Chapter 3

THE RESOLUTION COMPLEXITY OF GRAPH PROBLEMS

We are now ready to describe the technical contributions of this thesis in detail. We begin in this chapter with our main proof complexity results. These are for the resolution proof system and apply to the CNF formulations of three graph problems, namely, (the existence of) independent sets, vertex covers, and cliques.

An independent set in an undirected graph is a set of vertices no two of which share an edge. The problem of determining whether or not a given graph contains an independent set of a certain size is NP-complete as shown by Karp [69][1]. Consequently, the complementary problem of determining non-existence of independent sets of that size in the graph is co-NP-complete. This chapter studies the problem of providing a resolution proof of the non-existence of independent sets.

Any result that holds for nearly all graphs can be alternatively formalized as a result that holds with very high probability when a graph is chosen at random from a "fair" distribution. We use this approach and study the resolution complexity of the independent set problem in random graphs chosen from a standard distribution. Independent sets and many other combinatorial structures in random graphs have very interesting mathematical properties as discussed at length in the texts by Bollobás [26] and Janson, Łuczak, and Ruciński [66]. In particular, the size of the largest independent set can be described with high certainty and accuracy in terms of simple graph parameters.

This work proves that given almost any graph G and a number k, exponential-size resolution proofs are required to show that G does not contain an independent set of size k. In fact, when G has no independent set of size k, exponential-size resolution proofs are required to show that independent sets of even a much larger size $k' \gg k$ do not exist in G. This yields running time lower bounds for certain classes of algorithms for approximating the size of the largest independent sets in random graphs.

Closely related to the independent set problem are the problems of proving the non-existence of cliques or vertex covers of a given size. Our results for the independent set problem also lead to bounds for these problems. As the approximations for the vertex cover problem act differently from those for independent sets, we state the results in terms of vertex covers as well as independent sets. (Clique approximations are essentially identical to independent set approximations.)

[1]Karp actually proved the related problem of clique to be NP-complete.

Many algorithms for finding a maximum-size independent set have been proposed. Influenced by algorithms of Tarjan [105] and Tarjan and Trojanowski [106], Chvátal [34] devised a specialized proof system for the independent set problem. In this system he showed that with probability approaching 1, proofs of non-existence of large independent sets in random graphs with a linear number of edges must be exponential in size. Chvátal's system captures many backtracking algorithms for finding a maximum independent set, including those of Tarjan [105], Tarjan and Trojanowski [106], Jian [67], and Shindo and Tomita [100]. In general, the transcript of any *f-driven algorithm* [34] for independent sets running on a given graph can be translated into a proof in Chvátal's system.

Our results use the well-known resolution proof system for propositional logic rather than Chvátal's specialized proof system. Given a graph G and an integer k, we consider encoding the existence of an independent set of size k in G as a CNF formula and examine the proof complexity of such formulas in resolution. Resolution on one of the encodings we present captures the behavior of Chvátal's proofs on the corresponding graphs. For all our encodings, we show that given a randomly chosen graph G of moderate edge density, almost surely, the size of any resolution proof of the statement that G does not have an independent set of a certain size must be exponential in the number of vertices in G. This implies an exponential lower bound on the running time of many algorithms for searching for, or even approximating, the size of a maximum independent set or minimum vertex cover in G.

Although resolution is a relatively simple and well-studied proof system, one may find the concept of resolution proofs of graph theoretic problems somewhat unnatural. The tediousness of propositional encodings and arguments related to them contributes even more to this. Chvátal's proof system, on the other hand, is completely graph theoretic in nature and relates well to many known algorithms for the independent set problem. By proving that resolution can efficiently simulate Chvátal's proof system, we provide another justification for studying the complexity of resolution proofs of graph problems.

In the proof complexity realm, exponential bounds for specialized structured formulas and for unstructured random k-CNF formulas have previously been shown by several researchers including Haken [60], Urquhart [110], Razborov [95], Chvátal and Szemerédi [35], Beame et al. [17], and Ben-Sasson and Wigderson [23]. However, much less is known for large classes of structured formulas. Our results significantly extend the families of structured random formulas for which exponential resolution lower bounds are known beyond the graph coloring example recently shown by Beame et al. [14]. (Note that our results neither imply nor follow from those in [14]. Although the non-existence of an independent set of size n/K implies in a graph of n vertices implies that the graph is not K-colorable, the argument requires an application of the pigeonhole principle which is not efficiently provable in resolution [60].)

For obtaining our lower bounds, instead of looking at the general problem of disproving the existence of *any* large independent set in a graph, we focus on a restricted class of independent sets that we call *block-respecting independent sets*. We show that even ruling out this smaller class of independent sets requires exponential-size resolution proofs. These restricted independent sets are simply the ones obtained by dividing the n vertices of the given graph into k blocks of equal size (assuming k divides n) and choosing one vertex from each block. Since it is easier to rule out a smaller class of independent sets, the lower bounds we obtain for the restricted version are stronger in the sense that they imply lower bounds for the general problem. While block-respecting independent sets are a helpful tool in analyzing general resolution proofs, we are able to give better lower bounds for DPLL proofs by applying a counting argument directly to the general problem.

We show that our results extend the known lower bounds for Chvátal's system [34] to resolution and also extend them to graphs with many more than a linear number of edges, yielding bounds for approximation algorithms as well as for exact computation. More precisely, we show that no resolution-based technique can achieve polynomial-time approximations of independent set size within a factor of $\Delta/(6 \log \Delta)$. For the vertex cover problem, we show an analogous result for approximation factors better than $3/2$.

Recently, by computing a property related to the Lovász number of a random graph, more precisely its vector chromatic number, Coja-Oghlan [37] gave an expected polynomial time $O(\sqrt{\Delta}/\log \Delta)$-approximation algorithm for the size of the maximum independent set in random graphs of density Δ. Thus our results show that this new approach is provably stronger than that obtainable using resolution-based algorithms.

The proof our of main lower bound is based on the size-width relationship of resolution proofs discussed in Section 2.2.3. It uses the property that any proof of non-existence of an independent set of a certain size in a random graph is very likely to refer to a relatively large fraction of the vertices of the input graph, and that any clause capturing the properties of this large fraction of vertices must have large width.

More precisely, the proof can be broadly divided into two parts, both of which use the fact that random graphs are almost surely locally sparse. We first show that the minimum number s of input clauses that are needed for any refutation of the problem is large for most graphs. We then use combinatorial properties of independent sets in random graphs to say that any clause minimally implied by a relatively large subset of these s clauses has to be large. Here minimally implied means that implied by the size-s set of clauses under consideration but not by any proper subset of it. These two arguments together allow us to deduce that the width of any such refutation has to be large. The size-width relationship translates this into a lower bound on the refutation size.

We begin with basic properties of independent sets in Section 3.1. In Section 3.2 we describe three natural encodings of the independent set problem as CNF formulas and compare the proof sizes of the different encodings. In Sections 3.3 and 3.4 we compare these to proofs in Chvátal's proof system for independent sets and to the proof complexity of related graph theory problems, namely, vertex cover and clique. After giving some simple proof complexity upper bounds based on exhaustive backtracking algorithms in Section 3.5, we prove the main resolution lower bounds in Sections 3.6 to 3.8. Note that Sections 3.2 to 3.5 contain somewhat tedious details that the reader may want to skip during the first read. Finally, in Section 3.10 we prove a somewhat stronger lower bound that applies to exhaustive backtracking algorithms (as well as the DPLL procedure) and qualitatively matches our upper bounds for the same.

Remark 3.1. Although we described DPLL algorithms in Section 2.3 as working on propositional CNF formulas, they capture a much more general class of algorithms that are based on branching and backtracking. For instance, basic algorithms for finding a maximum independent set, such as that of Tarjan [105], branch on each vertex v by either including v in the current independent set and deleting it and all its neighbors from further consideration, or excluding v from the current independent set and recursively finding a maximum independent set in the remaining graph. This can be formulated as branching and backtracking on appropriate variables of a CNF formulation of the problem. In fact, more complicated algorithms, such as that of Tarjan and Trojanowski [106], branch in a similar manner not only on single vertices but on small subsets of vertices, reusing subproblems already solved. Such algorithms also fall under the category of resolution-based (not necessarily tree-like) algorithms and our lower bounds apply to them as well because of the following reasoning. The computation history of these algorithms can be translated into a proof in Chvátal's system by replacing each original branch in the computation with a small tree of single-vertex branches. We then resort to our result that resolution can efficiently simulate Chvátal's proof system.

3.1 Independent Sets in Random Graphs

For any undirected graph $G = (V, E)$, let $n = |V|$ and $m = |E|$. A *k-independent set* in G is a set of k vertices no two of which share an edge. We will describe several natural ways of encoding in clausal form the statement that G has a k-independent set. Their refutations will be proofs that G does *not* contain any k-independent set. We will be interested in size bounds for such proofs.

Combinatorial properties of random graphs have been studied extensively (see, for instance, [26, 66]). We use the standard model $\mathbb{G}(n, p)$ for graphs with n vertices where each of the $\binom{n}{2}$ edges is chosen independently at random with probability $p \in [0, 1]$. $G \sim \mathbb{G}(n, p)$ denotes a graph G chosen at random from this distribution. We will state most of our results in terms of parameters n and Δ, where $\Delta \overset{\text{def}}{=} np$ is (roughly) the average degree of G.

We will need both worst case and almost certain bounds on the size of the largest independent set in graphs of density Δ.

Proposition 3.1 (Turan's Theorem). *Every graph G with n vertices and average degree Δ has an independent set of size $\lfloor \frac{n}{\Delta+1} \rfloor$. In general, for any integer k satisfying $\Delta < \frac{n}{k-1} - 1$, G has an independent set of size k.*

For $\epsilon > 0$, let $k_{\pm\epsilon}$ be defined as follows[2]:

$$k_{\pm\epsilon} = \lfloor \frac{2n}{\Delta}(\log \Delta - \log\log \Delta + 1 - \log 2 \pm \epsilon) \rfloor$$

Proposition 3.2 ([66], Theorem 7.4). *For every $\epsilon > 0$ there is a constant C_ϵ such that the following holds. Let $\Delta = np$, $C_\epsilon \le \Delta \le n/\log^2 n$, and $G \sim \mathbb{G}(n,p)$. With probability $1 - o(1)$ in n, the largest independent set in G is of size between $k_{-\epsilon}$ and $k_{+\epsilon}$.*

This shows that while random graphs are very likely to have an independent set of size $k_{-\epsilon}$, they are very unlikely to have one of size $k_{+\epsilon}+1$. The *number* of independent sets of a certain size also shows a similar threshold behavior. While there are almost surely no independent sets of size $(2n/\Delta)\log\Delta$, the following lemma, which follows by a straightforward extension of the analysis in [66, Lemma 7.3], shows that there are exponentially many of size $(n/\Delta)\log\Delta$. We use this bound later to put a limit on the best one can do with exhaustive backtracking algorithms that systematically consider all potential independent sets of a certain size.

Lemma 3.1. *There is a constant $C > 0$ such that the following holds. Let $\Delta = np$, $\Delta \le n/\log^2 n$, and $G \sim \mathbb{G}(n,p)$. With probability $1 - o(1)$ in n, G contains at least $2^{C(n/\Delta)\log^2\Delta}$ independent sets of size $\lfloor (n/\Delta)\log\Delta \rfloor$.*

Proof. Let X_k be a random variable whose value is the number of independent sets of size k in $G = (V,E)$. The expected value of X_k is given by:

$$\begin{aligned}
\mathbb{E}[X_k] &= \sum_{S\subseteq V, |S|=k} \Pr[S \text{ is an independent set in } G] \\
&= \binom{n}{k}(1-p)^{\binom{k}{2}} \\
&\ge \left(\frac{n}{k}\right)^k e^{-cpk^2} \qquad \text{for } c > 1/2,\ p = o(1) \text{ in } n, \text{ and large enough } n \\
&= \left(\frac{n}{k}\, e^{-c\Delta k/n}\right)^k
\end{aligned}$$

[2]Throughout this thesis, logarithms denoted by log will have the natural base e and those denoted by $\log_2$ will have base 2.

Let $c = 0.55$ and $C = 0.05/\log 2$ so that $\Delta^{1-c}/\log\Delta \geq 2^{C\log\Delta}$. Setting $k = \lfloor (n/\Delta)\log\Delta \rfloor$ and observing that $\left((n/k)e^{-c\Delta k/n}\right)^k$ decreases with k,

$$\begin{aligned}\mathbb{E}\left[X_{\lfloor (n/\Delta)\log\Delta\rfloor}\right] &\geq \left(\frac{\Delta}{\log\Delta}e^{-c\log\Delta}\right)^{(n/\Delta)\log\Delta} \\ &\geq 2^{C(n/\Delta)\log^2\Delta}.\end{aligned}$$

We now use the standard second moment method to prove that X_k for $k = \lfloor (n/\Delta)\log\Delta \rfloor$ asymptotically almost surely lies very close to its expected value. We begin by computing the expected value of X_k^2 and deduce from it that the variance of X_k is small.

$$\begin{aligned}\mathbb{E}\left[X_k^2\right] &= \sum_{S\subseteq V,\ |S|=k} \Pr\left[S \text{ is independent}\right] \sum_{i=0}^{k} \sum_{T\subseteq V,\ |T|=k,\ |S\cap T|=i} \Pr\left[T \text{ is independent}\right] \\ &= \binom{n}{k}(1-p)^{\binom{k}{2}} \sum_{i=0}^{k} \binom{k}{i}\binom{n-k}{k-i}(1-p)^{\binom{k}{2}-\binom{i}{2}}\end{aligned}$$

$$\begin{aligned}\text{Therefore}\quad \frac{var\left[X_k\right]}{\mathbb{E}\left(\left[X_k\right]\right)^2} &= \frac{\mathbb{E}\left[X_k^2\right]}{\left(\mathbb{E}\left[X_k\right]\right)^2} - 1 \\ &= \frac{\binom{n}{k}(1-p)^{\binom{k}{2}} \sum_{i=0}^{k} \binom{k}{i}\binom{n-k}{k-i}(1-p)^{\binom{k}{2}-\binom{i}{2}}}{\left[\binom{n}{k}(1-p)^{\binom{k}{2}}\right]^2} - 1\end{aligned}$$

This is the same expression as equation (7.8) of [66, page 181]. Following the calculation of Lemma 7.3 of [66], we obtain that $var[X_k]/(\mathbb{E}[X_k^2])^2 \to 0$ for $k = \lfloor (n/\Delta)\log\Delta \rfloor$ as $n \to \infty$ when $\Delta \geq \sqrt{n}\log^2 n$. When $\Delta \leq \sqrt{n}\log^2 n$, an argument along the lines of Theorem 7.4 of [66] provides the same result. Applying the second moment method, this leads to the desired bound. □

3.2 Encoding Independent Sets as Formulas

In order to use a propositional proof system to prove that a graph does not have an independent set of a particular size, we first need to formulate the problem as a propositional formula. This is complicated by the difficulty of counting set sizes using CNF formulas.

