$ in which the expectation is taken over the randomness of the mechanism and the random types of other agents. This expectation can be computed via sampling. The main obstacle is to ensure that the utility of misreported types are also scaled by a factor less than or equal to γ .

¹⁷Note that the payment terms cancel out because the utility of the auctioneer is counted toward the social welfare of the mechanism

6 Single Agent Mechanisms

In this section, for each of the settings listed in Table 1, we present an approximately optimal single agent mechanism (and a corresponding benchmark) satisfying the requirements of one of the generic multi agent mechanisms of §5. Recall that plugging an α -approximate single agent mechanism into our generic construction yields a $\gamma_k\alpha$ -approximate multi agent mechanism.

For the rest of this section, we restrict the space of mechanisms (except in §6.4) to item pricing mechanisms with budget randomization as defined next.

Definition 11 (Item Pricing with Budget Randomization (IPBR)). *An item pricing mechanism is a possibly randomized mechanism that offers a menu of item prices to each agent and allows each agent to choose their favorite bundle. The menu of prices offered to each agent may depend on the reports of other agents. The payment of an agent is equal to the total price of the items in her purchased bundle. The prices offered to different agents do not need to be identical. In the presence of budget constraints, an agent is allowed to pay a fraction of the price of an item and receive the item with a probability equal to the paid fraction¹⁸. A mechanism is considered an item pricing mechanism if its outcome can be interpreted as such¹⁹.*

Item pricing mechanisms are simple and practical as opposed to optimal BIC mechanisms which often involve lotteries. Budget randomization allows us to get around the hardness of the knapsack problem faced by budgeted agents. However, assuming budgets are large compared to prices, budget randomization can be safely ignored since the optimal integral solution of the knapsack problem approaches its optimal fractional solution. For unit demand agents with no budget constraints, IPBR mechanisms are the same as deterministic mechanisms.

For each single agent mechanism presented in this section, the single agent benchmark function $R(x)$ is defined as the optimal value of some convex program of the following general form in which u is a concave function, g_j are convex functions, and \mathbb{Y} is a convex set (in the rest of this section we consider only a single agent, so we will omit the subscript i).

$$\begin{aligned} & \text{maximize} && u(y) && (\overline{OPT}_1) \\ & \text{subject to} && g_j(y) \leq \bar{x}_j, && \forall j \in [m] \\ & && y \in \mathbb{Y} \end{aligned}$$

Lemma 1. $R(\bar{x})$ is concave, i.e., the optimal value of a convex program of the form (\overline{OPT}_1) as a function of \bar{x} is always concave.

Proof. See §A. □

Note that we can replace each R_i in the multi agent benchmark convex program (\overline{OPT}) with the corresponding single agent benchmark convex program to obtain a combined convex program which can be solved efficiently. In particular, if each R_i is captured by a linear program, the combined multi agent program will also be a linear program.

¹⁸A utility maximizing agent, with submodular valuations and budget constraint, always pays the full price for any item she purchases, except potentially for the last item purchased, for which she must have run out of budget.

¹⁹I.e., an item pricing mechanism may collect all the reports and compute the final outcome along with agent specific prices, such that the outcome of each agent would be the same as if each agent purchased their favorite bundle according to her observed prices, and the prices observed by each agent should be independent of her report.

6.1 Single Item, Unit Demand, Budget Constraint

In this section, we consider a unit-demand agent with a publicly known budget B and one type of item (i.e., $m = 1$). The only private information of the agent is her valuation v for the item which is drawn from a publicly known distribution with CDF $F(v)$. To avoid complicating the proofs we assume F is continuous and strictly increasing in its domain²⁰. We present a single agent mechanism which is optimal among IPBR mechanisms. We start by defining the modified CDF function \tilde{F} as follows:

$$\tilde{F}(v) = \begin{cases} F(v), & v \leq B, \\ 1 - (1 - F(v))\frac{B}{v}, & v \geq B. \end{cases} \quad (\tilde{F})$$

Intuitively, $1 - \tilde{F}(p)$ is the probability of allocating the item to the agent if we offer the item at price p . Note that the agent only buys if her valuation is more than p which happens with probability $1 - F(p)$; if $p > B$, she pays her whole budget and receives the item with probability $\frac{B}{p}$, otherwise she pays the full price and receives the item with probability 1. To allocate the item with probability x we should set the price to $\tilde{F}^{-1}(1 - x)$ yielding an expected revenue of $x\tilde{F}^{-1}(1 - x)$. Define $\text{Rev}(x) = x\tilde{F}^{-1}(1 - x)$ and let $\widehat{\text{Rev}}(x)$ denote its concave closure (i.e., the smallest concave function that is an upper bound on $\text{Rev}(x)$ for every x). We will address the problem of efficiently computing $\widehat{\text{Rev}}(x)$ later in Lemma 2. Next, we show that the optimal value of the following convex program is equal to the expected revenue of the optimal single agent IPBR mechanism subject to \bar{x} ; therefore we will define the single agent benchmark function $R(\bar{x})$ as the optimal value of this program as a function of \bar{x} .

$$\begin{aligned} & \text{maximize} && \widehat{\text{Rev}}(x) && (\text{REV}_{\text{single}}) \\ & \text{subject to} && x \leq \bar{x}, \\ & && x \geq 0. \end{aligned}$$

Theorem 9. *The revenue of the optimal single agent IPBR mechanism, subject to an upper bound of \bar{x} on the ex ante allocation rule, is equal to the optimal value of the convex program $(\text{REV}_{\text{single}})$.*

Proof. We only prove that the optimal value of the convex program is an upper bound on the expected revenue of the optimal single agent IPBR mechanism. Later we propose a single agent IPBR mechanism whose expected revenue is equal to the optimal value of the above program which completes the proof.

Observe that any single agent IPBR mechanism can be specified as a distribution over prices. Let \mathcal{P} be the optimal price distribution. Then the optimal revenue is $\mathbf{E}_{p \sim \mathcal{P}}[p(1 - \tilde{F}(p))]$. Note that every price p corresponds to an allocation probability $q = 1 - \tilde{F}(p)$. So any probability distribution over p can be specified as a probability distribution over q . Let \mathcal{Q} denote the probability distribution over q that corresponds to price distribution \mathcal{P} , so we can write

$$\begin{aligned} (\text{Optimal Revenue}) &= \mathbf{E}_{q \sim \mathcal{Q}} [q\tilde{F}^{-1}(1 - q)] = \mathbf{E}_{q \sim \mathcal{Q}} [\text{Rev}(q)] \\ &\leq \mathbf{E}_{q \sim \mathcal{Q}} [\widehat{\text{Rev}}(q)] && \text{by definition of concave closure} \\ &\leq \widehat{\text{Rev}}(\mathbf{E}_{q \sim \mathcal{Q}}[q]) && \text{by Jensen's inequality} \\ &\leq R(\bar{x}) \end{aligned}$$

²⁰Our proofs can be modified to work without this assumption.

Notice that $x = \mathbf{E}_{q \sim \mathcal{Q}}[q]$ is a feasible assignment for the convex program because it is exactly the probability of allocation using the price distribution \mathcal{P} , so it must be no more than \bar{x} . So the revenue of the optimal IPBR mechanism is upper bounded by the optimal value of the convex program which is what $R(\bar{x})$ is defined as. \square

Next, we present an optimal single agent IPBR mechanism.

Definition 12 (Single Agent Mechanism).

- (I) Define the single agent benchmark $R(\bar{x})$ to be the optimal value of convex program ($\text{REV}_{\text{single}}$) as a function of \bar{x} .
- (II) Given \bar{x} , solve ($\text{REV}_{\text{single}}$) and let x^* be an optimal assignment.
- (III) If $\widehat{\text{Rev}}(x^*) = \text{Rev}(x^*)$, offer the item at price $p = \tilde{F}^{-1}(1 - x^*)$; otherwise compute $x^-, x^+, \theta \in [0, 1]$ such that $x^* = \theta x^- + (1 - \theta)x^+$ and $\widehat{\text{Rev}}(x^*) = \theta \text{Rev}(x^-) + (1 - \theta) \text{Rev}(x^+)$ (this is always possible by the definition of concave closure), and offer the item at price $p^- = \tilde{F}^{-1}(1 - x^-)$ with probability θ and price $p^+ = \tilde{F}^{-1}(1 - x^+)$ with probability $1 - \theta$.

Theorem 10. *The mechanism of Definition 12 is the optimal revenue maximizing single agent IPBR mechanism. Furthermore, this mechanism satisfies the requirements of γ -pre-rounding.*

Proof. We already proved that optimal value of convex program ($\text{REV}_{\text{single}}$) is an upper bound on the expected revenue of the optimal IPBR mechanism (Theorem 9). We now prove that the mechanism of Definition 12 obtains the same expected revenue and therefore is optimal. First, observe that the ex ante probability of allocation is exactly $\theta x^- + (1 - \theta)x^+ = x^*$ which is no more than \bar{x} and therefore is feasible. Furthermore, its expected revenue is exactly $\theta \text{Rev}(x^-) + (1 - \theta) \text{Rev}(x^+) = \widehat{\text{Rev}}(x^*)$ which is equal to the optimal value of the convex program. Also, the benchmark function, $R(x)$, is concave (this follows from Lemma 1) and has a trivial budget balanced cost sharing scheme (because there is only one item), therefore it meets the requirements of γ -pre-rounding. \square

Next, we address the problem of efficiently computing $\widehat{\text{Rev}}$. We assume all non-zero valuations of the agent are contained in a closed interval that is strictly separated from 0 (i.e., there exist a minimum non-zero valuation and a maximum valuation). This is a reasonable assumption given that the objective is to maximize revenue which is often measured in terms of a common currency.

Lemma 2. *Assume all non-zero valuations of the agent are contained in a closed interval that is strictly separated from 0; without loss of generality, we may assume this interval to be $[1, L]$, for some $L > 1$. For any $\epsilon > 0$, a $(1 + \epsilon)$ -approximation of $\widehat{\text{Rev}}$, which we denote by $\widehat{\text{Rev}}_\epsilon$, can be constructed using a piece-wise linear function with $\ell = \lfloor \frac{\log L}{\log(1 + \epsilon)} \rfloor$ pieces and in time $O(\ell \log \ell)$.*

Proof. For each $r = 0 \dots \ell$, consider the price $p_r = (1 + \epsilon)^{\ell - r}$ and compute the corresponding $x_r = 1 - \tilde{F}(p_r)$. Define $\widehat{\text{Rev}}_\epsilon$ to be the function obtained by taking the convex hull of the following points:

$(0, 0), (x_1, p_1 x_1), (x_2, p_2 x_2), \dots, (x_\ell, p_\ell x_\ell), (1, 0)$. The computation can be done in time $O(\ell \log \ell)$. Note that $\tilde{F}^{-1}(1 - x)$ is a decreasing function of x so for every $x \in [x_r, x_{r+1}]$, the corresponding price is $\tilde{F}^{-1}(x) \in [p_{r+1}, p_r]$, but $p_r = (1 + \epsilon)p_{r+1}$; therefore for all x we have $\widehat{\text{Rev}}_\epsilon(x) \leq \widehat{\text{Rev}}(x) \leq (1 + \epsilon)\widehat{\text{Rev}}_\epsilon(x)$ which completes the proof. \square

In order to use $\widehat{\text{Rev}}_\epsilon$ in the single agent mechanism of Definition 12, we need to replace $\widehat{\text{Rev}}$ with $(1 + \epsilon)\widehat{\text{Rev}}_\epsilon$ in the objective function of the convex program ($\text{REV}_{\text{single}}$) for computing the benchmark. Furthermore, the mechanism will be a $(1 - \epsilon)$ -approximation of the optimal single agent IPBR mechanism. Also notice that finding p^- and p^+ is trivial from $\widehat{\text{Rev}}_\epsilon$.