One natural way to encode the independent set problem is to have indicator variables that say which vertices are in the independent set and auxiliary variables that count the number of vertices in the independent set. This encoding is discussed in Section 3.2.1. The clauses in this encoding, although capturing the simple concept of counting, are somewhat involved. Moreover, the existence of two different types of variables makes this encoding difficult to reason about directly.

A second encoding, derived from the counting-based encoding, is described in Section 3.2.2. It is based on a mapping from the vertices of the graph to k additional nodes as an alternative to straightforward counting, and uses variables of only one type. This is essentially the same encoding as the one used by Bonet, Pitassi, and Raz [29] for the clique problem, except that in our case we need to add an extra set of clauses, called ordering clauses, to make the lower bounds non-trivial. (Otherwise, lower bounds trivially follow from known lower bounds for the pigeonhole principle [60] which have nothing to do with the independent set problem; in [29] this problem did not arise because the proof system considered was cutting planes where, as shown by Cook et al. [40], the pigeonhole principle has short proofs.)

Section 3.2.3 finally describes a much simpler encoding which is the one we analyze directly for our lower bounds. This encoding considers only a restricted class of independent sets that we call *block-respecting independent sets*, for which the problem of counting the set size is trivial. Hence, the encoding uses only one type of variable that indicates whether or not a given vertex is in the independent set. Refutation of this third encoding rules out the existence of the smaller class of block-respecting independent sets only. Intuitively, this should be easier to do than ruling out all possible independent sets. In fact, we show that the resolution and DPLL refutations of this encoding are bounded above in size by those of the mapping encoding and are at worst a small amount larger than those of the counting encoding. As a result, we can translate our lower bounds for this third encoding to each of the other encodings. Further, we give upper bounds for the two general encodings which also apply to the simpler block-respecting independent set encoding.

For the rest of this chapter, identify the vertex set of the input graph with $\{1, 2, \dots, n\}$. Each encoding will be defined over variables from one or more of the following three categories:

- $x_v, 1 \le v \le n$, which is TRUE iff vertex v is chosen by the truth assignment to be in the independent set,
- $y_{v,i}, 0 \le i \le v \le n$, $0 \le i \le k$, which is TRUE iff precisely i of the first v vertices are chosen in the independent set, and
- $z_{v,i}, 1 \le v \le n, 1 \le i \le k$, which is TRUE iff vertex v is chosen as the i^{th} node of the independent set.

A desirable property of all independent set encodings is their *monotonicity*, i.e., for $k' > k$, proving the non-existence of an independent set of size k' in that encoding must not be any harder than doing so for size k, up to a polynomial factor. This property indeed holds for each of the three encodings we consider below.

3.2.1 Encoding Based on Counting

The *counting encoding*, $\alpha_{count}(G,k)$, of the independent set problem is defined over variables x_v and $y_{v,i}$. As mentioned previously, this encoding is somewhat tedious in nature. It has the following three kinds of clauses:

(a) Edge Clauses: For each edge (u,v), $\alpha_{count}(G,k)$ has one clause saying that at most one of u and v is selected; $\forall (u,v) \in E, u < v : (\neg x_u \vee \neg x_v) \in \alpha_{count}(G,k)$

(b) Size-k Clause: There is a clause saying that the independent set chosen is of size k; $y_{n,k} \in \alpha_{count}(G,k)$

(c) Counting Clauses: There are clauses saying that variables $y_{v,i}$ correctly count the number of vertices chosen. For simplicity, we first write this condition not as a set of clauses but as more general propositional formulas. For the base case, $\alpha_{count}(G,k)$ contains $y_{0,0}$ and the clausal form of $(y_{v,0} \leftrightarrow (y_{v-1,0} \wedge \neg x_v))$ for $v \in \{1,\ldots n\}$. Further, $\forall i,v, 1 \leq i \leq v \leq n$, $1 \leq i \leq k$, $\alpha_{count}(G,k)$ contains the clausal form of $(y_{v,i} \leftrightarrow ((y_{v-1,i} \wedge \neg x_v) \vee (y_{v-1,i-1} \wedge x_v)))$, unless $i = v$, in which case $\alpha_{count}(G,k)$ contains the clausal form of the simplified formula $(y_{v,v} \leftrightarrow (y_{v-1,v-1} \wedge x_v))$.

Translated into clauses, these conditions take the following form. Formulas defining $y_{v,0}$ for $v \geq 1$ translate into $\{(\neg y_{v,0} \vee y_{v-1,0}), (\neg y_{v,0} \vee \neg x_v), (y_{v,0} \vee \neg y_{v-1,0} \vee x_v)\}$. Further, formulas defining $y_{v,i}$ for $v > i \geq 1$ translate into $\{(y_{v,i} \vee \neg y_{v-1,i} \vee x_v), (y_{v,i} \vee \neg y_{v-1,i-1} \vee \neg x_v), (\neg y_{v,i} \vee y_{v-1,i} \vee y_{v-1,i-1}), (\neg y_{v,i} \vee y_{v-1,i} \vee x_v), (\neg y_{v,i} \vee y_{v-1,i-1} \vee \neg x_v)\}$, whereas in the case $i = v$ they translate into $\{(\neg y_{v,v} \vee y_{v-1,v-1}), (\neg y_{v,v} \vee x_v), (\neg x_v \vee \neg y_{v-1,v-1} \vee y_{v,v})\}$.

Lemma 3.2. *For any graph G over n vertices and $k' > k$,*

$$\begin{aligned} \mathsf{RES}(\alpha_{count}(G,k')) &< n\ \mathsf{RES}(\alpha_{count}(G,k)) + 2n^2 \quad \textit{and} \\ \mathsf{DPLL}(\alpha_{count}(G,k')) &< n\ \mathsf{DPLL}(\alpha_{count}(G,k)) + 2n^2. \end{aligned}$$

Proof. If G contains an independent set of size k, then there are no resolution refutations of $\alpha_{count}(G,k)$. By our convention, $Res(\alpha_{count}(G,k)) = DPLL(\alpha_{count}(G,k)) = \infty$, and the result holds. Otherwise consider a refutation π of $\alpha_{count}(G,k)$. Using π, we construct a refutation π' of $\alpha_{count}(G,k')$ such that $size(\pi') \leq (n-k+1)\ size(\pi) + 2(k'-k)(n-k)$, which is less than $n\ size(\pi) + 2n^2$. Further, if π is a tree-like refutation, then so is π'.

$\alpha_{count}(G,k')$ contains all clauses of $\alpha_{count}(G,k)$ except the size-k clause, $y_{n,k}$. Therefore, starting with $\alpha_{count}(G,k')$ as initial clauses and using π modified not to use the clause $y_{n,k}$, we derive a subclause of $\neg y_{n,k}$. This clause, however, cannot be a strict subclause of $\neg y_{n,k}$ because $\alpha_{count}(G,k) \setminus \{y_{n,k}\}$ is satisfiable. Hence, we must obtain $\neg y_{n,k}$. Call this derivation D_n. By construction, $size(D_n) \leq size(\pi)$. Making a copy of D_n, we restrict it by setting $x_n \leftarrow \text{FALSE}, y_{n,k} \leftarrow y_{n-1.k}$ to obtain a derivation D_{n-1} of $\neg y_{n-1,k}$. Continuing this process, construct derivations D_p of $\neg y_{p,k}$ for $p \in \{n-1, n-2, \ldots, k\}$ by further setting $x_{p+1} \leftarrow \text{FALSE}, y_{p+1,k} \leftarrow y_{p,k}$. Again, by construction, $size(D_p) \leq size(\pi)$. Combining derivations $D_n, D_{n-1}, \ldots, D_k$ into π' gives a derivation of size at most $(n-k+1)size(\pi)$ of clauses $\neg y_{p,k}, k \leq p \leq n$, which is tree-like if π is.

Continuing to construct π', resolve the above derived clause $\neg y_{k,k}$ with the counting clause $(\neg y_{k+1,k+1} \vee y_{k,k})$ of $\alpha_{count}(G, k')$ to obtain $\neg y_{k+1,k+1}$. Now for v going from $k+2$ to n, resolve the already derived clauses $\neg y_{v-1,k+1}$ and $\neg y_{v-1,k}$ with the counting clause $(\neg y_{v,k+1} \vee y_{v-1,k+1} \vee y_{v-1,k})$ of $\alpha_{count}(G, k')$ to obtain $\neg y_{v,k+1}$. This gives a tree-like derivation of size less than $2(n-k)$ of clauses $\neg y_{p,k+1}, k+1 \leq p \leq n$, starting from clauses $\neg y_{q,k}, k \leq q \leq n$. Repeating this process $(k'-k)$ times gives a tree-like derivation of size less than $2(k'-k)(n-k)$ of clauses $\neg y_{p,k'}, k' \leq p \leq n$, starting from clauses $\neg y_{q,k}, k \leq q \leq n$, derived previously. In particular, $\neg y_{n,k'}$ is now a derived clause. Resolving it with the size-k' clause $y_{n,k'}$ of $\alpha_{count}(G, k')$ completes refutation π'. $\square$

3.2.2 Encoding Based on Mapping

This encoding, denoted $\alpha_{map}(G, k)$, uses a mapping from n vertices of G to k nodes of the independent set as an indirect way of counting the number of vertices chosen by a truth assignment to be in the independent set. It can be viewed as a set of constraints restricting the mapping (see Figure 3.1). The idea is to map the nodes of the independent set to the sequence $(1, 2, \ldots, k)$ in the increasing order of their index as vertices in the graph. This encoding is defined over variables $z_{v,i}$ and has the following five kinds of clauses:

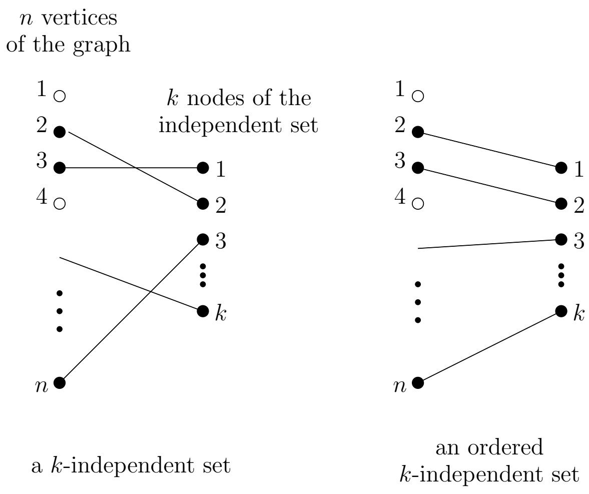
(a) Edge Clauses: For each edge (u, v), there are clauses saying that at most one of u and v is chosen in the independent set; $\forall (u, v) \in E, i, j, 1 \leq i < j \leq k$: $(\neg z_{u,i} \vee \neg z_{v,j}) \in \alpha_{map}(G, k)$

(b) Surjective Clauses: For each node i, there is a clause saying that some vertex is chosen as the i^{th} node of the independent set; $\forall i, 1 \leq i \leq k: \ (z_{1,i} \vee z_{2,i} \vee \ldots \vee z_{n,i}) \in \alpha_{map}(G, k)$

(c) Function Clauses: For each vertex v, there are clauses saying that v is not mapped to two nodes, i.e. it is not counted twice in the independent set; $\forall v, i, j, 1 \leq v \leq n, 1 \leq i < j \leq k: \ (\neg z_{v,i} \vee \neg z_{v,j}) \in \alpha_{map}(G, k)$

(d) 1-1 Clauses: For each node i, there are clauses saying no two vertices map to the i^{th} node of the independent set; $\forall i, u, v, 1 \leq i \leq k, 1 \leq u < v \leq n$: $(\neg z_{u,i} \vee \neg z_{v,i}) \in \alpha_{map}(G, k)$

(e) Ordering Clauses: For every pair of consecutive nodes, there are clauses saying that vertices are not mapped to these in the reverse order. This, by transitivity, implies that there is a unique mapping to k nodes once we have chosen k vertices to be in the independent set. $\forall u, v, i, 1 \leq u < v \leq n, 1 \leq i < k$: $(\neg z_{u,i+1} \vee \neg z_{v,i}) \in \alpha_{map}(G, k)$.



n vertices of the graph
k nodes of the independent set
1
2
3
4
n
1
2
3
k
a k-independent set
an ordered k-independent set

Figure 3.1: Viewing independent sets as a mapping from n vertices to k nodes

Lemma 3.3. *For any graph G and $k' \geq k$,*

$$\begin{aligned} \mathsf{RES}(\alpha_{map}(G, k')) &\leq \mathsf{RES}(\alpha_{map}(G, k)) \quad \textit{and} \\ \mathsf{DPLL}(\alpha_{map}(G, k')) &\leq \mathsf{DPLL}(\alpha_{map}(G, k)). \end{aligned}$$

Proof. If G contains an independent set of size k, then there are no resolution refutations of $\alpha_{map}(G, k)$. By our convention, $Res(\alpha_{map}(G, k)) = DPLL(\alpha_{map}(G, k)) = \infty$, and the result holds. Otherwise consider a refutation π of $\alpha_{map}(G, k)$. Observe that all clauses of $\alpha_{map}(G, k)$ are also clauses of $\alpha_{map}(G, k')$. Hence π is also a refutation of $\alpha_{map}(G, k')$, proving the desired bounds. $\square$

3.2.3 Encoding Using Block-respecting Independent Sets

Fix $b = n/k$ for the rest of the chapter and assume for simplicity that k divides n (denoted $k \mid n$). Arbitrarily partition the vertices of G into k subsets, called *blocks*, of size b each. A *block-respecting independent set* of size k in G under this partitioning is an independent set in G with precisely one vertex in each of the k blocks. Clearly, if a graph does not contain *any* k-independent set, then it certainly does not contain any block-respecting independent set of size k either. Note that the restriction $k \mid n$ is only to make the presentation simple. We can extend our arguments to all $k < n$ by letting each block have either b or $b+1$ vertices for $b = \lfloor n/k \rfloor$. The calculations are nearly identical to what we present here.

We now define a CNF formula $\alpha_{block}(G,k)$ over variables x_v that says that G contains a block-respecting independent set of size k. Assume without loss of generality that the first b vertices of G form the first block, the second b vertices form the second block, and so on. Henceforth, in all references to G, we will implicitly assume this fixed order of vertices and partition into k blocks. Since this order and partition are chosen arbitrarily, the bounds we derive hold for any partitioning of G into blocks.

The encoding $\alpha_{block}(G,k)$ contains the following three kinds of clauses:

(a) Edge Clauses: For each edge (u,v), there is one clause saying that not both u and v are selected; $\forall (u,v) \in E, u < v : \ (\neg x_u \vee \neg x_v) \in \alpha_{block}(G,k)$

(b) Block Clauses: For each block, there is one clause saying that at least one of the vertices in it is selected; $\forall\ i, 0 \le i < k :\ (x_{bi+1} \vee x_{bi+2} \vee \ldots \vee x_{bi+b}) \in \alpha_{block}(G,k)$

(c) 1-1 Clauses: For each block, there are clauses saying that at most one of the vertices in it is selected; $\forall\ i,p,q, 0 \le i < k, 1 \le p < q \le b :\ (\neg x_{bi+p} \vee \neg x_{bi+q}) \in \alpha_{block}(G,k)$

$\alpha_{block}(G,k)$ is satisfiable iff G has a block-respecting independent set of size k under the fixed order and partition of vertices implicitly assumed. Note that there is no exact analog of Lemmas 3.2 and 3.3 for the block encoding. In fact, if one fixes the order of vertices and division into blocks is based on this order, then the non-existence of a block-respecting independent set of size k doesn't even logically imply the non-existence of one of size k' for all $k' > k$. This monotonicity, however, holds when $k \mid k'$.

Lemma 3.4. *For any graph G, $k' \ge k$, $k \mid k'$, and $k' \mid n$,*

$$\begin{aligned} \mathsf{RES}(\alpha_{block}(G,k')) &\le \mathsf{RES}(\alpha_{block}(G,k)) \quad \textit{and} \\ \mathsf{DPLL}(\alpha_{block}(G,k')) &\le \mathsf{DPLL}(\alpha_{block}(G,k)). \end{aligned}$$

The result holds even when the 1-1 clauses are omitted from both encodings.

Proof. If G contains a block-respecting independent set of size k, then there is no resolution refutation of $\alpha_{block}(G,k)$. By our convention, $Res(\alpha_{block}(G,k)) = DPLL(\alpha_{block}(G,k)) = \infty$, and the result holds. Otherwise consider a refutation π of $\alpha_{block}(G,k)$. The two encodings, $\alpha_{block}(G,k)$ and $\alpha_{block}(G,k')$, are defined over the same set of variables and have identical edge clauses. We will apply a transformation

σ to the variables so that the block and 1-1 clauses of $\alpha_{block}(G, k)$ become a subset of the block and 1-1 clauses of $\alpha_{block}(G, k')$, respectively.

σ works as follows. Each block of vertices in $\alpha_{block}(G, k)$ consists exactly of k'/k blocks of vertices in $\alpha_{block}(G, k')$ because $k \mid k'$. σ sets all but the first n/k' vertices of each block of $\alpha_{block}(G, k)$ to FALSE. This shrinks all block clauses of $\alpha_{block}(G, k)$ to block clauses of $\alpha_{block}(G, k')$. Further, it trivially satisfies all 1-1 clauses of $\alpha_{block}(G, k)$ that are not 1-1 clauses of $\alpha_{block}(G, k')$. Hence $\pi|_\sigma$ is a refutation of $\alpha_{block}(G, k')$ which in fact uses only a subset of the original block and 1-1 clauses of the formula. $\square$

3.2.4 Relationships Among Encodings

For reasonable bounds on the block size, resolution refutations of the block encoding are essentially as efficient as those of the other two encodings. We state the precise relationship in the following lemmas.

Lemma 3.5. *For any graph G over n vertices, $k \mid n$, and $b = n/k$,*

$$\begin{aligned}\mathsf{RES}(\alpha_{block}(G, k)) &\le b^2\ \mathsf{RES}(\alpha_{count}(G, k)) \quad \textit{and} \\ \mathsf{DPLL}(\alpha_{block}(G, k)) &\le (2\ \mathsf{DPLL}(\alpha_{count}(G, k)))^{\log_2 2b}.\end{aligned}$$

Proof. Fix a resolution proof π of $\alpha_{count}(G, k)$. We describe a transformation ρ on the underlying variables such that for each initial clause $C \in \alpha_{count}(G, k)$, $C|_\rho$ is either TRUE or an initial clause of $\alpha_{block}(G, k)$. This lets us generate a resolution proof of $\alpha_{block}(G, k)$ from $\pi|_\rho$ of size not much larger than $size(\pi)$. ρ is defined as follows: for each $i \in \{0, 1, \ldots, k\}$, set $y_{bi,i} =$ TRUE and $y_{bi,j} =$ FALSE for $j \ne i$; set all $y_{v,i} =$ FALSE if vertex v does not belong to either block $i + 1$ or block i; finally, for $1 \le j \le b$, replace all occurrences of $y_{bi+j,i+1}$ and $\neg y_{bi+j,i}$ with $(x_{bi+1} \vee x_{bi+2} \vee \ldots \vee x_{bi+j})$, and all occurrences of $\neg y_{bi+j,i+1}$ and $y_{bi+j,i}$ with $(x_{bi+j+1} \vee x_{bi+j+2} \vee \ldots \vee x_{bi+b})$. Note that setting $y_{bi,i} =$ TRUE for each i logically implies the rest of the transformations stated above.

We first prove that ρ transforms initial clauses of $\alpha_{count}(G, k)$ as claimed. The edge clauses are the same in both encodings. The size-k clause $y_{n,k}$ and the counting clause $y_{0,0}$ of $\alpha_{count}(G, k)$ transform to TRUE. The following can also be easily verified by plugging in the substitutions for the y variables. The counting clauses that define $y_{v,0}$ for $v \ge 1$ are either satisfied or translate into the first block clause $(x_1 \vee \ldots \vee x_b)$. Further, the counting clauses that define $y_{v,i}$ for $v \ge 1, i \ge 1$ are either satisfied or transform into the i^{th} or the $(i + 1)^{st}$ block clause, i.e., into $(x_{b(i-1)+1} \vee \ldots \vee x_{b(i-1)+b})$ or $(x_{bi+1} \vee \ldots \vee x_{bi+b})$. Hence, all initial clauses of $\alpha_{count}(G, k)$ are either satisfied or transform into initial clauses of $\alpha_{block}(G, k)$.

We now describe how to generate a valid resolution proof of $\alpha_{block}(G, k)$ from this transformation. Note that the substitutions for $y_{bi+j,i+1}$ and $y_{bi+j,i}$ replace these variables by a disjunction of at most b positive literals. Any resolution step performed on these y's in the original proof must now be converted into a set of equivalent resolution steps, which will lengthen the transformed refutation. More specifically, a step resolving clauses $(y \vee A)$ and $(\neg y \vee B)$ on the literal y (where y is either $y_{bi+j,i+1}$ or $y_{bi+j,i}$) will now be replaced by a set of resolution steps deriving $(A' \vee B')$ from clauses $(x_{u_1} \vee \ldots \vee x_{u_p} \vee A')$ and $(x_{v_1} \vee \ldots \vee x_{v_q} \vee B')$ and any initial clauses of $\alpha_{block}(G,k)$, where all x's mentioned belong to the same block of G, $\{u_1, \ldots, u_p\}$ is disjoint from $\{v_1, \ldots, v_q\}$, $p + q = b$, and A' and B' correspond to the translated versions of A and B, respectively.

The obvious way of doing this is to resolve the clause $(x_{u_1} \vee \ldots \vee x_{u_p} \vee A')$ with all 1-1 clauses $(\neg x_{u_i} \vee \neg x_{v_1})$ obtaining $(\neg x_{v_1} \vee A')$. Repeating this for all x_{v_j}'s gives us clauses $(\neg x_{v_j} \vee A')$. Note that this reuses $(x_{u_1} \vee \ldots \vee x_{u_p} \vee A')$ q times and is therefore not tree-like. Resolving all $(\neg x_{v_j} \vee A')$ in turn with $(x_{v_1} \vee \ldots \vee x_{v_q} \vee B')$ gives us $(A' \vee B')$. This takes $pq + q < b^2$ steps. Hence the blow-up in size for general resolution is at most a factor of b^2. Note that this procedure is symmetric in A' and B'; we could also have chosen the clause $(\neg y \vee B)$ to start with, in which case we would need $qp + p < b^2$ steps.

The tree-like case is somewhat trickier because we need to replicate clauses that are reused by the above procedure. We handle this using an idea similar to the one used by Clegg et al. [36] for deriving the size-width relationship for tree-like resolution proofs. Let $newSize(s)$ denote the maximum over the sizes of all transformed tree-like proofs obtained from original tree-like proofs of size s by applying the above procedure and creating enough duplicates to take care of reuse. We prove by induction that $newSize(s) \leq (2s)^{\log_2 2b}$. For the base case, $newSize(1) = 1 \leq 2b = 2^{\log_2 2b}$. For the inductive step, consider the subtree of the original proof that derives $(A \vee B)$ by resolving $(y \vee A)$ and $(\neg y \vee B)$ on the literal y as above. Let this subtree be of size $s \geq 2$ and assume without loss of generality that the subtree deriving $(y \vee A)$ is of size $s_A \leq s/2$. By induction, the transformed version of this subtree deriving $(x_{u_1} \vee \ldots \vee x_{u_p} \vee A')$ is of size at most $newSize(s_A)$ and that of the other subtree deriving $(x_{v_1} \vee \ldots \vee x_{v_q} \vee B')$ is of size at most $newSize(s - s_A - 1)$. Choose $(x_{u_1} \vee \ldots x_{u_p} \vee A')$ as the clause to start the new derivation of $(A' \vee B')$ as described in the previous paragraph. The size of this refutation is at most $b \cdot newSize(s_A) + newSize(s - s_A - 1) + b^2$. Since this can be done for any original proof of size s, $newSize(s) \leq b \cdot newSize(s_A) + newSize(s - s_A - 1) + b^2$ for $s \geq 2$ and $s_A \leq s/2$. It can be easily verified that $newSize(s) = 2bs\ b^{\log_2 s} = (2s)^{\log_2 2b}$ is a solution to this. This proves the bound for the DPLL case. □

Lemma 3.6. *For any graph G over n vertices and $k \mid n$,*

$$\begin{aligned} \mathsf{RES}(\alpha_{block}(G,k)) &\leq \mathsf{RES}(\alpha_{map}(G,k)) \quad \textit{and} \\ \mathsf{DPLL}(\alpha_{block}(G,k)) &\leq \mathsf{DPLL}(\alpha_{map}(G,k)). \end{aligned}$$

Proof. In the general encoding $\alpha_{map}(G,k)$, a vertex v can potentially be chosen as the i^{th} node of the k-independent set for *any* $i \in \{1, 2, \ldots, k\}$. In the restricted encoding, however, vertex v belonging to block j can be thought of as either being selected as the j^{th} node of the independent set or not being selected at all. Hence, if we start with a resolution (or DPLL) refutation of $\alpha_{map}(G,k)$ and set $z_{v,i} = \textsc{false}$ for $i \neq j$, we get a simplified refutation where the only variables are of the form $z_{v,j}$, where vertex v belongs to block j. Renaming these $z_{v,j}$'s as x_v's, we get a refutation in the variables of $\alpha_{block}(G,k)$ that is no larger in size than the original refutation of $\alpha_{map}(G,k)$.

All we now need to do is verify that for every initial clause of $\alpha_{map}(G,k)$, this transformation either converts it into an initial clause of $\alpha_{block}(G,k)$ or satisfies it. The transformed refutation will then be a refutation of $\alpha_{block}(G,k)$ itself. This reasoning is straightforward:

(a) Edge clauses $(\neg z_{u,i} \vee \neg z_{v,j})$ of $\alpha_{map}(G,k)$ that represented edge $(u,v) \in E$ with u in block i and v in block j transform into the corresponding edge clause $(\neg x_u \vee \neg x_v)$ of $\alpha_{block}(G,k)$. If vertex u (or v) is not in block i (or j, resp.), then the transformation sets $z_{u,i}$ (or $z_{v,j}$, resp.) to FALSE and the clause is trivially satisfied.