6.2 Multi Item (Independent), Unit Demand

In this section, we consider a unit demand agent with independent private valuations for m items. We assume that for each item j , the agent's valuation v_j is distributed independently according to a publicly known distribution with CDF $F_j(v)$. We present a single agent mechanism which is a $\frac{1}{2}$ -approximation of the optimal deterministic revenue maximizing mechanism. To avoid complicating the proofs, we assume that each F_j is continuous and strictly increasing in its domain. Furthermore, we require the distributions to be regular. This mechanism can be used with γ -pre-rounding (Definition 7) to obtain a $\frac{1}{2}\gamma_k$ -approximate sequential posted pricing multi agent mechanism. The previous best approximation mechanism for this setting was a $\frac{1}{6.75}$ -approximate sequential posted pricing mechanism by Chawla et al. [2010]²¹.

We start by defining $\text{Rev}_j(x) = xF_j^{-1}(1 - x)$ for each item j . We proceed by showing that Rev_j is concave.

Lemma 3. *If F is the CDF of a regular distribution, then $\text{Rev}(x) = xF^{-1}(1 - x)$ is concave.*

Proof. It is enough to show that $\frac{\partial}{\partial x} \text{Rev}(x)$ is non-increasing in x . Observe that $\frac{\partial}{\partial x} \text{Rev}(x) = F^{-1}(1 - x) - \frac{x}{f(F^{-1}(1 - x))}$ in which f is the derivative of F . By substituting $x = 1 - F(p)$, it is enough to show that the resulting function is non-decreasing in p because x is itself non-increasing in p . However, by this substitution we get $\frac{\partial}{\partial x} \text{Rev}(x) = p - \frac{1 - F(p)}{f(p)}$ which is non-decreasing in p by definition of regularity. \square

Note that any deterministic mechanism for a unit demand agent can be interpreted as item pricing. Consequently, $\text{Rev}_j(x_j)$ is the maximum revenue that such a mechanism can obtain if item j is allocated with probability x_j . Next, we show that the optimal value of the following convex program is an upper bound on the optimal revenue:

$$\begin{aligned}
& \text{maximize} && \sum_j \text{Rev}_j(x_j) && (\text{REV}_{\text{unit}}) \\
& \text{subject to} && x_j \leq \bar{x}_j && \forall j \in [m], && (\lambda_j) \\
& && \sum_j x_j \leq 1 && (\tau) \\
& && x_j \geq 0 && \forall j \in [m]. && (\mu_j)
\end{aligned}$$

Theorem 11. *The expected revenue of the optimal deterministic single agent mechanism, subject to an upper bound of \bar{x} on the ex ante allocation rule, is no more than the optimal value of the convex program (REV_{unit}).*

Proof. Let x^* be the ex ante allocation rule of the optimal single agent deterministic mechanism. So the expected revenue obtained from each item j is upper bounded by $\text{Rev}_j(x_j^*)$ (proof of this claim is essentially the same as the proof of Theorem 9). Consequently, the optimal revenue cannot

²¹The mechanism of Chawla et al. [2010] needs regular distributions too despite the authors' claim. See the end of the current section for a detailed discussion of irregular distributions.

be more than $\sum_j \text{Rev}_j(x_j^*)$. Furthermore, the optimal mechanism never allocates more than one item, so $\sum_j x_j^* \leq 1$; also $x_j^* \leq \bar{x}_j$; therefore x^* is a feasible assignment for the convex program and the optimal revenue is upper bounded by the optimal value of the convex program. \square

Next, we present the single agent mechanism.

Definition 13 (Single Agent Mechanism).

- (I) Define the single agent benchmark $R(\bar{x})$ to be the optimal value of convex program $(\text{REV}_{\text{unit}})$ as a function of \bar{x} .
- (II) Given \bar{x} , solve $(\text{REV}_{\text{unit}})$ and let x^* denote an optimal assignment.
- (III) For each item j , set the price $p_j = F_j^{-1}(1 - x_j^*)$. Without loss of generality, assume that items are indexed in non-decreasing order of prices, i.e., $p_1 \leq \dots \leq p_m$.
- (IV) For each item j , define $r_j = \max(x_j^* p_j + (1 - x_j^*) r_{j+1}, r_{j+1})$ and define $r_{m+1} = 0$.
- (V) Let $S^* = \{j \in [m] \mid p_j > r_{j+1}\}$. Only offer the items in S^* at prices computed in the previous step (i.e., set the price of other items to infinity).

Theorem 12. *The mechanism of Definition 13 obtains at least $\frac{1}{2}$ fraction of the revenue of the optimal deterministic single agent mechanism in expectation. Furthermore, it satisfies the requirements of γ -pre-rounding.*

Proof. First, we show that this mechanism obtains in expectation at least $\frac{1}{2}$ of its benchmark $R(\bar{x})$, which by Theorem 11 is an upper bound on the optimal revenue. Observe that x_j^* is exactly the probability that the valuation of the agent for item j is at least p_j . Now consider an “adversarial replica” who has the same valuations as the original agent, but always buys the item that has the lowest price among all the items priced below her valuation. For any assignment of prices, the revenue obtained from the replica is a lower bound on the revenue obtained from the original agent. So it is enough to show that the mechanism obtains a revenue of at least $\frac{1}{2} \sum_j x_j^* p_j$ from the replica. Recall that x^* is an optimal assignment of the convex program yielding an objective value of $\sum_j x_j^* p_j = R(\bar{x})$. Observe that r_j is exactly the expected revenue obtained from the replica by offering the items in $S^* \cap \{j, \dots, m\}$ at computed prices. In particular, item j is included in S^* if $p_j > r_{j+1}$, in other words, item j is offered if its price is more than the expected revenue obtained from offering the items in $S^* \cap \{j+1, \dots, m\}$. Finally, the expected revenue of the mechanism from the replica is exactly r_1 . By invoking Lemma 4 we argue that $r_1 \geq \frac{1}{2} \sum_j x_j^* p_j$ which completes the proof of the first claim.

Next, we show that this mechanism satisfies the requirements of γ -pre-rounding. Observe that by Lemma 1 the optimal value of $(\text{REV}_{\text{unit}})$ is a concave function of \bar{x} , so $R(\bar{x})$ is concave. It only remains to show that $R(\bar{x})$ has a budget balanced cross monotonic cost sharing scheme. Let $x_j(\bar{x})$ denote the optimal assignment of variable x_j , in the convex program $(\text{REV}_{\text{unit}})$, as a function of \bar{x} . Define the cost share function

$$\xi(j, \bar{x}) = \text{Rev}_j(x_j(\bar{x})).$$

We show that ξ is budget balanced and cross monotonic (see Definition 8).

- **Budget balance.** We show that for any $\bar{x} \in [0, 1]^m$ and any $S \subseteq [m]$, $R(\bar{x}[S]) = \sum_{j \in S} \xi(j, \bar{x}[S])$. Observe that $R(\bar{x}[S]) = \sum_j \text{Rev}_j(x_j(\bar{x}[S])) = \sum_{j \in S} \xi(j, x_j(\bar{x}[S]))$ which proves that ξ is budget balanced. Note that $\text{Rev}_j(x_j(\bar{x}[S])) = 0$ for any $j \notin S$ because $x_j(\bar{x}[S])$ is forced to be 0.

- **Cross monotonicity.** We show that $\xi(j, \bar{x}[S]) \geq \xi(j, \bar{x}[S \cup S'])$ for any $\bar{x} \in [0, 1]^m$ and any $S, S' \subseteq [m]$. Let the Lagrangian of $(\text{REV}_{\text{unit}})$ be defined as follows:

$$L(x, \lambda, \tau, \mu) = - \sum_j \text{Rev}_j(x_j) + \sum_j \lambda_j (x_j - \bar{x}_j) + \tau \left(\sum_j x_j - 1 \right) - \sum_j \mu_j x_j.$$

The high level idea of the proof is as follows. We show that there is more pressure on the constraint associated with τ when the set of available items is $S \cup S'$ instead of S (i.e., τ is larger for $S \cup S'$); we then show that the optimal x_j can be determined from τ ; in particular, we show that, as the optimal τ increases, the optimal x_j decreases, and consequently $\xi(j, x)$ (which is equal to $\text{Rev}_j(x_j)$) decreases as well, which proves ξ is cross monotonic. Next we present the proof in detail.

By the KKT stationarity conditions the following holds for an optimal assignment:

$$\frac{\partial}{\partial x_j} L(x, \lambda, \tau, \mu) = - \frac{\partial}{\partial x_j} \text{Rev}_j(x_j) + \lambda_j + \tau - \mu_j = 0.$$

First we show that the optimal x_j , and consequently $\xi(j, \bar{x})$, can be determined from the optimal τ ; and they are both non-increasing in τ . Observe that (a) all dual variables must be non-negative, (b) by complementary slackness λ_j or μ_j may be non-zero only if $x_j = \bar{x}_j$ or $x_j = 0$ respectively, and (c) $\frac{\partial}{\partial x_j} \text{Rev}_j(x_j)$ is monotonically decreasing²²; therefore, if the optimal τ is given, the optimal x_j is uniquely determined by the above equation and the aforementioned complementarity slackness conditions. Let $x_j(\tau)$ denote the optimal assignment of x_j as a function of τ . Due to concavity of Rev_j , and the above KKT condition, we can argue that $x_j(\tau)$ is non-increasing in τ , which also implies that $\xi(j, \bar{x})$ is non-increasing in τ .

Next, we prove by contradiction that ξ is cross monotonic. Let $\tau(\bar{x})$ denote the optimal assignment of τ as a function of \bar{x} . By contradiction, suppose ξ is not cross monotonic, i.e. $\xi(j^*, \bar{x}[S \cup S']) > \xi(j^*, \bar{x}[S])$ for some item j^* ; therefore it must be that $\tau(\bar{x}[S]) > \tau(\bar{x}[S \cup S']) \geq 0$. Since $\tau(\bar{x}[S]) > 0$, the inequality associated with τ must be tight (by complementary slackness), so $\sum_j x_j(\tau(\bar{x}[S])) = 1$. On the other hand, for all j , $x_j(\tau(\bar{x}[S \cup S'])) \geq x_j(\tau(\bar{x}[S]))$, with the inequality being strict for $j = j^*$, which means $\sum_j x_j(\tau(\bar{x}[S \cup S'])) > 1$, which is a contradiction.

That completes the proof of the theorem. □

Lemma 4. Let p_1, \dots, p_m and x_1, \dots, x_m be two sequences of non-negative real numbers and suppose $\sum_j x_j \leq 1$. For each $j \in [m]$, define $r_j = \max(x_j p_j + (1 - x_j) r_{j+1}, r_{j+1})$ and let $r_{m+1} = 0$. Then $r_1 \geq \frac{1}{2} \sum_j x_j p_j$.

Proof. See §A. □

Irregular distributions. Unfortunately the single agent mechanism presented in this section does not work for irregular distributions. We show how this mechanism can be extended to irregular distributions, however the resulting mechanism can no longer be interpreted as a distribution over deterministic mechanisms, so it will be unjustified to compare its revenue against the revenue of the optimal deterministic mechanism.