(b) Surjective clauses of $\alpha_{map}(G,k)$ clearly transform to the corresponding block clauses of $\alpha_{block}(G,k)$ – for the i^{th} such clause, variables corresponding to vertices that do not belong to block i are set to FALSE and simply vanish, and we are left with the i^{th} block clause of $\alpha_{block}(G,k)$.

(c) It is easy to see that all function clauses and ordering clauses are trivially satisfied by the transformation.

(d) 1-1 clauses $(\neg z_{u,i} \vee \neg z_{v,i})$ of $\alpha_{map}(G,k)$ that involved vertices u and v both from block i transform into the corresponding 1-1 clause $(\neg x_u \vee \neg x_v)$ of $\alpha_{block}(G,k)$. If vertex u (or v) is not in block i, then the transformation sets $z_{u,i}$ (or $z_{v,i}$, resp.) to FALSE and the clause is trivially satisfied.

Thus, this transformed proof is a refutation of $\alpha_{block}(G,k)$ and the desired bounds follow. $\square$

3.3 *Simulating Chvátal's Proof System*

In this section, we show that resolution on $\alpha_{block}(G,k)$ can efficiently simulate Chvátal's proofs [34] of non-existence of k-independent sets in G. This indirectly provides bounds on the running time of various algorithms for finding a maximum independent set in a given graph. We begin with a brief description of Chvátal's proof system. Let (S,t) for $t \geq 1$ be the *statement* that the subgraph of G induced by a vertex subset S does not have an independent set of size t. $(\phi, 1)$ is given as an axiom and the goal is to derive, using a series of applications of one of two rules, the statement (V,k), where V is the vertex set of G and k is given as input. The two inference rules are

Branching Rule: for any vertex $v \in S$, from statements $(S \setminus N(v), t-1)$ and $(S \setminus \{v\}, t)$ one can infer (S, t), where $N(v)$ is the set containing v and all its neighbors in G;

Monotone Rule: from statement (S, t) one can infer any (S', t') that (S, t) *dominates*, i.e., $S \supseteq S'$ and $t \leq t'$.

For a graph G with vertex set $V(G)$, let $Chv(G, k)$ denote the size of the smallest proof in Chvátal's system of the statement $(V(G), k)$. Following our convention, $Chv(G, k) = \infty$ if no such proof exists. As an immediate application of the monotone rule, we have:

Proposition 3.3. *For $k' > k$, $Chv(G, k') \leq Chv(G, k) + 1$.*

Proposition 3.4. *Let G and G' be graphs with $V(G) = V(G')$ and $E(G) \subseteq E(G')$. For any k, $Chv(G', k) \leq 2 \cdot Chv(G, k)$ and the number of applications of the branching rule in the two shortest proofs is the same.*

Proof. Let π be a proof of $(V(G), k)$ in G. We convert π into a proof π' of $(V(G'), k)$ in G' by translating proof statements in the order in which they appear in π. The axiom statement translates directly without any change. For the derived statements, any application of a monotone inference can be applied equally for both graphs. For an application of the branching rule in π, some (S, t) is derived from $(S \setminus N(v), t-1)$ and $(S \setminus \{v\}, t)$. To derive (S, t) for G', the only difference is the replacement of $(S \setminus N(v), t-1)$ by $(S \setminus N'(v), t-1)$, where $N'(v)$ is the set containing v and all its neighbors in G'. If these two statements are different then since $N'(v) \supseteq N(v)$, the latter follows from the former by a single application of the monotone rule. In total, at most $size(\pi)$ additional inferences are added, implying $size(\pi') \leq 2 size(\pi)$. $\square$

The following lemma shows that by traversing the proof graph beginning with the axioms one can locally replace each inference in Chvátal's system by a small number of resolution inferences.

Lemma 3.7. *For any graph G over n vertices and $k \mid n$,*

$$\mathsf{RES}(\alpha_{block}(G, k)) \quad \leq \quad 4n\ Chv(G, k).$$

Proof. Let V denote the vertex set of G. Arbitrarily partition V into k blocks of equal size. Let G_{block} be the graph obtained by adding to G all edges (u, v) such that vertices u and v belong to the same block of G. In other words, G_{block} is G modified to contain a clique on each block so that every independent set of size k in G_{block} is block-respecting with respect to G. By Proposition 3.4, the shortest proof in Chvátal's system, say π_{Chv}, of (V, k) in G_{block} is at most twice in size as the shortest proof of

(V, k) in G. We will use π_{Chv} to guide the construction of a resolution refutation π_{RES} of $\alpha_{block}(G, k)$ such that $size(\pi_{RES}) \leq 2n\ size(\pi_{Chv})$, proving the desired bound.

Observe that without loss of generality, for any statement (S, t) in π_{Chv}, t is at least the number of blocks of G containing vertices in S. This is so because it is true for the final statement (V, k), and if it is true for (S, t), then it is also true for both $(S \setminus \{v\}, t)$ and $(S \setminus N(v), t-1)$ from which (S, t) is derived. Call (S, t) a *trivial* statement if t is strictly bigger than the number of blocks of G containing vertices in S. The initial statement $(\phi, 1)$ of the proof is trivial, whereas the final statement (V, k) is not. Furthermore, all statements derived by applying the monotone rule are trivial.

π_{RES} will have a clause associated with each non-trivial statement (S, t) occurring in π_{Chv}. This clause will be a subclause of the clause $C_S \overset{\text{def}}{=} (\bigvee_{u \in N_S} x_u)$, where N_S is the set of all vertices in $V \setminus S$ that are in blocks of G containing at least one vertex of S. π_{RES} will be constructed inductively, using the non-trivial statements of π_{Chv}. Note that the clause associated in this manner with (V, k) will be the empty clause, making π_{RES} a refutation.

Suppose (S, t) is non-trivial and is derived in π_{Chv} by applying the branching rule to vertex $v \in S$. Write the target clause C_S as $(C_S^b \vee C_S^r)$, where C_S^b is the disjunction of all variables corresponding to vertices of N_S that are in the same block as v, and C_S^r is the disjunction of all variables corresponding to vertices of N_S that are in the remaining blocks. Before deriving the desired subclause of C_S, derive two clauses Cl_1 and Cl_2 as follows depending on the properties of the inference that produced (S, t):

Case 1: Both $(S \setminus \{v\}, t)$ and $(S \setminus N(v), t-1)$ are trivial. It is easy to see that since (S, t) is non-trivial, if $(S \setminus \{v\}, t)$ is trivial then v is the only vertex of S in its block. Let Cl_1 be the initial block clause for the block containing v, which is precisely $(x_v \vee C_S^b)$. The fact that $(S \setminus N(v), t-1)$ is also trivial implies that the neighbors of v include not only every vertex of S appearing in the block containing v but also all vertices in $S \cap B$, where B is some other block that does not contain v. Resolving the block clause for block B with all edge clauses $(\neg x_v \vee \neg x_u)$ for $u \in S \cap B$ gives a subclause Cl_2 of $(\neg x_v \vee C_S^r)$.

Case 2: $(S \setminus \{v\}, t)$ is trivial but $(S \setminus N(v), t-1)$ is non-trivial. Set Cl_1 exactly as in case 1. Given that $(S \setminus N(v), t-1)$ is non-trivial, by the inductive assumption the prefix of π_{RES} constructed so far contains a subclause of $C_{S \setminus N(v)}$. Since the given proof applies to G_{block}, $N(v) \cup v$ contains every vertex in the block containing v as well as all neighbors of v in G that are not in v's block. Therefore, the subclause of $C_{S \setminus N(v)}$ we have by induction is a subclause of $(C_S^r \vee x_{u_1} \vee \ldots \vee x_{u_p})$, where each u_i is a neighbor of v in S in blocks other than v's block. Derive a new clause Cl_2 by resolving this clause with all edge clauses $(\neg x_v \vee \neg x_{u_i})$. Observe that Cl_2 is a subclause of $(\neg x_v \vee C_S^r)$.

Case 3: $(S \setminus \{v\}, t)$ is non-trivial but $(S \setminus N(v), t-1)$ is trivial. Set Cl_2 as in case 1. Since $(S \setminus \{v\}, t)$ is non-trivial, by the inductive assumption the prefix of π_{RES}

constructed so far contains a subclause Cl_2 of $C_{S\setminus\{v\}}$, i.e., a subclause of $(x_v \vee C_S)$.

Case 4: Both $(S \setminus \{v\}, t)$ and $(S \setminus N(v), t-1)$ are non-trivial. In this case, derive Cl_1 as in case 3 and Cl_2 as in case 2.

It is easy to verify that Cl_1 is a subclause of $(x_v \vee C_S)$ and Cl_2 is a subclause of $(\neg x_v \vee C^r_S)$. If either Cl_1 or Cl_2 does not mention x_v at all, then we already have the desired subclause of C_S. Otherwise resolve Cl_1 with Cl_2 to get a subclause of C_S. This completes the construction. Given any non-trivial statement in π_{Chv}, it takes at most $2n$ steps to derive the subclause associated with it in the resolution proof, given that we have already derived the corresponding subclauses for the two branches of that statement. Hence, $size(\pi_{RES}) \leq 2n\ size(\pi_{Chv})$. □

It follows that lower bounds on the complexity of α_{block} apply to Chvátal's system and hence also to many algorithms for finding a maximum independent set in a given graph that are captured by his proof system, such as those of Tarjan [105], Tarjan and Trojanowski [106], Jian [67], and Shindo and Tomita [100].

3.4 Relation to Vertex Cover and Coloring

This section discusses how the independent set problem relates to vertex covers and colorings of random graphs in terms of resolution complexity.

3.4.1 Vertex Cover

As for independent sets, for any undirected graph $G = (V, E)$, let $n = |V|$, $m = |E|$, and $\Delta = m/n$. A *t-vertex cover* in G is a set of t vertices that contains at least one endpoint of every edge in G. I is an independent set in G if and only if $V \setminus I$ is a vertex cover of G. Hence, the problem of determining whether or not G has a t-vertex cover is the same as that of determining whether or not it has a k-independent set for $k = n - t$. We use this correspondence to translate our bounds on the resolution complexity of independent sets to those on the resolution complexity of vertex covers.

Consider encoding in clausal form the statement that G has a t-vertex cover. The only defining difference between an independent set and a vertex cover is that the former requires at most one of the endpoints of every edge to be included, where as the latter requires at least one. Natural methods to count remain the same, that is, explicit counting variables, implicit mapping variables, or blocks. Similar to the independent set encoding variables, let $x'_v, 1 \leq v \leq n$, be a set of variables such that $x'_v = \text{TRUE}$ iff vertex v is chosen to be in the vertex cover. Let $y'_{v,i}, 1 \leq v \leq n, 1 \leq i \leq t$, denote the fact that exactly i of the first v vertices are chosen in the vertex cover. Let $z'_{v,i}, 1 \leq v \leq n, 1 \leq i \leq t$, represent that vertex v is mapped to the i^{th} node of the vertex cover.

The *counting encoding* of vertex cover, $VC_{count}(G, t)$, is defined analogous to $\alpha_{count}(G, k)$ except for the change that for an edge $(u, v) \in E$, the edge clause for vertex cover is $(x'_u \vee x'_v)$ and not $(\neg x'_u \vee \neg x'_v)$. The rest of the encoding is obtained by setting $k \leftarrow t, x_v \leftarrow x'_v, y_{v,i} \leftarrow y'_{v,i}$. The *mapping encoding* of vertex cover, $VC_{mapping}(G,t)$ is similarly defined analogous to $\alpha_{mapping}(G,k)$ by setting $k \leftarrow t, z_{v,i} \leftarrow z'_{v,i}$, except for the change in edge clauses for edges $(u,v) \in E$ from $(\neg z_{u,i} \vee \neg z_{v,i})$ to $(z'_{u,i} \vee z'_{v,i})$. For $b = n/(n-t)$, the *block encoding* of vertex cover over $(n-t)$ blocks of size b each, $VC_{block}(G,t)$, is also defined analogous to $\alpha_{block}(G,k)$ by setting $k \leftarrow (n-t), x_v \leftarrow \neg x'_v$. It says that each edge is covered, and exactly $b-1$ vertices from each block are selected in the vertex cover, for a total of $(n-t)(b-1) = t$ vertices. Note that the 1-1 clauses of α_{block} translate into "all-but-one" clauses of VC_{block}.

It is not surprising that the resolution complexity of various encodings of the vertex cover problem is intimately related to that of the corresponding encodings of the independent set problem. We formalize this in the following lemmas.

Lemma 3.8. *For any graph G over n vertices,*

$$\mathsf{RES}(VC_{count}(G,t)) \quad \leq \quad \mathsf{RES}(\alpha_{count}(G,n-t)) + 6nt^2.$$

Proof. If G has an independent set of size $n-t$, then there is no resolution refutation of $\alpha_{count}(G,n-t)$. Consequently, $Res(\alpha_{count}(G,n-t)) = DPLL(\alpha_{count}(G,n-t)) = \infty$, trivially satisfying the claimed inequalities. Otherwise, consider a refutation π of $\alpha_{count}(G,n-t)$. We use π to construct a refutation π' of $VC_{count}(G,t)$ that is not too big.

Recall that the variables of π are $x_u, 1 \leq u \leq n$, and $y_{v,i}, 0 \leq i \leq v \leq n, 0 \leq i \leq n-t$. The variables of π' will be $x'_u, 1 \leq u \leq n$, and $y'_{v,i}, 0 \leq i \leq v \leq n, 0 \leq i \leq t$. Notice that the number of independent set counting variables $y_{v,i}$ is not the same as the number of vertex cover counting variables $y'_{v,i}$. We handle this by adding dummy counting variables, transforming π, and removing extra variables. To obtain π', apply transforms σ_1, σ_2 and σ_3 defined below to π.