²²To avoid complicating the proof, we assume that the functions Rev_j are strictly concave, however this assumption is not necessary.

Both the single agent mechanism presented here (Definition 13) and the mechanism proposed by Chawla et al. [2010] crucially depend on the following assumption: *there should exist an ordering of the items such that if the items are offered one by one in that order, the agent's best strategy (assuming she also knows the prices of future items) is to buy the first item that is priced below her valuation.* We should emphasize that such an ordering may depend on both the valuations of the agent and the prices; however, both our proof and the proof of Chawla et al. [2010] merely need the existence of such an ordering, not its knowledge. We will show that for irregular distributions such an ordering may not exist unless we allow for truly randomized mechanisms (i.e., mechanisms that may not be interpreted as randomization over deterministic mechanisms).

We first explain how the mechanism can be modified to work for irregular distributions. We then explain a subtle modification which allows us to interpret the mechanism as randomization over deterministic mechanisms, however the modification actually breaks the proof in the same way that the mechanism of Chawla et al. [2010] fails for irregular distributions.

First, we claim that the expected revenue of the optimal deterministic mechanism is upper bounded by the optimal value of the following convex program, in which $\widehat{\text{Rev}}_j$ is the concave closure of Rev_j (for each j):

$$\begin{aligned}
 & \text{maximize} && \sum_j \widehat{\text{Rev}}_j(x_j) && (\text{REV}_{\text{unit-irreg}}) \\
 & \text{subject to} && x_j \leq \bar{x}_j && \forall j \in [m], \\
 & && \sum_j x_j \leq 1, \\
 & && x_j \geq 0 && \forall j \in [m].
 \end{aligned}$$

The proof of the above claim is essentially the same as the proof of Theorem 11. Next, we present

the extended mechanism.

Definition 14 (Single Agent Mechanism (Irregular Distributions)).

- (I) Define the single agent benchmark $R(\bar{x})$ to be the optimal value of convex program $(\text{REV}_{\text{unit-irreg}})$ as a function of \bar{x} .
- (II) Given \bar{x} , solve $(\text{REV}_{\text{unit-irreg}})$ and let x^* be an optimal assignment.
- (III) For each item $j \in [m]$ compute the following:
 - If $\widehat{\text{Rev}}_j(x_j^*) = \text{Rev}_j(x_j^*)$, set $p_j, p_j^-, p_j^+ \leftarrow \tilde{F}_j^{-1}(1 - x_j^*)$ and set $\theta_j \leftarrow 0$.
 - Otherwise compute $x_j^-, x_j^+, \theta_j \in [0, 1]$ such that $x_j^* = \theta_j x_j^- + (1 - \theta_j) x_j^+$ and $\widehat{\text{Rev}}_j(x_j^*) = \theta_j \text{Rev}_j(x_j^-) + (1 - \theta_j) \text{Rev}_j(x_j^+)$ (this is always possible by the definition of concave closure), and set $p_j^- \leftarrow \tilde{F}_j^{-1}(1 - x_j^-)$ and $p_j^+ \leftarrow \tilde{F}_j^{-1}(1 - x_j^+)$ and $p_j \leftarrow \frac{\widehat{\text{Rev}}_j(x_j^*)}{x_j^*}$.
- Without loss of generality, assume that items are indexed in non-decreasing order of p_j , i.e., $p_1 \leq \dots \leq p_m$.
- (IV) For each item j , define $r_j = \max(x_j^* p_j + (1 - x_j^*) r_{j+1}, r_{j+1})$ and define $r_{m+1} = 0$.
- (V) Let $S^* = \{j \in [m] \mid p_j > r_{j+1}\}$. Reveal to the agent only the items in S^* along with the computed values of p_j, p_j^-, p_j^+ , and θ_j for all $j \in S^*$.
- (VI) Ask the agent to pick one item from S^* and let j^* denote that item. Offer item j^* at price p^- with probability θ_j and price p^+ with probability $1 - \theta_j$. The agent will not be allowed to pick another item regardless of whether or not she purchases j^* .

Observe that for regular distributions the above mechanism behaves exactly the same way as the mechanism of Definition 13. A crucial property of the above mechanism is that price randomization takes place only after the agent has picked an item. Note that the agent may pick an item and then refuse to buy it once the price is realized, however the agent will not be given a second chance to pick another item. The proof of revenue guarantee and other properties of the above mechanism is essentially the same as the proof of Theorem 12, except that the mechanism cannot be interpreted as a distribution over deterministic mechanisms. Observe that the agent may *regret* her choice after the price is realized (i.e., she would have picked a different item if she knew the random bits of the mechanism in advance), and therefore the mechanism cannot be interpreted as randomization over deterministic mechanisms.

Now suppose we do the price randomization before the agent is asked to pick an item. It is easy to see that the proof of revenue guarantee now fails because the agent may now decide whether to purchase or to skip to another item based on the realized price. Note that, prior to price randomization, there might be no ordering of items according to which the agent buys the first item priced below her valuation. The problem is illustrated by the following example.

Suppose the agent has i.i.d valuations for m items with the following distributions in which $1 < h < 2m$:

$$v_j = \begin{cases} h & \text{with probability } \frac{1}{2m}, \\ 1 & \text{with probability } 1 - \frac{1}{2m}. \end{cases}$$

It easy to see that $\widehat{\text{Rev}}_j$ is given by the following piecewise linear function:

$$\widehat{\text{Rev}}_j(x_j) = \begin{cases} x_j h, & x_j \leq \frac{1}{2m}, \\ \frac{1-x_j}{1-\frac{1}{2m}} \left(\frac{h}{2m} - 1 \right) + 1, & x_j > \frac{1}{2m}. \end{cases}$$

Furthermore, an optimal assignment for the convex program $(\text{REV}_{\text{unit-irreg}})$ is given by $x_j = \frac{1}{m}$ (for all j). Therefore, for each item j , the mechanism of Definition 14 offers items j at price h with probability $1 - \frac{1}{2m-1}$ and at price 1 with probability $\frac{1}{2m}$. Also it is easy to see that $S^* = \{1, \dots, m\}$. Now imagine an agent who has high valuation (e.g., h) for a subset $S' \subset [m]$ of items, and has low valuation (e.g., 1) for all other items. When faced with mechanism of Definition 14, the optimal strategy of the agent is to pick any item $j^* \in S'$ and purchase it regardless of its realized price. On the other hand, if the prices are realized before the agent makes a decision, then the agent obviously purchases the item $j^* \in S'$ that has the lowest price. The mechanism proposed by Chawla et al. [2010] also suffers from the same problem.

6.3 Multi Item (Independent), Additive, Budget Constraint

In this section, we consider an agent with a publicly known budget of B who has private independent and additive valuations for m items (i.e., her valuation for a bundle of items is the sum of her valuations for individual items in the bundle). We assume the agent's valuation for each item j is distributed independently according to a publicly known distribution with CDF F_j . To avoid complicating the proofs, we assume that each F_j is continuous and strictly increasing in its domain²³. We present a single agent mechanism which is a $(1 - \frac{1}{e})$ -approximation of the optimal revenue maximizing IPBR mechanism. This mechanism can be used with γ -pre-rounding (Definition 7) to yield a $(1 - \frac{1}{e})\gamma_k$ -approximate sequential posted pricing multi agent mechanism. The previous best approximation mechanism for this setting was an $O(1)$ -approximate²⁴ sequential posted pricing mechanism by Bhattacharya et al. [2010]. We should note that Bhattacharya et al. [2010] considers a more general setting as they allow agents to have demand constraints as well. Also they consider strictly deterministic mechanisms (i.e., no budget randomization).

As in §6.1, we start by defining the modified CDF function \tilde{F}_j for each item j as follows:

$$\tilde{F}_j(v) = \begin{cases} F_j(v), & v \leq B, \\ 1 - (1 - F_j(v)) \frac{B}{v}, & v \geq B. \end{cases} \quad (\tilde{F}_j)$$

Furthermore, for each item j , let $\text{Rev}_j(x) = x\tilde{F}_j^{-1}(1-x)$ and let $\widehat{\text{Rev}}_j$ be its concave closure as defined in §6.1. We show that the optimal value of the following convex program is an upper bound on the expected revenue of the optimal single agent IPBR mechanism:

$$\begin{aligned} & \text{maximize} && \min \left(\sum_j \widehat{\text{Rev}}_j(\bar{x}_j), B \right) && (\text{REV}_{\text{add}}) \\ & \text{subject to} && x_j \leq \bar{x}_j && \forall j \in [m], \\ & && x_j \geq 0 && \forall j \in [m]. \end{aligned}$$

Theorem 13. *The revenue of the optimal single agent IPBR mechanism, subject to an upper bound of \bar{x} on the ex ante allocation rule, is no more than the optimal revenue of convex program $(\text{REV}_{\text{add}})$.*

²³The proofs can be modified to work without this assumption.

²⁴ $\frac{1}{96}$

Proof. First, notice that if the optimal value of the convex program is B , then it is obviously an upper bound on the optimal revenue (because the agent cannot pay more than B at any instance). Otherwise, if the optimal value of the convex program is less than B , it must be $\min(\sum_j \widehat{\text{Rev}}_j(\bar{x}_j), B) = \sum_j \widehat{\text{Rev}}_j(\bar{x}_j)$, so we can fully decompose the convex program (REV_{add}) to m disjoint convex programs and solve each one separately, i.e., for each $j \in [m]$ we solve the following program:

$$\begin{aligned} & \text{maximize} && \widehat{\text{Rev}}_j(x_j) \\ & \text{subject to} && x_j \leq \bar{x}_j, \\ & && x_j \geq 0. \end{aligned}$$

By Theorem 9, the optimal value of the above convex program for each item j is an upper bound on the expected revenue of any single agent *IPBR* mechanism for item j in the absence of other items. It is easy to see that the presence of other items may only reduce the revenue obtained by an *IPBR* mechanism from item j . So assuming x_j (for each j) is an optimal assignment for the convex program of item j , we argue that x is also an optimal assignment for the joint convex program (REV_{add}) , yielding an optimal value of $\sum_j R_j(x_j)$ which is an upper bound on the expected revenue of the optimal *IPBR* mechanism. \square

Next, we present a $(1 - \frac{1}{e})$ -approximate revenue maximizing single agent *IPBR* mechanism.

Definition 15 (Mechanism).

- (I) Define the single agent benchmark $R(\bar{x})$ to be the optimal value of convex program (REV_{add}) as a function of \bar{x} .
- (II) Given \bar{x} , solve the convex program (REV_{add}) and let x^* denote an optimal assignment.
- (III) For each item j , if $\widehat{\text{Rev}}_j(x_j^*) = \text{Rev}_j(x_j^*)$, offer item j at price $p_j = \tilde{F}_j^{-1}(1 - x_j^*)$, otherwise compute $x_j^-, x_j^+, \theta_j \in [0, 1]$ such that $x_j^* = \theta_j x_j^- + (1 - \theta_j) x_j^+$ and $\widehat{\text{Rev}}_j(x_j^*) = \theta_j \text{Rev}_j(x_j^-) + (1 - \theta_j) \text{Rev}_j(x_j^+)$ (this is always possible by the definition of concave closure), and offer the item at price $p_j^- = \tilde{F}_j^{-1}(1 - x_j^-)$ with probability θ_j and price $p_j^+ = \tilde{F}_j^{-1}(1 - x_j^+)$ with probability $1 - \theta_j$. Note that the price randomization must be performed for each item independently.

Theorem 14. The mechanism of Definition 15 obtains at least $1 - \frac{1}{e}$ fraction of the revenue of the optimal single agent *IPBR* mechanism in expectation. Furthermore, this mechanism satisfies the requirements of γ -pre-rounding.

Proof. First, we show that the expected revenue of the mechanism is at least $1 - \frac{1}{e}$ fraction of $R(\bar{x})$ which by Theorem 13 is an upper bound on the optimal revenue. Consider an imaginary agent who has exactly the same valuations as the original agent, but has a separate budget B for each item. We call this imaginary agent the *super replica*. Furthermore, suppose that any payment made by the super replica with a total amount beyond B is lost (i.e., if the super replica pays Z , the mechanism receives only $\min(Z, B)$). Observe that for any assignment of prices, the payment received from the original agent and the payment received from the super replica are exactly the same because either (a) the original agent has not hit his budget limit, so both the original agent and the super replica buy the same set of items and pay the exact same amount, or (b) the original agent has reached his budget limit, so the mechanism receives exactly B from both the original agent and the super replica. Consequently, we only need to show that the revenue obtained by the mechanism from the super replica is at least $(1 - \frac{1}{e})R(\bar{x})$. Observe that from the view point of the super replica

there is no connection between different items, so the super replica makes a decision for each item independently. Let Z_j be the random variable corresponding to the payment made by the super replica for item j . By Theorems 9 and 10 we can argue that $\mathbf{E}[Z_j] = \widehat{\text{Rev}}_j(x_j^*)$. Furthermore, the total revenue received by the mechanism is $Z = \min(\sum_j Z_j, B)$. Notice that Z_1, \dots, Z_m are independent random variables that take values in the range $[0, B]$. By invoking Lemma 5, we can argue that $\mathbf{E}[\min(\sum_j Z_j, B)] \geq (1 - \frac{1}{e}) \min(\sum_j \mathbf{E}[Z_j], B) = (1 - \frac{1}{e})R(\bar{x})$ which proves our claim.

Next, we show that the mechanism satisfies the requirements of γ -pre-rounding. First observe that $R(\bar{x})$ is concave by Lemma 1. Let $x_j^* = \arg \max_{x' \in [0, \bar{x}_j]} \widehat{\text{Rev}}_j(x')$ for each j . Observe that $R(\bar{x}[S]) = \min(\sum_{j \in S} \widehat{\text{Rev}}_j(x_j^*), B)$ for all $S \subseteq [m]$. It is easy to see that $R(\bar{x}[S])$ is a budget additive submodular function in S , and therefore it has a cross monotonic budget balanced cost share scheme. \square

Lemma 5. *Let B be an arbitrary positive number and let Z_1, \dots, Z_m be independent random variables such that $Z_j \in [0, B]$ for all j . Then the following inequality holds:*

$$\mathbf{E} \left[\min \left(\sum_j Z_j, B \right) \right] \geq \left(1 - \frac{1}{e^{(\sum_j \mathbf{E}[Z_j])/B}} \right) B \geq \left(1 - \frac{1}{e} \right) \min \left(\sum_j \mathbf{E}[Z_j], B \right).$$

Proof. See §A. \square

6.4 Multi Item (Correlated), Additive, Budget and Matroid Constraints

In this section, we consider an agent with publicly known budget B who has private correlated additive valuations for m items; furthermore, a bundle of items can be allocated to the agent only if it is an independent set of a matroid $\mathcal{M} = ([m], \mathcal{I})$, where \mathcal{M} is publicly known; alternatively, instead of treating \mathcal{M} as a constraint on the allocation, we may assume that the agent has matroid valuations (Definition 10). We assume that the agent has a discrete type space T . Let $v_t \in \mathbb{R}_+^m$ denote the agent's valuation vector corresponding to type $t \in T$, and let $f(t)$ denote the probability of the agent having type t . We assume that f is represented explicitly as a part of the input, i.e., by enumerating all types along with their respective probabilities. The only private information of the agent is her type. We present an optimal single agent randomized mechanism. This mechanism can be used with γ -post-rounding (Definition 9) to obtain a γ_k -approximate multi agent BIC mechanism. Recall that γ_k is at least $\frac{1}{2}$, and approaches 1 as $k \rightarrow \infty$, which means the resulting multi agent mechanism approaches the optimal multi agent mechanism as $k \rightarrow \infty$.

Prior to the preliminary version of this work, the best approximation for this setting was a $\frac{1}{4}$ -approximate BIC mechanism by Bhattacharya et al. [2010]²⁵. At the time of writing the current version, Henzinger and Vidali [2011] has also presented a $\frac{1}{2}$ -approximate BIC mechanism for the same setting. Also recently, an algorithm has been proposed by Cai et al. [2012b] to compute a $1 - \epsilon$ -approximate BIC mechanism in time polynomial in $\frac{1}{\epsilon}$ and the total number of types of all agents. The mechanism of Cai et al. [2012b] applies to more general settings, however it needs to solve an optimization problem with exponentially many inter agent linear constraints, inevitably requiring the use of the ellipsoid method. On the other hand, the mechanism proposed in this paper needs to solve an optimization problem (i.e., convex program (\overline{OPT})) that has only k inter agent linear constraints which is more practical. Also, the approximation factor of our mechanism is γ_k which is close to optimal for relatively large k . Note that all of the aforementioned mechanisms

²⁵The mechanism in Bhattacharya et al. [2010] considers demand constraint, which is a special case of matroid constraints.

(including the one presented here) have running times polynomial only in $|T|$, which means their running time may not be polynomial in the input size if $|T|$ is exponential in n and m , and f has a compact representation.

Consider the following linear (LP) program in which $x_t \in [0, 1]^m$ represents the marginal allocation probabilities for type $t \in T$, and p_t represents the corresponding payment. Also let $r_{\mathcal{M}} : 2^m \rightarrow \{0, \dots, m\}$ denote the rank function of \mathcal{M} . The optimal value of this LP is obviously an upper bound on the optimal revenue:

$$\begin{array}{ll}
\text{maximize} & \sum_{t \in T} f(t) p_t \quad (\text{REV}_{\text{corr}}) \\
\text{subject to} & \sum_{t \in T} f(t) x_{tj} \leq \bar{x}_j \quad \forall j \in [m], \\
& \sum_{j \in S} x_{tj} \leq r_{\mathcal{M}}(S) \quad \forall t \in T, \forall S \subseteq [m], \\
& v_t \cdot x_t - p_t \geq v_t \cdot x_{t'} - p_{t'} \quad \forall t, t' \in T, \\
& x_t \in [0, 1]^m \quad \forall t \in T, \\
& p_t \in [0, B] \quad \forall t \in T.
\end{array}$$

Even though the above LP has exponentially many constraints, it can be solved in polynomial time²⁶. Also for special matroids (such as uniform matroids or partition matroids) only polynomially many constraints are needed. Next, we present a mechanism whose expected revenue is equal to the optimal value of the above LP, which also implies it is an optimal mechanism.

Definition 16 (Single Agent Mechanism).

- (I) Define the optimal benchmark $R(\bar{x})$ to be the optimal value of $(\text{REV}_{\text{corr}})$ as a function of \bar{x} .
- (II) Given \bar{x} , solve the LP of $(\text{REV}_{\text{corr}})$ and let x^* and p^* be an optimal assignment.
- (III) Let t be the agent's reported type. Allocate a random subset $X \subseteq [m]$ of items such that X is an independent set of \mathcal{M} and each item $j \in [m]$ is included in X with a marginal probability of exactly x_{tj}^* . This can be achieved by rounding x_t^* to a vertex of the matroid polytope using dependent randomized rounding (see Chekuri et al. [2010] and references therein). Also charge a payment of p_t^* .

Theorem 15. *The mechanism of Definition 16 is an optimal revenue maximizing single agent mechanism which is truthful in expectation, subject to an upper bound of \bar{x} on the ex ante allocation rule. Furthermore, it satisfies the requirements of γ -post-rounding.*

Proof. The proof of truthfulness and optimality trivially follows from $(\text{REV}_{\text{corr}})$. So, we only prove this mechanism satisfies the requirements of Theorem 8. First, observe that the benchmark function, $R(\bar{x})$, is concave (this follows from Lemma 1). Second, observe that the matroid constraints can be incorporated into a matroid valuation function for the agent. Third, notice that the exact ex ante allocation rule can be readily computed from the LP solution, i.e., $\hat{x}_j = \sum_t f(t) x_{tj}$ is the exact probability of allocating item j . Finally, the space of feasible allocations is downward closed. Therefore, the mechanism satisfies the requirements of γ -post-rounding. \square

²⁶See Schrijver [2003] for optimization over a matroid polytope.

Remark 1. *Observe that by replacing the objective function of $(\text{REV}_{\text{corr}})$ with $\sum_{t \in T} f(t)v_t \cdot x_t$, we obtain a welfare maximizing single agent mechanism that is truthful in expectation and also satisfies the requirements of γ -post-rounding.*

7 The Generalized Magician's Problem

This section presents a generalization of the magician's problem along with a near-optimal solution and the corresponding proofs. The original problem was introduced by Alaei [2011] (i.e., a preliminary version of the current paper), while the generalization presented here was introduced by Alaei et al. [2013].

Definition 17 (The Generalized Magician's Problem). *A magician is presented with a sequence of boxes one by one in an online fashion. There is a prize hidden in one of the boxes. The magician has a magic wand that can be used to open the boxes. The wand has k units of mana²⁷. If the wand is used on box i and has at least 1 unit of mana, the box opens, but the wand loses a random amount of mana $X_i \in [0, 1]$ drawn from a distribution specified on the box by its cumulative distribution function F_{X_i} (i.e., the magician learns F_{X_i} upon seeing box i). The magician wishes to maximize the probability of obtaining the prize, but unfortunately the sequence of boxes, the distributions written on the boxes, and the box containing the prize have been arranged by a villain; the magician has no prior information (not even the number of boxes); however, it is guaranteed that $\sum_i \mathbf{E}[X_i] \leq k$, and that the villain has to prepare the sequence of boxes in advance (i.e., cannot make any changes once the process has started).*

The magician could fail to open a box either because (a) he might choose to skip the box, or (b) his wand might run out of mana before getting to the box. Note that once the magician fixes his strategy, the best strategy for the villain is to put the prize in the box which, based on the magician's strategy, has the lowest ex ante probability of being opened. Therefore, in order for the magician to obtain the prize with a probability of at least γ , he has to devise a strategy that guarantees an ex ante probability of at least γ for opening each box. Notice that allowing the prize to be split among multiple boxes does not affect the problem. We present an algorithm parameterized by a probability $\gamma \in [0, 1]$ which guarantees a minimum ex-ante probability of γ for opening each box while trying to minimize the mana used. We show that for $\gamma \leq 1 - \frac{1}{\sqrt{k}}$ this

²⁷ "Mana is an indigenous Pacific islander concept of an impersonal force or quality that resides in people, animals, and inanimate objects. . . . Modern fantasy fiction, computer and role-playing games have adopted mana as a term for magic points, an expendable (and most often rechargeable) resource out of which magic users form their magical spells." Wikipedia [2012]

algorithm never requires more than k units of mana.