σ_1 simply creates new counting variables $y_{v,i}, 0 \leq v \leq n, (n-t+1) \leq i \leq v$, and adds counting clauses corresponding to these variables as unused initial clauses of π. σ_2 sets $x_u \leftarrow \neg x'_u, y_{v,i} \leftarrow y'_{v,v-i}$. Intuitively, σ_2 says that i of the first v vertices being in the independent set is equivalent to exactly $v-i$ of the first v vertices being in the vertex cover. σ_3 sets $y'_{v,i} \leftarrow$ FALSE for $0 \leq v \leq n, (t+1) \leq i \leq v$. Since σ_1, σ_2 and σ_3 only add new clauses, rename literals or set variables, their application transforms π into another, potentially simpler, refutation on a different set of variables and initial clauses. Call the resulting refutation π''.

The initial edge clauses $(\neg x_u \vee \neg x_v)$ of π transform into edge clauses $(x'_u \vee x'_v)$ of $VC_{count}(G,t)$. The initial size-$(n-t)$ clause of π transforms into the initial size-t clause of $VC_{count}(G,t)$. Finally, the initial counting clauses of π, including those corresponding to the variables added by σ_1, transform into counting clauses of $VC_{count}(G,t)$ and n extra clauses. To see this, note that σ_2 transforms counting formulas $y_{0,0}$

into $y'_{0,0}$, $(y_{v,0} \leftrightarrow (y_{v-1,0} \wedge \neg x_v))$ into $(y'_{v,v} \leftrightarrow (y'_{v-1,v-1} \wedge x'_v))$, for $i \geq 1 : (y_{v,i} \leftrightarrow ((y_{v-1,i} \wedge \neg x_v) \vee (y_{v-1,i-1} \wedge x_v)))$ into $(y'_{v,v-i} \leftrightarrow ((y'_{v-1,v-i-1} \wedge x'_v) \vee (y'_{v-1,v-i} \wedge \neg x'_v)))$, and $(y_{v,v} \leftrightarrow (y_{v-1,v-1} \wedge x_v))$ into $(y'_{v,0} \leftrightarrow (y_{v-1,0} \wedge \neg x'_v))$. Applying σ_3 to set $y'_{v,i} \leftarrow$ FALSE for $(t+1) \leq i \leq v$ removes all but the initial counting clauses of $VC_{count}(G,t)$ and the counting formulas corresponding to the variables $y'_{v,t+1}, t+1 \leq v \leq n$, that simplify to $(\neg y'_{v-1,t} \vee \neg x'_v)$. Call this extra set of $n-t$ clauses $Bdry(G,t)$, or *boundary* clauses for (G,t).

At this stage, we have a refutation π'' of size at most $size(\pi)$ starting from clauses $VC_{count}(G,t) \cup Bdry(G,t)$. The boundary clauses together say that no more than t vertices are chosen in the vertex cover. This, however, is implied by the rest of the initial clauses. Using this fact, we first give a derivation π_{Bdry} of every boundary clause starting from the clauses of $VC_{count}(G,t)$. Appending π'' to π_{Bdry} gives a refutation π' of $VC_{count}(G,t)$.

Let $S_i = \bigvee_{i'=0}^{\min\{i,t\}} y'_{n-i,t-i'}$ for $0 \leq i \leq n-t$. Let $R_{v,i,j} = (\neg y'_{v,i} \vee \neg y'_{v,j})$ for $0 \leq i < j \leq v \leq n$ and $j \leq t$. We first give a derivation of these S and R clauses, and then say how to derive the boundary clauses from these. $S_0 = y'_{n,t}$ is an initial clause, and $S_i, i \geq 1$, is obtained by sequentially resolving S_{i-1} with the counting clauses $(\neg y'_{n-i+1,t-i'} \vee y'_{n-i,t-i'} \vee y'_{n-i,t-i'-1})$ for $0 \leq i' < \min\{i,t\}$. Similarly, when $i = 0$, $R_{v,0,v}$ is derived by resolving counting clauses $(\neg y'_{v,0} \vee \neg x'_v)$ and $(\neg y'_{v,v} \vee x'_v)$ on x'_v, clauses $R_{v,0,j}$ for $0 < j < v$ are derived by sequentially resolving $R_{v-1,0,j}$ with the counting clauses $(\neg y'_{v,j} \vee y'_{v-1,j} \vee x'_v)$ and $(\neg y'_{v,0} \vee \neg x'_v)$. Note that $R_{v,0,v}$ and $R_{v,0,j}$ are defined and derived only when $j \leq t$. When $i > 0$, $R_{v,i,v}$ is derived by sequentially resolving $R_{v-1,i-1,v-1}$ with the counting clauses $(\neg y'_{v,i} \vee y'_{v-1,i-1} \vee \neg x'_v)$ and $(\neg y'_{v,v} \vee y'_{v-1,v-1} \vee \neg x'_v)$, and resolving the result on x'_v with the counting clause $(\neg y'_{v,v} \vee x'_v)$. Finally, $R_{v,i,j}$ for $j < v$ is derived by resolving $R_{v-1,i,j}$ with the counting clauses $(\neg y'_{v,i} \vee y'_{v-1,i} \vee x'_v)$ and $(\neg y'_{v,j} \vee y'_{v-1,j} \vee x'_v)$, resolving $R_{v-1.i-1.j-1}$ with the counting clauses $(\neg y'_{v,i} \vee y'_{v-1,i-1} \vee \neg x'_v)$ and $(\neg y'_{v,j} \vee y'_{v-1,j-1} \vee \neg x'_v)$, and resolving the result of the two on x'_v.

To derive the boundary clause $(\neg y'_{v-1,t} \vee \neg x'_v)$ for any v, resolve each pair of clauses $(\neg y'_{v,t-i'} \vee y'_{v-1,t-i'-1} \vee \neg x'_v)$ and $R_{v-1,t-i'-1,t}$ for $0 \leq i' \leq \min\{n-v,t\}$, and resolve all resulting clauses with S_{n-v}. Note that when $\min\{n-v,t\} = t$, there is no $R_{v-1,t-i'-1,t}$, but the corresponding counting clause itself is of the desired form, $(\neg y'_{v,0} \vee \neg x'_v)$. This finishes the derivation π_{Bdry} of all clauses in $Bdry(G,t)$. As stated before, appending π'' to π_{Bdry} gives a refutation π' of $VC_{count}(G,t)$.

For general resolution, $size(\pi') = size(\pi'') + size(\pi_{Bdry}) \leq size(\pi) + size(\pi_{Bdry})$. Each S_i in π_{Bdry}, starting with $i = 0$, is derived in $\min\{i,t\}$ resolution steps from previous clauses, and each $R_{v,i,j}$, starting with $i = 0, v = j = 1$, requires at most 5 resolution steps from previous clauses. Hence, $size(\pi_{Bdry}) \leq nt + 5nt^2 \leq 6nt^2$ for large enough n, implying that $size(\pi') \leq size(\pi) + 6nt^2$. Note that this approach doesn't quite work for tree-like resolution proofs because π_{Bdry} itself becomes exponential in size due to the heavy reuse of clauses involved in the derivation of the $R_{v,i,j}$'s. $\square$

Given that the encodings $\alpha_{count}(G, n-t)$ and $VC_{count}(G, t)$ are duals of each other, the argument made for the Lemma above can also be made the other way, immediately giving us the following reverse result:

Lemma 3.9. *For any graph G over n vertices,*

$$\mathsf{RES}(\alpha_{count}(G,k)) \;\leq\; \mathsf{RES}(VC_{count}(G,n-k)) + 6nk^2.$$

Lemma 3.10. *For any graph G over n vertices and $(n-t)\,|\,n$,*

$$\begin{aligned}\mathsf{RES}(VC_{block}(G,t)) &= \mathsf{RES}(\alpha_{block}(G,n-t)) \quad \textit{and}\\ \mathsf{DPLL}(VC_{block}(G,t)) &= \mathsf{DPLL}(\alpha_{block}(G,n-t)).\end{aligned}$$

This result also holds without the 1-1 clauses of α_{block} and the corresponding all-but-one clauses of VC_{block}.

Proof. If G has an independent set of size $n-t$, then it also has a vertex cover of size t. In this case, there are no resolution refutations of $VC_{block}(G,t)$ or $\alpha_{block}(G,n-t)$, making the resolution complexity of both infinite and trivially satisfying the claim.

Otherwise, consider a refutation π of $\alpha_{block}(G,n-t)$. We use π to construct a refutation π' of $VC_{block}(G,t)$, which is of the same size and is tree-like if π is. π' is obtained from π by simply applying the transformation $x_v \leftarrow \neg x'_v, 1 \leq v \leq n$. Since this is only a 1-1 mapping between literals, π' is a legal refutation of size exactly $size(\pi)$. All that remains to argue is that the initial clauses of π' are the clauses of $VC_{block}(G,t)$. This, however, follows immediately from the definition of $VC_{block}(G,t)$.

Given the duality of the encodings $VC_{block}(G,t)$ and $\alpha_{block}(G,n-t)$, we can repeat the argument above to translate any refutation of the former into one of the latter. Combining this with the above, the resolution complexity of the two formulas is exactly the same. ☐

3.4.2 Coloring

A K-*coloring* of a graph $G = (V,E)$ is a function $col : V \to \{1,2,\ldots,K\}$ such that for every edge $(u,v) \in E$, $col(u) \neq col(v)$. For a random graph G chosen from a distribution similar to $\mathbb{G}(n,p)$, the resolution complexity of the formula $\chi(G,K)$ saying that G is K-colorable has been addressed by Beame et al. [14].

Suppose G is K-colorable. Fix a K-coloring col of G and partition the vertices into color classes $V_i, 1 \leq i \leq K$, where $V_i = \{v \in V : col(v) = i\}$. Each color class, by definition, must be an independent set, with the largest of size at least n/K. Thus, non-existence of a $k \stackrel{\text{def}}{=} n/K$ size independent set in G implies the non-existence of a K-coloring of G.

Let $\alpha(G,k)$ be an encoding of the k-independent set problem on graph G. The correspondence above can be used to translate properly encoded resolution proofs of

$\alpha(G,k)$ into those of $\chi(G,K)$. A lower bound on $\mathsf{RES}(\chi(G,K))$, such as the one in [14], would then imply a lower bound on $\mathsf{RES}(\alpha(G,k))$. However, such a translation between proofs must involve a resolution counting argument showing that K sets of vertices, each of size less than n/K, cannot cover all n vertices. This argument itself is at least as hard as PHP^{n}_{n-K}, the (weak) pigeonhole principle on n pigeons and $n-K$ holes, for which exponential lower bound has been shown by Raz [94]. This makes any translation of a proof of $\alpha(G,k)$ into one of $\chi(G,K)$ necessarily large, ruling out any interesting lower bound for independent sets as a consequence of [14].

On the other hand, non-existence of a K-coloring does not imply the non-existence of a k-independent set. In fact, there are very simple graphs with no K-coloring but with an independent set as large as $n-K$ (e.g. a clique of size $K+1$ along with $n-K-1$ nodes of degree zero). Consequently, our lower bounds for independent sets do not give any interesting lower bounds for K-coloring.

3.5 Upper Bounds

Based on a very simple exhaustive backtracking strategy, we give upper bounds on the DPLL (and hence resolution) complexity of the independent set and vertex cover encodings we have considered.

Lemma 3.11. *There is a constant C_0 such that if G is a graph over n vertices with no independent set of size k, then*

$$\mathsf{DPLL}(\alpha_{map}(G,k)) \;\leq\; 2^{C_0 k \log(ne/k)}.$$

This bound also holds when $\alpha_{map}(G,k)$ does not include 1-1 clauses.

Proof. A straightforward way to disprove the existence of a k-independent set is to go through all $\binom{n}{k}$ subsets of vertices of size k and use as evidence an edge from each subset. We use this strategy to construct a refutation of $\alpha_{map}(G,k)$.

To begin with, apply transitivity to derive all ordering clauses of the form $(\neg z_{u,j} \vee \neg z_{v,i})$ for $u<v$ and $i<j$. If $j=i+1$, this is simply one of the original ordering clauses. For $j=i+2$, derive the new clause $(\neg z_{u,i+2} \vee \neg z_{v,i})$ as follows. Consider any $w \in \{1,2,\ldots,n\}$. If $u<w$, we have the ordering clause $(\neg z_{w,i+1} \vee \neg z_{u,i+2})$, and if $u \geq w$, then $v>w$ and we have the ordering clause $(\neg z_{v,i} \vee \neg z_{w,i+1})$. Resolving these n ordering clauses (one for each w) with the surjective clause $(z_{1,i+1} \vee \ldots \vee z_{n,i+1})$ gives the new ordering clause $(\neg z_{u,i+2} \vee \neg z_{v,i})$ associated with u and v. This clearly requires only n steps and can be done for all $u<v$ and $j=i+2$. Continue to apply this argument for $j=i+3,i+4,\ldots,k$ and derive all new ordering clauses in n steps each.

We now construct a tree-like refutation starting with the initial clauses and the new ordering clauses we derived above. We claim that for any $i \in \{1,2,\ldots,k\}$ and for any $1 \leq v_i < v_{i+1} < \ldots < v_k \leq n$, a subclause of $(\neg z_{v_i,i} \vee \neg z_{v_{i+1},i+1} \vee \ldots \vee \neg z_{v_k,k})$

can be derived. We first argue why this claim is sufficient to obtain a refutation. For $i = k$, the claim says that a subclause of $\neg z_{v_k,k}$ can be derived for all $1 \leq v_k \leq n$. If any one of these n subclauses is a strict subclause of $\neg z_{v_k,k}$, it has to be the empty clause, resulting in a refutation. Otherwise, we have $\neg z_{v_k,k}$ for every v_k. Resolving all these with the surjective clause $(z_{1,k} \vee \ldots \vee z_{n,k})$ results in the empty clause.