Definition 18 (Generalized γ -Conservative Magician). *The magician adaptively computes a sequence of thresholds $\theta_1, \theta_2, \dots \in \mathbb{R}_+$ and makes a decision about each box as follows: let W_i denote the amount of mana lost prior to seeing the i^{th} box; the magician makes a decision about box i by comparing W_i against θ_i ; if $W_i < \theta_i$, it opens the box; if $W_i > \theta_i$, it does not open the box; and if $W_i = \theta_i$, it randomizes and opens the box with some probability (to be defined). The magician chooses the smallest threshold θ_i for which $\Pr[W_i \leq \theta_i] \geq \gamma$ where the probability is computed ex ante (i.e., not conditioned on X_1, \dots, X_{i-1}). Note that γ is a parameter that is given. Let $F_{W_i}(w) = \Pr[W_i \leq w]$ denote the ex ante CDF of random variable W_i , and let S_i be the indicator random variable which is 1 iff the magician chooses to open box i . Formally, the probability with which the magician should open box i condition on W_i is computed as follows:²⁸*

$$\Pr[S_i = 1 | W_i] = \begin{cases} 1, & W_i < \theta_i \\ (\gamma - F_{W_i}^-(\theta_i)) / (F_{W_i}(\theta_i) - F_{W_i}^-(\theta_i)), & W_i = \theta_i \\ 0, & W_i > \theta_i, \end{cases} \quad (S)$$

$$\theta_i = \inf\{w | F_{W_i}(w) \geq \gamma\}. \quad (\theta)$$

In the above definition, $F_{W_i}^-$ is the left limit of F_{W_i} , i.e., $F_{W_i}^-(w) = \Pr[W_i < w]$.

Note that $F_{W_{i+1}}$ and $F_{W_{i+1}}^-$ are fully determined by F_{W_i} and F_{X_i} and the choice of γ (see Theorem 17). Observe that θ_i is in fact computed before seeing box i itself.

In order to prove that a γ -conservative magician does not fail for a given choice of γ , we must show that the thresholds θ_i are no more than $k - 1$. The following theorem states a condition on γ that is sufficient to guarantee that $\theta_i \leq k - 1$ for all i .

Theorem 16 (Generalized γ -Conservative Magician). *For any $\gamma \leq 1 - \frac{1}{\sqrt{k}}$, a generalized γ -conservative magician with k units of mana opens each box with an ex ante probability of γ exactly. If all X_i are Bernoulli random variables (i.e., $X_i \in \{0, 1\}$ for all i), then for any $\gamma \leq 1 - \frac{1}{\sqrt{k+3}}$, a generalized γ -conservative magician with k units of mana opens each box with an ex ante probability of exactly γ . In the latter case, if F_{X_i} only stochastically dominates the actual CDF of X_i (for each i), then the magician opens each box with an ex ante probability of at least γ .*

Proof. See §7.1. □

Definition 19 (γ_k^*). *We define γ_k^* to be the largest probability such that for any $k' \geq k$ and any instance of the magician's problem with k' units of mana, the thresholds computed by a generalized γ_k^* -conservative magician are no more than $k' - 1$. In other words, γ_k^* is the optimal choice of γ which works for all instances with $k' \geq k$ units of mana. By Theorem 16, γ_k^* must be²⁹ at least $1 - \frac{1}{\sqrt{k}}$.*

By definition, both γ_k^* and γ_k are non-decreasing in k and by Theorem 16 they both approach 1 as $k \rightarrow \infty$. However, $\gamma_1^* = 0$ whereas $\gamma_1 = \frac{1}{2}$. In fact we show that both of these bounds are exact for $k = 1$.

Proposition 1. *For the generalized magician's problem with $k = 1$, no algorithm for the magician (online or offline³⁰) can guarantee a constant positive probability for opening each box.*

²⁸ Assume $W_0 = 0$.

²⁹ Because for any $k' \geq k$ obviously $1 - \frac{1}{\sqrt{k}} \leq 1 - \frac{1}{\sqrt{k'}}$.

³⁰ Offline means the magician can see the whole sequence of boxes in advance and can pick the boxes he wants to open in any order.

Proof. Suppose there is an algorithm for the magician that is guaranteed to open each box with a probability of at least $\gamma \in (0, 1]$. We construct an instance in which the algorithm fails. Let $n = \lceil \frac{1}{\gamma} \rceil + 1$. Suppose all X_i are (independently) drawn from the following distribution:

$$X_i = \begin{cases} \frac{1}{2n} & \text{with probability } 1 - \frac{1}{2n} \\ 1 & \text{with probability } \frac{1}{2n} \end{cases} \quad \forall i \in [n].$$

As soon as the magician opens a box, the remaining mana will be less than 1, so he will not be able to open any other box, i.e., the magician can open only one box at every instance. Let S_i denote the indicator random variable which is 1 iff the magician opens box i . Since $\sum_i S_i \leq 1$, it must be $\sum_i \mathbf{E}[S_i] \leq 1$. On the other hand, $\mathbf{E}[S_i] \geq \gamma$ because the magician has guaranteed to open each box with a probability of at least γ . However $\sum_i \mathbf{E}[S_i] \geq n\gamma > 1$ which is a contradiction. Note that $\sum_i \mathbf{E}[X_i] < 1$ so it satisfies the requirement of Definition 17. \square

Proposition 2. *For the generalized magician's problem for $k = 1$, when all X_i are Bernoulli random variables, no algorithm for the magician can guarantee a probability of more than $\frac{1}{2}$ for opening each box.*

Proof. Suppose there is an algorithm for the magician that is guaranteed to open each box with a probability of at least $\gamma \in (0.5, 1]$. We construct an instance in which the algorithm fails. Pick any $\delta \in (\frac{1}{2\gamma}, 1]$. Suppose there are two boxes with the following distributions:

$$X_1 = \begin{cases} 1 & \text{with probability } \delta, \\ 0 & \text{otherwise,} \end{cases}$$

$$X_2 = \begin{cases} 1 & \text{with probability } 1 - \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the algorithm must open the first box with probability at least γ , so the probability that there is enough mana left for the second box is at most $1 - \gamma\delta < \frac{1}{2}$; therefore the algorithm will not be able to open the second box with a probability of $\frac{1}{2}$ or more. Note that $\sum_i \mathbf{E}[X_i] = 1$, so it satisfies the requirement of Definition 17. \square

Computation of $F_{W_i}(\cdot)$. For every $i \in [n]$, the equation $W_{i+1} = W_i + S_i X_i$ relates the distribution of W_{i+1} to those of W_i and X_i ³¹. The following theorem shows that the distribution of W_{i+1} is fully determined by the information available to the magician before seeing box $i + 1$.

Theorem 17. *In the algorithm of the generalized γ -conservative magician (Definition 18), the choice of γ and the distributions of X_1, \dots, X_i fully determine the distribution of W_{i+1} for every $i \in [n]$. In particular, $F_{W_{i+1}}$ can be recursively defined as follows:*

$$F_{W_{i+1}}(w) = F_{W_i}(w) - G_i(w) + \mathbf{E}_{X_i \sim F_{X_i}} [G_i(w - X_i)] \quad \forall i \in [n], \forall w \in \mathbb{R}_+, \quad (F_w)$$

$$G_i(w) = \min(F_{W_i}(w), \gamma) \quad \forall i \in [n], \forall w \in \mathbb{R}_+. \quad (G)$$

Proof. See §7.1. The claim follows directly from Lemma 6. \square

As a corollary of Theorem 17, we show how F_{W_i} can be computed using dynamic programming, assuming X_i can only take discrete values.

³¹Note that the distribution of S_i is dependent on/determined by W_i .

Corollary 2. *If all X_i are proper multiples of $\frac{1}{D}$ for some $D \in \mathbb{N}$, then F_{w_i} can be computed recursively for all $i \in [n]$ and $w \in \mathbb{R}_+$ as follows:*

$$F_{w_{i+1}}(w) = \begin{cases} F_{w_i}(w) - G_i(w) + \sum_{\ell=0}^D \mathbf{Pr}[X_i = \frac{\ell}{D}] G_i(w - \frac{\ell}{D}), & i \geq 1, w \geq 0, \\ 1, & i = 0, w \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the γ -conservative magician makes a decision for each box in $O(D)$ time.

Note that it is enough to compute F_{w_i} only for proper multiples of $\frac{1}{D}$ because $F_{w_i}(w) = F_{w_i}(\frac{\lfloor Dw \rfloor}{D})$ for any $w \in \mathbb{R}_+$.

7.1 Analysis of Generalized γ -Conservative Magician

We present the proof of Theorems 16 and 17. We start by proving Theorem 16 in two parts. In the first part, we show that the thresholds computed by the generalized γ -conservative magician indeed guarantee that each box is opened with an ex ante probability of γ , assuming there is unlimited mana. In the second part, we show that for any $\gamma \leq 1 - \frac{1}{\sqrt{k+3}}$ (in case of Bernoulli X_i , for any $\gamma \leq 1 - \frac{1}{\sqrt{k}}$), the thresholds θ_i are no more than $k - 1$ for all i , which implies that the magician never requires more than k units of mana. It can be shown that a non-adaptive algorithm cannot guarantee a probability of more than $1 - O(\frac{\sqrt{\ln k}}{\sqrt{k}})$ for opening each box.

Below, we repeat the formulation of the threshold based strategy of the magician. Recall that W_i denotes the amount of mana lost prior to seeing the i^{th} box, and S_i is the indicator random variable corresponding to opening box i . Also recall that the strategy of the magician can be specified as follows:

$$\mathbf{Pr}[S_i = 1 | W_i] = \begin{cases} 1, & W_i < \theta_i, \\ (\gamma - F_{w_i}^-(\theta_i)) / (F_{w_i}(\theta_i) - F_{w_i}^-(\theta_i)), & W_i = \theta_i, \\ 0, & W_i > \theta_i, \end{cases} \quad (S_i)$$

$$\theta_i = \inf\{w | F_{w_i}(w) \geq \gamma\}. \quad (\theta)$$

Part 1. We show that the thresholds computed by a generalized γ -conservative magician guarantee that each box is opened with an ex ante probability of γ (i.e., $\mathbf{Pr}[S_i = 1] = \gamma$), assuming there is unlimited mana:

$$\begin{aligned} \mathbf{Pr}[S_i \leq w] &= \mathbf{Pr}[S_i = 1 \cap W_i < \theta_i] + \mathbf{Pr}[S_i = 1 \cap W_i = \theta_i] + \mathbf{Pr}[S_i = 1 \cap W_i > \theta_i] \\ &= \mathbf{Pr}[W_i < \theta_i] + \frac{\gamma - F_{w_i}^-(\theta_i)}{F_{w_i}(\theta_i) - F_{w_i}^-(\theta_i)} \mathbf{Pr}[W_i = \theta_i] \\ &= \gamma. \end{aligned}$$

In the case of all X_i being Bernoulli random variables, we show that the thresholds computed by a generalized γ -conservative magician guarantee that each box is opened with an ex ante probability at least γ , assuming F_{X_i} stochastically dominates the actual CDF of X_i for all i and assuming there is unlimited mana. Let $x_i = \mathbf{E}_{X'_i \sim F_{X_i}}[X'_i]$; thus x_i is an upper bound on $\mathbf{Pr}[X_i = 1]$ (recall that $X_i \in \{0, 1\}$). Also as a shorthand notation we define $s_i^\ell = \mathbf{Pr}[S_i = 1 | W_i = \ell]$.³² The proof is as follows.

³²Note that W_i may only take integral values when X_i are Bernoulli random variables.

- (a) First we prove that $\Pr[W_i \leq \ell] \geq F_{W_i}(\ell)$ for all i and ℓ which implies that $F_{W_i}(\ell)$ stochastically dominates the actual CDF of W_i . The proof is by induction on i . The base case (i.e., $i = 1$) is trivial. Supposing the inequality holds for i , we prove it for $i + 1$ as follows:

$$\begin{aligned}
\Pr[W_{i+1} \leq \ell] &\geq \Pr[W_i \leq \ell - 1] + \Pr[W_i = \ell](1 - s_i^\ell x_i) \\
&= \Pr[W_i \leq \ell - 1] s_i^\ell x_i + \Pr[W_i \leq \ell](1 - s_i^\ell x_i) \\
&\geq F_{W_i}(\ell - 1) s_i^\ell x_i + F_{W_i}(\ell)(1 - s_i^\ell x_i) && \text{by induction hypothesis} \\
&= F_{W_{i+1}}(\ell) && \text{by } (F_w).
\end{aligned} \tag{1.a}$$

- (b) Next, we show that each box is opened with probability at least γ . We will show that $\Pr[S_i = 1] \geq \gamma$:

$$\begin{aligned}
\Pr[S_i = 1] &= \sum_{\ell} \Pr[S_i = 1 | W_i = \ell] \Pr[W_i = \ell] \\
&= \sum_{\ell=0}^{\theta_i} s_i^\ell \Pr[W_i = \ell] \\
&= \Pr[W_i < \theta_i] + s_i^{\theta_i} \Pr[W_i = \theta_i] && \text{because } s_i^\ell = 1 \text{ for } \ell < \theta_i \\
&= (1 - s_i^{\theta_i}) \Pr[W_i < \theta_i] + s_i^{\theta_i} \Pr[W_i \leq \theta_i] \\
&\geq (1 - s_i^{\theta_i}) F_{W_i}(\theta_i - 1) + s_i^{\theta_i} F_{W_i}(\theta_i) && \text{by (1.a)} \\
&= F_{W_i}(\theta_i - 1) + s_i^{\theta_i} (F_{W_i}(\theta_i) - F_{W_i}(\theta_i - 1)) \\
&= \gamma && \text{by substituting } s_i^{\theta_i} \text{ from (S).}
\end{aligned}$$

Part 2. Assuming $\gamma \leq 1 - \frac{1}{\sqrt{k}}$ (or $\gamma \leq 1 - \frac{1}{\sqrt{k+3}}$ in the case of all X_i being Bernoulli), we show that the thresholds computed by a generalized γ -conservative magician are no more than $k - 1$ (i.e., $\theta_i \leq k - 1$ for all i). First, we present an interpretation of how F_{W_i} evolves in i .

Definition 20 (Sand Waves). *Consider one unit of infinitely divisible sand which is initially at position 0 on the real line. The sand is gradually moved to the right and distributed over the real line in n waves. Let $F_{W_i}(w)$ denote the total amount of sand in the interval $[0, w]$ at the beginning of the i^{th} wave. The following takes place at each wave $i \in [n]$.*

- (I) *The leftmost γ -fraction of the sand is selected by first identifying the smallest threshold $\theta_i \in \mathbb{R}_+$ such that $F_{W_i}(\theta_i) \geq \gamma$ and then selecting all the sand in the interval $[0, \theta_i)$ and selecting a fraction of the sand at position θ_i itself such that the total amount of selected sand is equal to γ . Formally, for any $w \in \mathbb{R}_+$ the total amount of sand selected from the interval $[0, w]$ is given by $G_i(w) = \min(F_{W_i}(w), \gamma)$. Note that only a fraction of the sand at position θ_i itself might be selected; however, all the sand to the left of position θ_i is selected.*
- (II) *The selected sand is moved to the right as follows. We are given a random variable $X_i \in [0, 1]$ with CDF F_{X_i} . For every point $w \in \mathbb{R}_+$ and every distance $\delta \in [0, 1]$, a fraction proportional to $dF_{X_i}(\delta)$ is taken out of the sand selected from position w and moved to position $w + \delta$. In other words, for every w , all the selected sand at position w is spread in the interval $[w, w + 1]$ according to the distribution of X_i .*

It is easy to see that θ_i and $F_{W_i}(w)$ resulting from the above process are exactly the same as those resulting from the generalized γ -conservative magician.

Lemma 6. *At the end of the i^{th} wave, the total amount of sand in the interval $[0, w]$ is given by the following equation:*

$$F_{w_{i+1}}(w) = F_{w_i}(w) - G_i(w) + \mathbf{E}_{X_i \sim F_{X_i}} [G_i(w - X_i)] \quad \forall i \in [n], \forall w \in \mathbb{R}_+. \quad (F_w)$$

Proof. According to the definition of the sand displacement process, $F_{w_{i+1}}(w)$ can be defined as follows:

$$\begin{aligned} F_{w_{i+1}}(w) &= (F_{w_i}(w) - G_i(w)) + \iint_{\sigma + \delta \leq w} dG_i(\sigma) dF_{X_i}(\delta) \\ &= F_{w_i}(w) - G_i(w) + \int G_i(w - \delta) dF_{X_i}(\delta) \\ &= F_{w_i}(w) - G_i(w) + \mathbf{E}_{X_i \sim F_{X_i}} [G_i(w - X_i)]. \end{aligned}$$

□

Consider a conceptual barrier which is at position $\theta_i + 1$ at the beginning of each wave $i \in [n]$ and is moved to position $\theta_{i+1} + 1$ before the next wave. It is easy to verify (i.e., by induction) that the sand never crosses to the right side of the barrier (i.e., $F_{w_{i+1}}(\theta_i + 1) = 1$). The following theorem implies that the sand remains concentrated near the barrier throughout the process.

Theorem 18 (Sand). *Throughout the process of Definition 20, the following inequality holds at the beginning of each wave $i \in [n]$:*

$$F_{w_i}(w) < \gamma F_{w_i}(w + 1) \quad \forall i \in [n], \forall w \in [0, \theta_i). \quad (F_w\text{-ineq})$$

Furthermore, at the beginning of wave $i \in [n]$, the average distance of the sand from the barrier, denoted by d_i , is upper bounded by the following inequalities in which the first inequality is strict except for $i = 1$ (recall that $\{z\} = z - \lfloor z \rfloor$ for all $z \in \mathbb{R}$):

$$d_i \leq (1 - \{\theta_i\}) \frac{1 - \gamma^{\lfloor \theta_i \rfloor + 1}}{1 - \gamma} + \{\theta_i\} \frac{1 - \gamma^{\lceil \theta_i \rceil + 1}}{1 - \gamma} \leq \frac{1 - \gamma^{\lceil \theta_i \rceil + 1}}{1 - \gamma} < \frac{1}{1 - \gamma} \quad \forall i \in [n]. \quad (d)$$

Proof. We start by proving the inequality (F_w -ineq). The proof is by induction on i . The case of $i = 1$ is trivial because all the sand is at position 0 and so $\theta_1 = 0$. Suppose the inequality holds at the beginning of wave i for all $w \in [0, \theta_i)$; we show that it holds at the beginning of wave $i + 1$ for all $w \in [0, \theta_{i+1})$. Note that $\theta_i \leq \theta_{i+1} \leq \theta_i + 1$, so it could be either of the following two possible cases:

Case 1. $w \in [0, \theta_i)$. Observe that $G_i(w) = F_{w_i}(w)$ for all such w , so

$$\begin{aligned} F_{w_{i+1}}(w) &= F_{w_i}(w) - G_i(w) + \mathbf{E}_{X_i} [G_i(w - X_i)] && \text{by } (F_w) \\ &= \mathbf{E}_{X_i} [F_{w_i}(w - X_i)] && \text{by } G_i \text{ being equal to } F_{w_i} \text{ over } [0, \theta_i) \\ &< \mathbf{E}_{X_i} [\gamma F_{w_i}(w - X_i + 1)] && \text{by induction hypothesis} \\ &= \gamma \mathbf{E}_{X_i} [F_{w_i}(w - X_i + 1) - G_i(w - X_i + 1) + G_i(w - X_i + 1)] \\ &\leq \gamma (F_{w_i}(w + 1) - G_i(w + 1) + \mathbf{E}_{X_i} [G_i(w - X_i + 1)]) && \text{by monotonicity of } F_{w_i} - G_i \\ &= \gamma F_{w_{i+1}}(w + 1) && \text{by } (F_w). \end{aligned}$$

Case 2. $w \in [\theta_i, \theta_{i+1}]$. We prove the claim by showing $F_{w_{i+1}}(w) < \gamma$ and $F_{w_{i+1}}(w + 1) = 1$. Observe that $F_{w_{i+1}}(w) < \gamma$ because $w < \theta_{i+1}$ and because of how θ_{i+1} is defined in (θ).

Furthermore, $F_{w_{i+1}}(\theta_i + 1) = 1$ (because after wave i all the sand is still contained in the interval $[0, \theta_i + 1]$), and $w \geq \theta_i$, so $F_{w_{i+1}}(w + 1) = 1$.

Next, we prove inequality (d), which gives an upper bound on the average distance of the sand from the barrier at the beginning of wave $i \in [n]$:

$$\begin{aligned}
d_i &= \int_0^{\theta_i+1} (\theta_i + 1 - w) dF_{w_i}(w) \\
&= \int_0^{\theta_i+1} F_{w_i}(w) dw && \text{by integration by part} \\
&= \sum_{\ell=0}^{\lfloor \theta_i \rfloor} \int_{\theta_i-\ell}^{\theta_i+1-\ell} F_{w_i}(w) dw \\
&\leq \sum_{\ell=0}^{\lfloor \theta_i \rfloor} \int_{\theta_i}^{\theta_i+1} \gamma^\ell F_{w_i}(w) dw + \int_{\lfloor \theta_i \rfloor+1}^{\theta_i+1} \gamma^{\lfloor \theta_i \rfloor} F_{w_i}(w) dw && \text{by } (F_w\text{-ineq}) \\
&\leq \sum_{\ell=0}^{\lfloor \theta_i \rfloor} \gamma^\ell + \{\theta_i\} \gamma^{\lfloor \theta_i \rfloor} && \text{by } F_{w_i}(w) \leq 1 \\
&= (1 - \{\theta_i\}) \sum_{\ell=0}^{\lfloor \theta_i \rfloor} \gamma^\ell + \{\theta_i\} \sum_{\ell=0}^{\lfloor \theta_i \rfloor} \gamma^\ell \\
&= (1 - \{\theta_i\}) \frac{1 - \gamma^{\lfloor \theta_i \rfloor+1}}{1 - \gamma} + \{\theta_i\} \frac{1 - \gamma^{\lfloor \theta_i \rfloor+1}}{1 - \gamma} \\
&\leq \frac{1 - \gamma^{\lfloor \theta_i \rfloor+1}}{1 - \gamma}.
\end{aligned}$$

The last inequality follows because $(1 - \beta)L + \beta H \leq H$ for any $\beta \in [0, 1]$ and any L, H with $L \leq H$. Note that at least one of the first two inequalities is strict except for $i = 1$, which proves the claim. \square

Theorem 19 (Barrier). *If $\sum_{i=1}^n \mathbf{E}_{X_i \sim F_{X_i}}[X_i] \leq k$ for some $k \in \mathbb{N}$, and $\gamma \leq 1 - \frac{1}{\sqrt{k}}$ (or $\gamma \leq 1 - \frac{1}{\sqrt{k+3}}$ and all X_i are Bernoulli random variables), then the distance of the barrier from the origin is no more than k throughout the process, i.e., $\theta_i \leq k - 1$ for all $i \in [n]$.*