We now prove the claim by induction on i. For the base case, fix $i = 1$. For any given k vertices $v_1 < v_2 < \ldots < v_k$, choose an edge (v_p, v_q) that witnesses the fact that these k vertices do not form an independent set. The corresponding edge clause $(\neg z_{v_p,p} \vee \neg z_{v_q,q})$ works as the required subclause.

For the inductive step, fix $v_{i+1} < v_{i+2} < \ldots < v_k$. We will derive a subclause of $(\neg z_{v_{i+1},i+1} \vee \neg z_{v_{i+2},i+2} \vee \ldots \neg z_{v_k,k})$. By induction, derive a subclause of $(\neg z_{v_i,i} \vee \neg z_{v_{i+1},i+1} \vee \ldots \vee z_{v_k,k})$ for any choice of $v_i < v_{i+1}$. If for some such v_i, $\neg z_{v_i,i}$ does not appear in the corresponding subclause, then the same subclause works here for the inductive step and we are done. Otherwise, for every $v_i < v_{i+1}$, we have a subclause of $(\neg z_{v_i,i} \vee \neg z_{v_{i+1},i+1} \vee \ldots \vee \neg z_{v_k,k})$ that contains $\neg z_{v_i,i}$. Resolving all these subclauses with the surjective clause $(z_{1,i} \vee z_{2,i} \vee \ldots \vee z_{n,i})$ results in the clause $(z_{v_{i+1},i} \vee \ldots \vee z_{v_k,i} \vee \neg z_{u_1,j_1} \vee \ldots \vee \neg z_{u_p,j_p})$, where each z_{u_c,j_c} lies in $\{z_{v_{i+1},i+1}, \ldots, z_{v_k,k}\}$. Observe that for each positive literal $z_{v_q,i}, i+1 \leq q \leq k$, in this clause, $(\neg z_{v_q,i} \vee \neg z_{v_{i+1},i+1})$ is either a 1-1 clause or an ordering clause. Resolving with all these clauses finally gives $(\neg z_{v_{i+1},i+1} \vee \neg z_{u_1,j_1} \vee \ldots \vee \neg z_{u_p,j_p})$, which is the kind of subclause we wanted to derive. This proves the claim.

Associate each subclause obtained using the iterative procedure above with the tuple $(v_i, v_{i+1}, \ldots, v_k)$ for which it was derived, giving a total of $\sum_{i=1}^{k} \binom{n}{i} \leq (ne/k)^k$ subclauses. Each of these subclauses is used at most once in the proof. Further, the derivation of each such subclause uses at most n new ordering clauses, each of which can be derived in at most n^2 steps. Thus, with enough copies to make the refutation tree-like, the size of the proof is $O(n^3(ne/k)^k)$, which is at most $2^{C_0 k \log(ne/k)}$ for a large enough constant C_0. ☐

Lemma 3.12. *There is a constant C'_0 such that if G is graph over n vertices with no independent set of size k, then*

$$\mathsf{DPLL}(\alpha_{count}(G,k)) \quad \leq \quad 2^{C'_0 k \log(ne/k)}.$$

Proof. As in the proof of Lemma 3.11, we construct a refutation by looking at each size k subset of vertices and using as evidence an edge from that subset.

For every i, v such that $0 \leq i \leq v < n$, first derive a new counting clause $(\neg y_{v+1,i+1} \vee y_{v,i} \vee y_{v-1,i} \vee \ldots \vee y_{i,i})$ by resolving original counting clauses $(\neg y_{u+1,i+1} \vee y_{u,i+1} \vee y_{u,i})$ for $u = v, v-1, \ldots, i+1$ together, and resolving the result with the counting clause $(\neg y_{i+1,i+1} \vee y_{i,i})$. Next, for any edge (i,j), $i > j$, resolve the edge clause $(\neg x_i \vee \neg x_j)$ with the counting clauses $(\neg y_{i,i} \vee x_i)$ and $(\neg y_{j,j} \vee x_j)$ to get the clause $(\neg y_{i,i} \vee \neg y_{j,j})$. Call this new clause $E_{i,j}$. We now construct a tree-like refutation using the initial clauses, these new counting clauses, and the new $E_{i,j}$ clauses.

We claim that for any $i \in \{1, 2, \ldots, k\}$ and for any $1 \leq v_i < v_{i+1} < \ldots < v_k \leq n$ with $v_j \geq j$ for $i \leq j \leq k$, we can derive a subclause of $(\neg y_{v_i,i} \vee y_{v_i-1,i} \vee \neg y_{v_{i+1},i+1} \vee y_{v_{i+1}-1,i+1} \vee \ldots \vee \neg y_{v_k,k} \vee y_{v_k-1,k})$ such that if $y_{v_j-1,j}$ occurs in the subclause for some j, then so does $\neg y_{v_j,j}$. Note that for $v_j = j$, the variable $y_{v_j-1,j}$ does not even exist and will certainly not appear in the subclause. Given this claim, we can derive for $i = k$ a subclause B_j of $(\neg y_{j,k} \vee y_{j-1,k})$ for each $j \in \{k+1, \ldots, n\}$ and a subclause B_k of $\neg y_{k,k}$. If any of these B_j's is the empty clause, the refutation is complete. Otherwise every B_j contains $\neg y_{j,k}$. Let j' be the largest index such that $B_{j'}$ does not contain $y_{j'-1,k}$. Since B_k has to be the clause $\neg y_{k,k}$, such a j' must exist. Resolving all B_j's for $j \in \{j', \ldots, k\}$ with each other gives the clause $y_{n,k}$. Resolving this with the size-k clause $y_{n,k}$ gives the empty clause.

We now prove that the claim holds by induction on i. For the base case $i = 1$, fix $1 \leq v_1 < v_2 < \ldots < v_k \leq n$. Choose an edge (v_p, v_q) that witnesses the fact that these v_i's do not form an independent set. Resolve the corresponding edge clause $(\neg x_{v_p} \vee \neg x_{v_q})$ with the counting clauses $(\neg y_{v_p,p} \vee y_{v_p-1,p} \vee x_p)$ and $(\neg y_{v_q,q} \vee y_{v_q-1,q} \vee x_q)$ to get $(\neg y_{v_p,p} \vee y_{v_p-1,p} \vee \neg y_{v_q,q} \vee y_{v_q-1,q})$, which is a subclause of the desired form.

For the inductive step, fix $v_{i+1} < v_{i+2} < \ldots < v_k$. By induction, derive a subclause C_j of $(\neg y_{j,i} \vee y_{j-1,i} \vee \neg y_{v_{i+1},i+1} \vee y_{v_{i+1}-1,i+1} \vee \ldots \vee \neg y_{v_k,k} \vee y_{v_k-1,k})$ for any j in $\{i, i+1, \ldots, v_{i+1} - 1\}$. If for some such j, neither $\neg y_{j,i}$ nor $y_{j-1,i}$ appears in C_j, then this subclause also works here for the inductive step and we are done. Otherwise for every j, C_j definitely contains $\neg y_{j,i}$, possibly $y_{j-1,i}$, and other positive or negative occurrences of variables of the form $y_{v',i'}$ where $i' > i$. Now use these C_j's to derive clauses C'_j's such that C'_j contains $\neg y_{j,i}$ but not $y_{j-1,i}$. The other variables appearing in C'_j will all be of the form $y_{v',i'}$ for $i' > i$.

If $\{v_{i+1}, \ldots, v_k\}$ is not an independent set, then there is an edge (v_p, v_q) witnessing this. In this case, simply use $E_{p,q}$ as the desired subclause and the inductive step is over. Otherwise there must be an edge (i, v_q) from vertex i touching this set. Let C'_i be the clause E_{i,v_q}. For j going from $i+1$ to k, do the following iteratively. If $y_{j-1,i}$ does not appear in C_j, then set $C'_j = C_j$. Otherwise set C'_j to be the clause obtained by resolving C_j with C'_{j-1}. If C'_{j-1} does not contain $\neg y_{j,i}$, then it can be used as the desired subclause for this inductive step and the iteration is stopped here, otherwise it continues onto the next value of j. If desired subclause is not derived somewhere along this iterative process, then we end up with all C'_j's containing $\neg y_{j,i}$ but not $y_{j-1,i}$. Resolving all these with the new counting clause $(\neg y_{v_{i+1},i+1} \vee y_{v_{i+1}-1,i} \vee y_{v_{i+1}-2,i} \vee \ldots \vee y_{i,i})$ finally gives a subclause of the desired form. This proves the claim.

Associate each subclause obtained using the iterative procedure above with the tuple $(v_i, v_{i+1}, \ldots, v_k)$ for which it was derived, giving a total of $\sum_{i=1}^{k} \binom{n}{i} \leq (ne/k)^k$ subclauses. Each of these subclauses is used at most once in the proof. Further, the derivation of each such subclause uses one new counting clause and one new clause $E_{i,j}$, each of which can be derived in at most n steps. Thus, with enough copies to make the refutation tree-like, the size of the proof is $O(n(ne/k)^k)$, which is at most $2^{C_0' k \log(ne/k)}$ for a large enough constant C_0'. □

Theorem 3.1 (Independent Set Upper Bounds). *There are constants c_0, c_0' such that the following holds. Let $\Delta = np$, $\Delta \leq n/\log^2 n$, and $G \sim \mathbb{G}(n,p)$. Let k be such that G has no independent set of size k. With probability $1 - o(1)$ in n,*

$$\begin{aligned}
\mathsf{DPLL}(\alpha_{map}(G,k)) &\leq 2^{c_0 (n/\Delta) \log^2 \Delta}, \\
\mathsf{DPLL}(\alpha_{count}(G,k)) &\leq 2^{c_0' (n/\Delta) \log^2 \Delta}, \quad \textit{and} \\
\mathsf{DPLL}(\alpha_{block}(G,k)) &\leq 2^{c_0 (n/\Delta) \log^2 \Delta}.
\end{aligned}$$

The bounds also holds when 1-1 clauses are removed from $\alpha_{map}(G,k)$ or $\alpha_{block}(G,k)$. The block encoding bound holds when $k \mid n$.

Proof. By Proposition 3.1, $n/(\Delta+1) < k \leq n$. Hence $k \log(ne/k) \leq n \log(e(\Delta+1))$. We will use this fact when Δ is a relatively small constant.

Fix any $\epsilon > 0$ and let C_ϵ be the corresponding constant from Proposition 3.2. When $\Delta < C_\epsilon$, the desired upper bounds in this theorem are of the form $2^{O(n)}$. Moreover, the upper bounds provided by Lemmas 3.11 and 3.12 for the mapping and counting encodings, respectively, are exponential in $k \log(ne/k) \leq n \log(e(\Delta+1))$, and thus also of the form $2^{O(n)}$ when $\Delta < C_\epsilon$. Hence, for large enough constants c_0 and c_0', the claimed bounds hold with probability 1 for the mapping and counting encodings when $\Delta < C_\epsilon$. Lemma 3.6 extends this to the block encoding as well.

Assume for the rest of this proof that $C_\epsilon \leq \Delta \leq n/\log^2 n$. Let $k_{min} \leq k$ be the smallest integer such that G does not have an independent set of size k_{min}. By Proposition 3.2, with probability $1 - o(1)$ in n, $k_{min} \leq k_{+\epsilon} + 1$.

For the mapping-based encoding,

$$\begin{aligned}
\mathsf{DPLL}(\alpha_{map}(G,k)) &\leq \mathsf{DPLL}(\alpha_{map}(G,k_{min})) && \text{by Lemma 3.3} \\
&\leq 2^{C_0 k_{min} \log(n/k_{min})} && \text{by Lemma 3.11} \\
&\leq 2^{C_0 (k_{+\epsilon}+1) \log(n/(k_{+\epsilon}+1))} && \text{almost surely} \\
&\leq 2^{(c_0 n/\Delta) \log^2 \Delta} && \text{for large enough } c_0.
\end{aligned}$$

The bound for $\alpha_{block}(G,k)$ follows immediately from this bound for $\alpha_{map}(G,k)$ and Lemma 3.6. Further, Lemma 3.11 implies that these bounds hold even when the corresponding 1-1 clauses are removed from the mapping and block encodings. For the counting-based encoding,

$$\begin{aligned}
\mathsf{DPLL}(\alpha_{count}(G,k) &\le n\ \mathsf{DPLL}(\alpha_{count}(G,k_{min}) + 2n^2 && \text{by Lemma 3.2}\\
&\le n2^{C_0' k_{min}\log(n/k_{min})} + 2n^2 && \text{by Lemma 3.12}\\
&\le n2^{C_0' k_{min}\log(n/k_{min})} + 2n^2 && \\
&\le n2^{C_0'(k_{+\epsilon}+1)\log(n/(k_{+\epsilon}+1))} + 2n^2 && \text{almost surely}\\
&\le 2^{(c_0' n/\Delta)\log^2\Delta} && \text{for a large enough constant } c_0'.
\end{aligned}$$

This finishes the proof. □

Corollary 3.1 (Vertex Cover Upper Bounds). *There are constants c_0, c_0'' such that the following holds. Let $\Delta = np$, $\Delta \le n/\log^2 n$, and $G \sim \mathbb{G}(n,p)$. Let t be such that G has no vertex cover of size t. With probability $1 - o(1)$ in n,*

$$\begin{aligned}
\mathsf{RES}(VC_{count}(G,t)) &\le 2^{c_0''(n/\Delta)\log^2\Delta}, \quad \textit{and}\\
\mathsf{DPLL}(VC_{block}(G,t)) &\le 2^{c_0(n/\Delta)\log^2\Delta}.
\end{aligned}$$

The bounds also holds when all-but-one clauses are removed from $VC_{block}(G,t)$. The block encoding bound holds when $(n-t) \,|\, n$.