Proof. At the beginning of each wave $i \in [n]$, let d_i and d'_i denote the average distance of the sand from the barrier and from the origin respectively. Recall that the barrier is defined to be at position $\theta_i + 1$ at the beginning of wave i . Observe that $d_i + d'_i = \theta_i + 1$. Furthermore, $d'_{i+1} = d'_i + \gamma \mathbf{E}[X_i]$, i.e., the average distance of the sand from the origin is increased exactly by $\gamma \mathbf{E}[X_i]$ during wave i (because the amount of selected sand is exactly γ and the sand selected from every position $w \in [0, \theta_i]$ is moved to the right by an expected distance of $\mathbf{E}[X_i]$). By applying Theorem 18 we get the following inequality for all $i \in [n]$:

$$\begin{aligned}
\theta_i + 1 &= d'_i + d_i \\
&< \gamma \sum_{r=1}^{i-1} \mathbf{E}[X_r] + d_i \\
&\leq \gamma k + (1 - \{\theta_i\}) \frac{1 - \gamma^{\lfloor \theta_i \rfloor+1}}{1 - \gamma} + \{\theta_i\} \frac{1 - \gamma^{\lfloor \theta_i \rfloor+1}}{1 - \gamma}.
\end{aligned} \tag{\Gamma}$$

In order to show that the distance of the barrier from the origin is no more than k throughout the process, it is enough to show that the above inequality cannot hold for $\theta_i > k - 1$. In fact it is just enough to show that it cannot hold for $\theta_i = k - 1$; alternatively, it is enough to show that the complement of the above inequality holds for $\theta_i = k - 1$:

$$k \geq \gamma k + \frac{1 - \gamma^k}{1 - \gamma}.$$

Consider the stronger inequality $k \geq \gamma k + \frac{1}{1 - \gamma}$; this inequality is quadratic in γ and can be solved to get a bound of $\gamma \leq 1 - \frac{1}{\sqrt{k}}$.

Next, consider the case in which all X_i are Bernoulli (i.e., $X_i \in \{0, 1\}$ for all i). Observe that the barrier will take only integer positions; therefore, to show that the distance of the barrier from the origin is no more than k , it is enough to show that inequality (Γ) cannot hold for $\theta_i = k$; alternatively, it is enough to show that the complement of that inequality holds for $\theta_i = k$:

$$k + 1 \geq \gamma k + \frac{1 - \gamma^{k+1}}{1 - \gamma}.$$

Consider the stronger inequality $k + 1 \geq \gamma k + \frac{1}{1 - \gamma}$, which is quadratic in γ and yields a bound of $\gamma \leq 1 - \frac{1}{1/2 + \sqrt{k+1/4}}$; this bound in fact imposes a looser constraint than $\gamma \leq 1 - \frac{1}{\sqrt{k+3}}$ when $k \geq 7$. Furthermore it can be verified that the inequality holds for $k < 7$ and $\gamma \leq 1 - \frac{1}{\sqrt{k+3}}$. That completes the proof. \square

Notice that Theorem 19 immediately implies that a generalized γ -conservative magician requires no more than k units of mana, assuming that $\gamma \leq 1 - \frac{1}{\sqrt{k}}$ (or assuming $\gamma \leq 1 - \frac{1}{\sqrt{k+3}}$ and all X_i are Bernoulli). That completes the proof of Theorem 16.

8 Multi Unit Demands

In this section, we show that the more general model, in which each agent may need more than one unit but no more than $\frac{1}{k}$ of all units of each item, can be reduced to the simpler model in which there are at least k units of every item and no agent demands more than 1 unit of each item.

Definition 21 (Multi Unit Demand Market Transformation). *Let k_j denote the number of units of item j . Define $c_j = \lfloor \frac{k_j}{k} \rfloor$ and divide the units of item j almost equally into c_j bins (i.e., each bin will contain either c_j or $c_j + 1$ units). Create a new item type for each bin (i.e., units from the same bin have the same type, but units from different bins are treated as different types of item).*

Theorem 20. *Let \mathbf{M} be the space of feasible mechanisms, in the original (multi unit demand) market, which allocate no more than $\frac{1}{k}$ of all units of each item to any single agent. Similarly, in the transformed market, let $\mathbf{M}^{(1)}$ be the space of feasible mechanisms which allocate no more than one unit of each item to any single agent. Any mechanism in \mathbf{M} can be interpreted as a mechanism in $\mathbf{M}^{(1)}$, and vice versa, with the same allocations/payments. Therefore, an optimal mechanism in the transformed market can be easily converted to an optimal mechanism in the original market, and vice versa.*

Proof. First, we show that any mechanism in $M \in \mathbf{M}^{(1)}$ can be interpreted as a mechanism in \mathbf{M} . That is trivially true because M allocates to each agent at most one unit from each bin, which is

at most c_j units of each item j of the original market, which is no more than $\frac{1}{k}$ of all units of item j .

Next, we show that any mechanism $M \in \mathbf{M}$ can be interpreted as a mechanism in $\mathbf{M}^{(1)}$. For every item j of the original market, we create a list L_j of all the bins of item j . L_j is initially sorted in decreasing order of the size of the bins. Let X_{ij} be the number of units of item j allocated to agent i by M . We determine the allocations in the transformed market as follows. For each agent i we repeat the following for X_{ij} times: Allocate one unit from the bin that is first in the list L_j and then move the bin to the end of the list. It is easy to see that no two units from the same bin are allocated to the same agent, so any feasible allocation in the original market corresponds to a feasible allocation in the transformed market. \square

Note that by Theorem 20, any mechanism in the original market is equivalent to a mechanism in the transformed market with the exact same allocations/payments from the perspective of agents. Therefore, without loss of generality, we only need to compute an optimal mechanism in the transformed market. However, to use our generic multi agent mechanisms in the transformed market, the underlying single agent mechanisms should be capable of handling correlated valuations, because units of the same item, even when labeled as different types, are perfect substitutes from the view point of an agent. Among the single agent mechanisms presented in this paper only the mechanism explained in §6.4 can handle correlated valuations.

9 Conclusion

In this paper, we presented an approximate reduction from multi agent problems to single agent problems in the context of Bayesian combinatorial auctions. From our results we draw the following high level conclusions.

- **Market size.** As the ratio of the maximum demand to supply (i.e., $\frac{1}{k}$) decreases, an approximately optimal mechanism requires less coordination on the decisions it makes for different agents; i.e., as $\frac{1}{k} \rightarrow 0$, the mechanism treats each agent almost independently of other agents. Observe that all of the approximation factors in this paper depend only on k (i.e., the single agent approximation factor is just multiplied by γ_k in the resulting multi agent mechanism) and not on n . It suggests that the right parameter to consider for characterizing asymptotic properties of such markets is perhaps the ratio of the maximum demand of a single buyer to the supply. In particular, notice that the number of agents is irrelevant in the approximation factor.
- **Computational hardness.** For mechanism design problems in a variety of settings, the difficulty of making coordinated optimal decisions for multiple agents can be avoided by losing a small constant factor in the objective (i.e., losing only a $\frac{1}{\sqrt{k+3}}$ fraction of the objective); therefore the main difficulty of constructing constant factor approximation mechanisms in multi dimensional settings stems from the difficulty of designing single agent mechanisms, which ultimately stems from the incentive compatibility constraints in the single agent problem.

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A Omitted Proofs

Proof of Theorem 5. We create n boxes to be presented to a magician with the probability of breaking a wand on box $i \in [n]$ being exactly $x_i = \frac{k}{n}$. We first show that the expected number of broken wands for any magician cannot be more than $(1 - \frac{k^k}{e^k k!})k$ asymptotically as $n \rightarrow \infty$. Let X_i be the indicator random variable which is 1 iff opening box i breaks a wand. X_1, \dots, X_n are independent random variables. We assume, without loss of generality, that every X_i is realized at the beginning (before the magician even sees the first box); however, X_i is revealed only if the magician chooses to open box i . The expected number of broken wands for any magician is at most $\mathbf{E}[\min(\sum_i X_i, k)]$. It is easy to show that $\mathbf{E}[\min(\sum_i X_i, k)] \approx (1 - \frac{k^k}{e^k k!})k$ asymptotically as $n \rightarrow \infty$. In particular, for any positive ϵ there is a large enough n such that $\mathbf{E}[\min(\sum_i X_i, k)] < (1 - \frac{k^k}{e^k k!} + \epsilon)k$.

Supposing there is a magician who can guarantee that each box is opened with probability at least $\gamma = 1 - \frac{k^k}{e^k k!} + \epsilon$, the expected number of broken wands for this magician will be at least $\sum_i \gamma \mathbf{E}[X_i] = k\gamma$, which contradicts our claim from the previous paragraph. Therefore it is not possible for any magician to make such a guarantee. \square

Proof of Theorem 7. First, we show that for each $i \in [n]$ and each $j \in [m]$, S_i includes item j with probability at least γ . Observe that for each item j , a sequence of n boxes are presented to the j^{th} magician with probabilities $\bar{x}_{1j}, \dots, \bar{x}_{nj}$ written on them. Since $\sum_i \bar{x}_{ij} \leq k_j$ and $\gamma \in [0, \gamma_k]$, we can argue that each box is opened with probability at least γ (see Theorem 4 and Definition 4). Therefore S_i includes each item j with probability at least γ .

Next, we show that the expected objective value of γ -pre-rounding is at least $\gamma\alpha$ fraction of the expected objective value of the optimal mechanism in \mathbf{M} . Recall that by Theorem 1 the expected objective value of the optimal mechanism in \mathbf{M} is upper bounded by the optimal value of (\overline{OPT}) , which is $\sum_i R_i(\bar{x}_i)$; therefore, it is enough to show that $\mathbf{E}_{S_i}[R_i(\bar{x}_i[S_i])] \geq \gamma\alpha R_i(\bar{x}_i)$; i.e., the expected objective value that $M_i\langle\bar{x}_i[S_i]\rangle$ obtains from agent i is at least $\gamma\alpha R_i(\bar{x}_i)$. Let ξ_i be a budget balanced cross monotonic cost share function for R_i . Then

$$\begin{aligned}
\mathbf{E}_{S_i}[R_i(\bar{x}_i[S_i])] &= \mathbf{E}_{S_i}\left[\sum_{j \in S_i} \xi_i(j, \bar{x}_i[S_i])\right] && \text{because } \xi_i \text{ is budget balanced} \\
&\geq \mathbf{E}_{S_i}\left[\sum_{j \in S_i} \xi_i(j, \bar{x}_i[\{1, \dots, m\}])\right] && \text{because } \xi_i \text{ is cross monotonic} \\
&= \sum_{j \in [m]} \Pr[j \in S_i] \xi_i(j, \bar{x}_i) \\
&\geq \sum_{j \in [m]} \gamma \xi_i(j, \bar{x}_i) \\
&= \gamma R_i(\bar{x}_i) && \text{because } \xi_i \text{ is budget balanced.}
\end{aligned}$$

Next, we show that the multi agent mechanism based on γ -pre-rounding is in \mathbf{M} and is dominant strategy incentive compatible (DSIC). The fact that this mechanism is in \mathbf{M} follows from Assumption A5 and the fact that for each item j the corresponding magician breaks no more than k_j wands, which means no more than k_j units of item j are allocated at any instance. To show that the mechanism is DSIC we argue that the only way the reports of other agents could affect the outcome of agent i is through S_i , yet $M_i\langle\bar{x}_i[S_i]\rangle$ is incentive compatible for all choices of S_i ; therefore the mechanism is incentive compatible even if each agent has full information of other

agents' types; thus the mechanism is DSIC. Observe that this mechanism also preserves all of the ex post properties of each single agent mechanism M_i (e.g., individual rationality, etc). \square

Proof of Theorem 8. First, we show that for each $i \in [n]$ and each $j \in [m]$, S_i includes item j with probability exactly γ . Observe that for each item j a sequence of n boxes are presented to the j^{th} magician with probabilities $\hat{x}_{1j}, \dots, \hat{x}_{nj}$ written on them. Since $\gamma \in [0, \gamma_k]$ and $\sum_i \hat{x}_{ij} \leq k_j$ and because each $M_i \langle \bar{x}_i \rangle$ allocates each item j with probability exactly \hat{x}_{ij} , we can argue that each box is opened with probability exactly γ (see Theorem 4 and Definition 4); therefore S_i includes item j with probability exactly γ .