Proof. Apply Theorem 3.1 with k set to $n-t$ and use Lemmas 3.8 and 3.10 to translate the result of the Theorem to encodings of vertex cover. Note that $\mathsf{RES}(\alpha_{count}(G,n-t)) \le \mathsf{DPLL}(\alpha_{count}(G,n-t))$. □

3.6 Key Concepts for Lower Bounds

This section defines key concepts that will be used in the lower bound argument given in the next section. Fix a graph G and a partition of its n vertices into k subsets of size b each. For any edge (u,v) in G, call it an *inter-block edge* if u and v belong to different blocks of G, and an *intra-block edge* otherwise.

Definition 3.1. A truth assignment to variables of $\alpha_{block}(G,k)$ is *critical* if it sets exactly one variable in each block to TRUE.

Critical truth assignments satisfy all block, 1-1 and intra-block edge clauses but may leave some inter-block edge clauses unsatisfied.

Definition 3.2. The block multi-graph of G, denoted $B(G)$, is the multi-graph obtained from G by identifying all vertices that belong to the same block and removing any self-loops that are thus generated.

$B(G)$ contains exactly k nodes and possibly multiple edges between pairs of nodes. The degree of a node in $B(G)$ is the number of inter-block edges touching the corresponding block of G. Given the natural correspondence between G and $B(G)$, we will write *nodes of* $B(G)$ and *blocks of* G interchangeably. For a subgraph H of G, $B(H)$ is obtained analogously by identifying all vertices of H that are in the same block of G and removing self-loops.

Definition 3.3. Let S be a set of blocks of G. H is *block induced by* S if it is the subgraph of G induced by all vertices present in the blocks S. H is a *block induced subgraph* of G if there exists a subset S of blocks such that H is block induced by S.

If H is block induced by S, then $B(H)$ is induced by S in $B(G)$. The reverse, however, may not be true. If H is a block induced subgraph, then there is a *unique* minimal block set S such that H is block induced by S. This S contains exactly those blocks that have non-zero degree in $B(H)$. With each block induced subgraph, associate such a *minimal* S and say that the subgraph is induced by $|S|$ blocks. Note that every block in any such minimal S must have non-zero degree.

Definition 3.4. The *block width* of a clause C with respect to G, denoted $w^G_{block}(C)$, is the number of different blocks of G the variables appearing in C come from.

Clearly, $w(C) \geq w^G_{block}(C)$. For a block induced subgraph H of G, let $E(H)$ denote the conjunction of the edge clauses of $\alpha_{block}(G,k)$ that correspond to the edges of H. Let H be induced by the block set S.

Definition 3.5. *H critically implies* a clause C, denoted $H \xrightarrow{c} C$, if $E(H) \rightarrow C$ evaluates to true for all critical truth assignments to the variables of $\alpha_{block}(G,k)$.

Definition 3.6. *H minimally implies* C, denoted $H \xrightarrow{m} C$, if $H \xrightarrow{c} C$ and for every subgraph H' of G induced by a proper subset of S, $H' \overset{c}{\not\rightarrow} C$.

Note that "minimally implies" should really be called "minimally critically implies," but we use the former phrase for brevity. Note further that if $H \xrightarrow{m} C$, then every block of H has non-zero degree.

Definition 3.7. The *complexity* of a clause C, denoted $\mu_G(C)$, is the minimum over the sizes of subsets S of blocks of G such that the subgraph of G induced by S critically implies C.

Proposition 3.5. *Let G be a graph and Λ denote the empty clause.*

(a) For $C \in \alpha_{block}(G,k)$, $\mu_G(C) \leq 2$.

(b) $\mu_G(\Lambda)$ is the number of blocks in the smallest block induced subgraph of G that has no block-respecting independent set.

(c) Subadditive property*: If clause C is a resolvent of clauses C_1 and C_2, then $\mu_G(C) \leq \mu_G(C_1) + \mu_G(C_2)$.*

Proof. Each initial clause is either an edge clause, a block clause or a 1-1 clause. Any critical truth assignment, by definition, satisfies all block, 1-1 and intra-block edge clauses. Further, an edge clause corresponding to an inter-block edge (u, v) is implied by the subgraph induced by the two blocks to which u and v belong. Hence, complexity of an initial clause is at most 2, proving part (a).

Part (b) follows from the definition of μ_G. Part (c) follows from the simple observation that if G_1 critically implies C_1, G_2 critically implies C_2, and both G_1 and G_2 are block induced subgraphs, then $G_{1\cup 2}$, defined as the block graph induced by the union of the blocks G_1 and G_2 are induced by, critically implies both C_1 and C_2, and hence critically implies C. □

3.7 Proof Sizes and Graph Expansion

This section contains the main ingredients of our lower bound results and is technically the most interesting and challenging part at the core of this chapter. We use combinatorial properties of block graphs and independent sets to obtain a lower bound on the size of resolution refutations for a given graph in terms of its expansion properties. Next, we argue that random graphs almost surely have good expansion properties. Section 3.8 combines these two to obtain an almost certain lower bound for random graphs.

The overall argument in a little more details is as follows. We define the notion of "boundary" for block induced subgraphs as a measure of the number of blocks in it that have an isolated vertex and thus contribute trivially to any block-respecting independent set. Lemmas 3.13 and 3.14 relate this graph-theoretic concept to resolution refutations. The main lower bound follows in three steps from here. First, Lemma 3.16 argues that one must almost surely consider a large fraction of the blocks of a graph to prove the non-existence of a block-respecting independent set in it. Second, Lemma 3.17 shows that almost all subgraphs induced by large fractions of blocks must have large boundary. Finally, Lemma 3.18 combines these two to obtain an almost certain lower bound on the width of any refutation.

We begin by defining the notion of boundary.

Definition 3.8. The *boundary* of a block induced subgraph H, denoted $\beta(H)$, is the set of blocks of H that have at least one isolated vertex.

3.7.1 Relating Proof Size to Graph Expansion

We first derive a relationship between the width of clauses and the boundary size of block-induced subgraphs that minimally imply them.

Lemma 3.13. *Let C be a clause in the variables of $\alpha_{block}(G,k)$ and H be a block induced subgraph of G. If $H \xrightarrow{m} C$, then $w^G_{block}(C) \geq |\beta(H)|$.*

Proof. We use a *toggling property* of block-respecting independent sets (Figure 3.2) to show that each boundary block of H contributes at least one literal to C.

Let H be induced by the set of blocks S. Fix a boundary block $B \in S$. Let H_B be the subgraph induced by $S \setminus \{B\}$. By minimality of H, $H_B \overset{c}{\not\to} C$. Therefore, there exists a critical truth assignment γ such that $\gamma(E(H_B)) = \text{TRUE}$ but $\gamma(C) = \text{FALSE}$. Since $\gamma(C) = \text{FALSE}$ and $H \xrightarrow{c} C$, it follows that $\gamma(E(H)) = \text{FALSE}$. Further, since $\gamma(E(H_B)) = \text{TRUE}$, $\gamma(E(H) \setminus E(H_B))$ must be FALSE, implying that γ violates the edge clause corresponding to an inter-block edge $(v,w), v \in B, w \notin B$. In particular, $\gamma(v) = \text{TRUE}$.

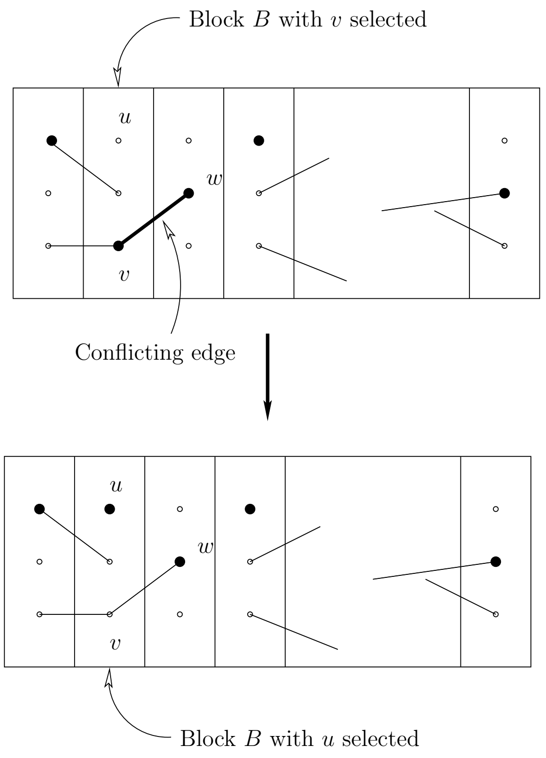


Figure 3.2: Toggling property of block-respecting independent sets; selected vertices are shown in bold

Fix an isolated vertex $u \in B$. Create a new critical truth assignment $\bar{\gamma}$ as follows: $\bar{\gamma}(v) = \text{FALSE}$, $\bar{\gamma}(u) = \text{TRUE}$, and $\bar{\gamma}(x) = \gamma(x)$ for every other vertex x in H. By construction, $\bar{\gamma}(E(H_B)) = \gamma(E(H_B)) = \text{TRUE}$. Further, since u does not have any inter-block edges and γ is critical, even $\bar{\gamma}(E(H))$ is TRUE. It follows from $H \xrightarrow{c} C$

that $\bar{\gamma}(C) = \text{TRUE}$. Recall that $\gamma(C) = \text{FALSE}$. This is what we earlier referred to as the toggling property. Since γ and $\bar{\gamma}$ differ only in their assignment to variables in block B, clause C must contain at least one literal from B. □

The subgraph of G induced by the empty set of blocks clearly has a block-respecting independent set while the subgraph induced by all blocks does not. This motivates the following definition. Let $s+1$ denote the minimum number of blocks such that some subgraph of G induced by $s+1$ blocks does not have a block respecting independent set.

Definition 3.9. The *sub-critical expansion*, $e(G)$, of G is the maximum over all $t, 2 \leq t \leq s$, of the minimum boundary size of any subgraph H of G induced by t' blocks, where $t/2 < t' \leq t$.

Lemma 3.14. *Any resolution refutation of $\alpha_{block}(G,k)$ must contain a clause of width at least $e(G)$.*

Proof. Let t be chosen as in the definition of $e(G)$ and π be a resolution refutation of $\alpha_{block}(G,k)$. By Proposition 3.5 (b), $\mu_G(\Lambda) = s+1$. Further, Proposition 3.5 (a) says that any initial clause has complexity at most 2. Therefore for $2 < t \leq s$ there exists a clause C in π such that $\mu_G(C) > t \geq 2$ and no ancestor of C has complexity greater than t.

Since $\mu_G(C) > 2$, C cannot be an initial clause. It must then be a resolvent of two parent clauses C_1 and C_2. By Proposition 3.5 (c) and the fact that no ancestor of C has complexity greater than t, one of these clauses, say C_1, must have $\mu_G(C_1)$ between $(t+1)/2$ and t. If H is a block induced subgraph that witnesses the value of $\mu_G(C_1)$, then by Lemma 3.13, $w^G_{width}(C_1) \geq |\beta(H)|$. Hence, $w(C_1) \geq |\beta(H)|$. By definition of $e(G)$, $|\beta(H)| \geq e(G)$. Thus $w(C_1) \geq e(G)$ as required. □

Corollary 3.2. *Let $c = 1/(9 \log 2)$ and $k \,|\, n$. For any graph G with its n vertices partitioned into k blocks of size $b = n/k$ each,*

$$\begin{aligned} \mathsf{RES}(\alpha_{block}(G,k)) &\geq 2^{c(e(G)-b)^2/n} \quad \textit{and} \\ \mathsf{DPLL}(\alpha_{block}(G,k)) &\geq 2^{e(G)-b}. \end{aligned}$$

Proof. This follows immediately from Lemma 3.14 and Propositions 2.2 and 2.1 by observing that the initial width of $\alpha_{block}(G,k)$ is b. □

3.7.2 Lower Bounding Sub-critical Expansion

Throughout this section, the probabilities are with respect to the random choice of a graph G from the distribution $\mathbb{G}(n,p)$ for some fixed parameters n and p. Let $B(G)$ be a block graph corresponding to G with block size b. For the rest of this chapter, we will fix b to be 3, which corresponds to the largest independent set size ($k = n/3$) for which the results in this section hold. Although the results can be generalized to any $b \geq 3$, our best bounds are obtained for the simpler case of $b = 3$ that we present.

Definition 3.10. $B(G)$ is (r,q)-*dense* if some subgraph of G induced by r blocks (i.e., some subgraph of $B(G)$ with r nodes) contains at least q edges.

The following lemma shows that for almost all random graphs G, the corresponding block graph $B(G)$ is locally sparse.

Lemma 3.15. *Let $G \sim \mathbb{G}(n,p)$ and $B(G)$ be a corresponding block graph with block size* 3*. For $r, q \geq 1$,*

$$\Pr[B(G) \text{ is } (r,q)\text{-dense}] < \left(\frac{ne}{3r}\right)^r \left(\frac{9er^2p}{2q}\right)^q.$$

Proof. Let H be a subgraph of G induced by r blocks. H contains $3r$ vertices. For $G \sim \mathbb{G}(n,p)$, the number of edges contained in H has the binomial distribution with parameters $\binom{3r}{2}$ and p. Therefore,

$$\Pr\left[H \text{ has at least } q \text{ edges}\right] \leq \binom{\binom{3r}{2}}{q} p^q < \binom{\frac{9r^2}{2}}{q} p^q \leq \left(\frac{9er^2p}{2q}\right)^q.$$

Summing this over all $\binom{n/3}{r} \leq (ne/3r)^r$ subgraphs H induced by r blocks gives the desired bound. □

We use this local sparseness property of the block graphs of almost all random graphs to prove that the smallest pair-induced subgraph one needs to consider for proving that G does not have a paired vertex cover is almost surely large.