Next, we show that γ -post-rounding obtains in expectation at least $\gamma\alpha$ fraction of the expected objective value of the optimal mechanism in \mathbf{M} . Recall that by Theorem 1 the expected objective value of the optimal mechanism in \mathbf{M} is upper bounded by the optimal value of convex program (\overline{OPT}) , which is $\sum_i R_i(\bar{x}_i)$; therefore, it is enough to show that $\mathbf{E}_{t_i, X_i, P_i}[\text{OBJ}_i(t_i, X_i, P_i)] \geq \gamma\alpha R_i(\bar{x}_i)$; i.e., the expected objective value that γ -post-rounding obtains from agent i is at least $\gamma\alpha R_i(\bar{x}_i)$. Let ξ_i be a budget balanced cross monotonic cost share function for OBJ_i as in Assumption A'2. Then

$$\begin{aligned}
\mathbf{E}_{t_i, X_i, P_i}[\text{OBJ}_i(t_i, X_i, P_i)] &= \mathbf{E}_{t_i, X_i, P_i}[\text{OBJ}_i(t_i, X_i, 0) + c_i P_i] && \text{by Assumption A'2} \\
&= \mathbf{E}_{t_i, X_i, P_i} \left[\sum_{j \in X_i} \xi_i(j, t_i, X_i) + c_i P_i \right] && \text{because } \xi_i \text{ is budget balanced} \\
&\geq \mathbf{E}_{t_i, X_i, P_i} \left[\sum_{j \in X_i} \xi_i(j, t_i, X'_i) + c_i P_i \right] && \text{because } \xi_i \text{ is cross monotonic} \\
&= \mathbf{E}_{t_i, X'_i, P'_i, S_i} \left[\sum_{j \in X'_i} \Pr[j \in S_i] \xi_i(j, t_i, X'_i) + c_i \gamma P'_i \right] \\
&= \mathbf{E}_{t_i, X'_i, P'_i} \left[\sum_{j \in X'_i} \gamma \xi_i(j, t_i, X'_i) + c_i \gamma P'_i \right] \\
&= \gamma \mathbf{E}_{t_i, X'_i, P'_i}[\text{OBJ}_i(t_i, X'_i, P'_i)] \\
&\geq \gamma\alpha R_i(\bar{x}_i).
\end{aligned}$$

Note that the last step follows because $\mathbf{E}_{t_i, X'_i, P'_i}[\text{OBJ}_i(t_i, X'_i, P'_i)]$ is exactly the expected objective value of $M_i \langle \bar{x}_i \rangle$, which is at least $\alpha R_i(\bar{x}_i)$.

Next, we show that γ -post-rounding is Bayesian incentive compatible (BIC) and does not over allocate any item. Consider any arbitrary agent i . Observe that each item $j \in X'_i$ is included in X_i with probability exactly γ ; furthermore, by Assumption A'4, agent i must have matroid valuations; without loss of generality, we assume that X'_i is always an independent set of this matroid³³; therefore, the valuation of the agent for the items in X'_i is additive; consequently, her expected valuation for X_i is exactly γ times her valuation for X'_i . Observe that $M_i \langle \bar{x}_i \rangle$ is incentive compatible and both the expected valuation and the expected payment of agent i are scaled by γ for any outcome of $M_i \langle \bar{x}_i \rangle$. Therefore the final mechanism is also incentive compatible, however it is only Bayesian incentive compatible because S_i depends on the private types/reports of agents

³³Otherwise, we could replace X'_i by a maximum weight independent subset of X'_i .

other than i ³⁴. Furthermore, the mechanism never overallocates, because allocating a unit of item j corresponds to breaking one of the k_j wands of the j^{th} magician. \square

Proof of Lemma 1. The proof is very similar to the proof of Theorem 2. To show that $R(\bar{x})$ is concave it is enough to show that $R(\beta\bar{x} + (1 - \beta)\bar{x}') \geq \beta R(\bar{x}) + (1 - \beta)R(\bar{x}')$ for any \bar{x} and \bar{x}' and any $\beta \in [0, 1]$. Let y and y' be optimal assignments for the convex program subject to \bar{x} and \bar{x}' respectively. Then $y'' = \beta y + (1 - \beta)y'$ is also a feasible assignment for the convex program subject to $\beta\bar{x} + (1 - \beta)\bar{x}'$; therefore, $R(\beta\bar{x} + (1 - \beta)\bar{x}')$ must be at least $u(\beta y + (1 - \beta)y')$; on the other hand $u(\cdot)$ is concave, so $u(\beta y + (1 - \beta)y') \geq \beta u(y) + (1 - \beta)u(y') = \beta R(\bar{x}) + (1 - \beta)R(\bar{x}')$. That proves the claim. \square

Proof of Lemma 5. Let $\mu = \sum_j \mathbf{E}[Z_j]$. Define the random variables $Y_j = \max(Y_{j-1} - Z_j, 0)$ and $Y_0 = B$. Observe that $Y_j = \max(B - \sum_{r=1}^j Z_r, 0)$, and therefore $\min(\sum_{r=1}^j Z_r, B) + Y_j = B$. Thus $\mathbf{E}[\min(\sum_{r=1}^j Z_r, B)] + \mathbf{E}[Y_j] = B$ and to prove the lemma it is enough to show that $\mathbf{E}[Y_m] \leq \frac{1}{e^{\mu/B}} B$. We claim that

$$\mathbf{E}[Y_j] \leq \left(1 - \frac{\mathbf{E}[Z_j]}{B}\right) \mathbf{E}[Y_{j-1}] \quad \forall j \in [m], \quad (Y_j)$$

which immediately implies

$$\mathbf{E}[Y_m] \leq B \prod_{j=1}^m \left(1 - \frac{\mathbf{E}[Z_j]}{B}\right) \leq B \frac{1}{e^{\mu/B}}.$$

The last inequality follows because $\prod_{j=1}^m (1 - \frac{\mathbf{E}[Z_j]}{B})$ takes its maximum when $m \rightarrow \infty$ and $\mathbf{E}[Z_j] = \frac{\mu}{m}$ (for all j).

To prove the second inequality in the statement of the lemma, we use the fact that $(1 - x^a) \geq (1 - x)a$ (for all $a \leq 1$ and $x \geq 0$) to conclude

$$\left(1 - \frac{1}{e^{\mu/B}}\right) B \geq \left(1 - \frac{1}{e^{\min(\mu, B)/B}}\right) B \geq \left(1 - \frac{1}{e}\right) \frac{\min(\mu, B)}{B} B = \left(1 - \frac{1}{e}\right) \min(\mu, B).$$

To complete the proof, we prove inequality (Y_j) as follows:

$$\begin{aligned} \mathbf{E}[Y_j] &= \mathbf{E}[\max(Y_{j-1} - Z_j, 0)] \\ &\leq \mathbf{E}\left[\max\left(Y_{j-1} - Z_j \frac{Y_{j-1}}{B}, 0\right)\right] && \text{because } \frac{Y_{j-1}}{B} \leq 1 \\ &= \mathbf{E}\left[Y_{j-1} - Z_j \frac{Y_{j-1}}{B}\right] && \text{because } \frac{Z_j}{B} \leq 1 \\ &= \mathbf{E}[Y_{j-1}] - \frac{1}{B} \mathbf{E}[Z_j Y_{j-1}] \\ &\leq \mathbf{E}[Y_{j-1}] - \frac{1}{B} \mathbf{E}[Z_j] \mathbf{E}[Y_{j-1}] && \text{because } Z_j \text{ and } Y_{j-1} \text{ are independent} \\ &= \left(1 - \frac{\mathbf{E}[Z_j]}{B}\right) \mathbf{E}[Y_{j-1}]. \end{aligned}$$

\square

³⁴I.e., $\mathbf{Pr}[j \in S_i]$ is equal to γ only when the probability is taken over random types of other agents.

Proof of Lemma 4. To prove the claim, it is enough to show that $\frac{r_1}{\sum_j x_j p_j} \geq \frac{1}{2}$. Without loss of generality, we assume that $\sum_j p_j x_j = 1$ since we can scale p_1, \dots, p_m by a constant $c = \frac{1}{\sum_j x_j p_j}$ and that also scales r_1, \dots, r_m by the same constant c , so their ratio is not affected.

Observe that x_j, p_j , and r_j , as defined in the statement of the lemma, form a feasible assignment for the following LP (p_j and r_j are variables and everything else is constant):

$$\begin{array}{llll}
\text{minimize} & r_1 & & \\
\text{subject to} & r_j \geq x_j p_j + (1 - x_j) r_{j+1} & \forall j \in [m], & (\alpha_j) \\
& r_j \geq r_{j+1} & \forall j \in [m], & (\beta_j) \\
& \sum_{j=1}^m x_j p_j \geq 1, & & (\gamma) \\
& p_j \geq 0 & \forall j \in [m], & \\
& r_j \geq 0 & \forall j \in [m+1]. &
\end{array}$$

We construct a feasible assignment for the dual of the above LP, obtaining a value of $\frac{1}{2}$, which implies that the optimal objective value of the above LP is bounded below by $\frac{1}{2}$; therefore, any feasible assignment yields an objective value of at least $\frac{1}{2}$ which proves the lemma. The dual LP is as follows:

$$\begin{array}{llll}
\text{maximize} & \gamma & & \\
\text{subject to} & \gamma \leq \alpha_j & \forall j \in [m], & (p_j) \\
& \alpha_1 + \beta_1 \leq 1, & & (r_1) \\
& \alpha_j + \beta_j \leq (1 - x_{j-1}) \alpha_{j-1} + \beta_{j-1} & \forall j \in \{2, \dots, m\}, & (r_j) \\
& 0 \leq (1 - x_m) \alpha_m + \beta_m, & & (r_{m+1}) \\
& \alpha_j \geq 0, \beta_j \geq 0, \gamma \geq 0 & \forall j \in [m]. &
\end{array}$$

We construct an assignment for the dual LP as follows. We set $\alpha_j = \gamma$ and $\beta_j = \beta_{j-1} - x_{j-1} \gamma$ for all j , except that for $j = 1$ we set $\beta_1 = 1 - \gamma$. From this assignment we get $\beta_j = 1 - \gamma - \gamma \sum_{\ell=1}^{j-1} x_\ell$. Observe that we get a feasible assignment as long as every β_j resulting from this assignment is non-negative. Furthermore, it is easy to see that $\beta_j \geq 1 - \gamma - \gamma \sum_{\ell=1}^m x_\ell \geq 1 - 2\gamma$ because $\sum_j x_j \leq 1$. Therefore, by setting $\gamma = \frac{1}{2}$, all β_j are non-negative and we always get a feasible assignment for the dual LP with an objective value of $\frac{1}{2}$, which completes the proof. \square