Lemma 3.16. *There is a constant C such that the following holds. Let $\Delta = np$ and $s < Cn/\Delta^3$. The probability that $G \sim \mathbb{G}(n,p)$ contains a subgraph induced by at most s blocks that has no block-respecting independent set is $o(1)$ in s.*

Proof. The probability that G contains a subgraph induced by at most s blocks that has no block-respecting independent set is the same as the probability that there is some *minimal* subgraph H of G induced by $r \leq s$ blocks that has no block-respecting independent set. By minimality, H has no isolated vertices and hence no boundary blocks. Consequently, each of the r blocks that induce H must have at least 3 inter-block edges. Hence, the subgraph of $B(G)$ with the r nodes corresponding to the r blocks that induce H must have at least $3r/2$ edges.

Thus, the probability that G contains such a block induced subgraph H is at most

$$\sum_{r=1}^{s} \Pr[B(G) \text{ is } (r, 3r/2)\text{-dense}].$$

By Lemma 3.15, we have $\Pr[B(G)$ is $(r, 3r/2)$-dense$] < D(r)$ where

$$\begin{aligned} D(r) &= \left(\frac{ne}{3r}\right)^r (3erp)^{3r/2} \\ &= \left(\frac{ne}{3}(3ep)^{3/2} r^{1/2}\right)^r \\ &= \left(Q(n,p)\ r^{1/2}\right)^r \end{aligned}$$

for $Q(n,p) = (ne/3)(3ep)^{3/2}$. Now

$$\begin{aligned} \frac{D(r+1)}{D(r)} &= \frac{\left(Q(n,p)\ (r+1)^{1/2}\right)^{r+1}}{\left(Q(n,p)\ r^{1/2}\right)^r} \\ &= Q(n,p)\ (r+1)^{1/2}\ \left(\frac{r+1}{r}\right)^{r/2} \\ &\le Q(n,p)\ (r+1)^{1/2}\ e^{1/2} \\ &\le \frac{ne}{3}\left(\frac{3e\Delta}{n}\right)^{3/2} e^{1/2}(r+1)^{1/2} \\ &= \left(\frac{3e^6\Delta^3(r+1)}{n}\right)^{1/2} \end{aligned}$$

This quantity, and hence $D(r+1)/D(r)$, is at most $1/2$ for $1 \le r < Cn/\Delta^3$, where $C \overset{\text{def}}{=} 1/(12e^6)$ is a constant. Let $s+1 = Cn/\Delta^3$. It follows that the probability that G contains such a block induced subgraph H is bounded above by a geometric series in r with common ratio $1/2$. It is therefore at most twice the largest term of the series which is less than $D(1)$. Now

$$D(1) = Q(n,p) = \frac{ne}{3}\left(\frac{3e\Delta}{n}\right)^{3/2} = \left(\frac{3e^5\Delta^3}{n}\right)^{1/2} = \left(\frac{3e^5C}{s+1}\right)^{1/2}.$$

Therefore, $D(1)$ is $o(1)$ in s as claimed. $\square$

We again use the local sparseness property to prove that any subgraph induced by not too many blocks has large boundary for almost all random graphs G. The intuition is that for sparse subgraphs, most blocks have degree less than 3 and thus belong to the boundary.

Lemma 3.17. *There is a constant c such that the following holds. Let $\Delta = np$, $0 < \epsilon \le 1/6$, $b' = 3(1-\epsilon)$, $t \le cn/\Delta^{\frac{b'}{b'-2}}$, and $G \sim \mathbb{G}(n,p)$. The probability that there exists $r \in (t/2, t]$ such that G has a subgraph H induced by r blocks with $\beta(H) \le \epsilon r$ is $o(1)$ in t.*

Proof. Fix b', ϵ, and t satisfying the conditions of the Lemma. Let H be a subgraph of G induced by r blocks. By definition, all r blocks inducing H must have non-zero degree in $B(H)$. Moreover, if H has at most ϵr boundary blocks, the other $(1-\epsilon)r$ blocks of non-zero degree inducing it must have degree at least 3. Hence, the r nodes of $B(G)$ that induce H form a subgraph with at least $(1-\epsilon)r3/2 = b'r/2$ edges. Therefore, H has at most ϵr boundary blocks only if $B(G)$ is $(r, b'r/2)$-dense. Thus, by Lemma 3.15, the probability that such an H exists is at most

$$\begin{aligned}
\Pr[B(G) \text{ is } (r, b'r)\text{-dense}] &< \left(\frac{ne}{3r}\right)^r \left(\frac{9erp}{b'}\right)^{b'r/2} \\
&= \left(\frac{e}{3}\left(\frac{3e\Delta}{1-\epsilon}\right)^{b'/2}\left(\frac{r}{n}\right)^{(b'-2)/2}\right)^r
\end{aligned}$$

For $r > t/2$, it suffices to obtain an upper bound on this probability that is exponentially small in r. Rearranging the terms in the expression above, $\Pr[B(G)$ is $(r, b'r)$-dense$] \leq 2^{-r}$ when

$$\begin{aligned}
\frac{r}{n} &\leq \left(\frac{3}{2e}\right)^{2/(b'-2)} \left(\frac{1-\epsilon}{3e\Delta}\right)^{b'/(b'-2)} \\
&= \left(\frac{1-\epsilon}{2e^2}\right)^{2/(b'-2)} \frac{1-\epsilon}{3e\Delta^{b'/(b'-2)}}.
\end{aligned}$$

Note that $\epsilon \leq 1/6$ and $b' = 3(1-\epsilon) \geq 5/2$. Hence $(1-\epsilon)/(2e^2)^{2/(b'-2)}$ is at least $(5/(12e^2))^4$ and it suffices to have

$$\frac{r}{n} \leq \left(\frac{5}{12e^2}\right)^4 \frac{5}{18e\Delta^{b'/(b'-2)}} = \frac{c}{\Delta^{b'/(b'-2)}}$$

for a constant $c \overset{\text{def}}{=} 5^5/(12^4 18e^9)$. Therefore, the probability that $B(G)$ is $(r, b'r)$-dense is at most 2^{-r} for $r \leq cn/\Delta^{b'/(b'-2)}$. It follows that the probability that there exists such an H with $r \in (t/2, t]$ is at most $\sum_{r=\lceil (t+1)/2\rceil}^{t} 2^{-r}$. This sum is $o(1)$ in t as required. □

Lemmas 3.16 and 3.17 combine to give the following lower bound on sub-critical expansion:

Lemma 3.18. *For each $\epsilon \in (0, 1/6]$ there is a constant c_ϵ such that the following holds. Let $\Delta = np$, $b' = 3(1-\epsilon)$, $W = n/\Delta^{b'/(b'-2)}$, and $G \sim \mathbb{G}(n,p)$. The probability that $e(G) < c_\epsilon W$ is $o(1)$ in W.*

Proof. Let C be the constant from Lemma 3.16 and c be the one from Lemma 3.17. Let $s+1$ be the minimum number of blocks such that some subgraph of G induced by $s+1$ blocks does not have a block induced independent set. By Lemma 3.16, $s \le Cn/\Delta^3$ with probability $o(1)$ in n. Now let $t = \min(C,c)W$. Conditioned on $s > Cn/\Delta^3$ and because $b' < 3$, we have that $t \le s$ as in the definition of $e(G)$. By Lemma 3.17, the probability that some subgraph of G induced by r blocks with $t/2 < r \le t \le s$ has less than $\epsilon r > \epsilon t/2 = c_\epsilon W$ boundary blocks is $o(1)$ in n, where $c_\epsilon = (\epsilon/2)\min(C,c)$. It follows from a union bound on the two bad events (s is small or some subgraph has small boundary) that $e(G) < c_\epsilon W$ with probability $o(1)$ in n. □

3.8 Lower Bounds for Resolution and Associated Algorithms

We now use the ideas developed in Sections 3.6 and 3.7, and bring the pieces of the argument together in a general technical result from which our resolution complexity lower bounds follow.

Lemma 3.19. *For each $\delta > 0$ there are constants $C_\delta, C'_\delta > 0$ such that the following holds. Let $\Delta = np$ and $G \sim \mathbb{G}(n,p)$. With probability $1-o(1)$ in n,*

$$\begin{aligned} \mathsf{RES}(\alpha_{block}(G,n/3)) &\ge 2^{C_\delta n/\Delta^{6+2\delta}} \quad \textit{and} \\ \mathsf{DPLL}(\alpha_{block}(G,n/3)) &\ge 2^{C'_\delta n/\Delta^{3+\delta}}. \end{aligned}$$

Proof. Observe that the expressions $n/\Delta^{6+2\delta}$ and $n/\Delta^{3+\delta}$ in the desired bounds decrease as δ increases. Hence, it suffices to prove the bounds for $\delta \in (0,2]$, and for $\delta > 2$, simply let $C_\delta = C_2$ and $C'_\delta = C'_2$.

Let $\epsilon = \delta/(6+3\delta)$, $b' = 3(1-\epsilon)$, and $W = n/\Delta^{b'/(b'-2)}$. For $\delta \in (0,2]$, we have that $\epsilon \in (0,1/6]$. From Lemma 3.18, there is a constant c_ϵ such that with probability $1-o(1)$ in n, $e(G) \ge c_\epsilon W$. It follows from Corollary 3.2 that for $c = 1/(9\log 2)$ and with probability $1-o(1)$ in n,

$$\begin{aligned} \mathsf{RES}(\alpha_{block}(G,n/3)) &\ge 2^{c(c_\epsilon W-3)^2/n} \quad \text{and} \\ \mathsf{DPLL}(\alpha_{block}(G,n/3)) &\ge 2^{c_\epsilon W-3}. \end{aligned}$$

Given the relationship between ϵ and δ, there are constants $C_\delta, C'_\delta > 0$ depending only on δ such that $c(c_\epsilon W-3)^2 \ge C_\delta W^2$ and $c_\epsilon W - 3 \ge C'_\delta W$. Note also that $b'/(b'-2) = (3-3\epsilon)/(1-3\epsilon) = 3+\delta$. Hence,

$$\log_2(\mathsf{RES}(\alpha_{block}(G,n/3))) \ge C_\delta W^2/n = C_\delta n/\Delta^{\frac{2b'}{b'-2}} = C_\delta n/\Delta^{6+2\delta} \quad \text{and}$$

$$\log_2(\mathsf{DPLL}(\alpha_{block}(G,n/3))) \ge C'_\delta W = C'_\delta n/\Delta^{\frac{b'}{b'-2}} = C'_\delta n/\Delta^{3+\delta}.$$

This finishes the proof. □

Theorem 3.2 (Independent Set Lower Bounds). *For each $\delta > 0$ there are constants C_δ, C'_δ, C''_δ, C'''_δ, $C''''_\delta > 0$ such that the following holds. Let $\Delta = np$, $k \leq n/3$, $k \mid n$, and $G \sim \mathbb{G}(n,p)$. With probability $1 - o(1)$ in n,*

$$\begin{array}{rcl} \mathsf{RES}(\alpha_{map}(G,k)) & \geq & 2^{C_\delta n/\Delta^{6+2\delta}}, \\ \mathsf{DPLL}(\alpha_{map}(G,k)) & \geq & 2^{C'_\delta n/\Delta^{3+\delta}}, \\ \mathsf{RES}(\alpha_{count}(G,k)) & \geq & 2^{C''_\delta n/\Delta^{6+2\delta}}, \\ \mathsf{DPLL}(\alpha_{count}(G,k)) & \geq & 2^{C'''_\delta n/\Delta^{3+\delta}}, \\ \mathsf{RES}(\alpha_{block}(G,k)) & \geq & 2^{C_\delta n/\Delta^{6+2\delta}}, \\ \mathsf{DPLL}(\alpha_{block}(G,k)) & \geq & 2^{C'_\delta n/\Delta^{3+\delta}}, \quad \text{and} \\ Chv(G,k) & \geq & 2^{C''''_\delta n/\Delta^{6+2\delta}}. \end{array}$$

The bounds for the block encoding require $k \mid (n/3)$.

Proof. All of the claimed bounds follow by applying monotonicity of the encoding at hand, using its relationship with the block encoding, and applying Lemma 3.19. Let C_δ and C'_δ be the constants from Lemma 3.19. For the mapping-based encoding,

$$\begin{aligned} \mathsf{RES}(\alpha_{map}(G,k)) &\geq \mathsf{RES}(\alpha_{map}(G,n/3)) && \text{by Lemma 3.3} \\ &\geq \mathsf{RES}(\alpha_{block}(G,n/3)) && \text{by Lemma 3.6} \\ &\geq 2^{C_\delta n/\Delta^{6+2\delta}} && \text{by Lemma 3.19,} \end{aligned}$$

$$\begin{aligned} \mathsf{DPLL}(\alpha_{map}(G,k)) &\geq \mathsf{DPLL}(\alpha_{map}(G,n/3)) && \text{by Lemma 3.3} \\ &\geq \mathsf{DPLL}(\alpha_{block}(G,n/3)) && \text{by Lemma 3.6} \\ &\geq 2^{C'_\delta n/\Delta^{3+\delta}} && \text{by Lemma 3.19.} \end{aligned}$$

For the counting-based encoding,

$$\begin{aligned} \mathsf{RES}(\alpha_{count}(G,k) &\geq \frac{1}{n}\left(\mathsf{RES}(\alpha_{count}(G,n/3)) - 2n^2\right) && \text{by Lemma 3.2} \\ &\geq \frac{1}{n}\left(\frac{1}{2}\mathsf{RES}(\alpha_{block}(G,n/3)) - 2n^2\right) && \text{by Lemma 3.5} \\ &\geq 2^{C''_\delta n/\Delta^{6+2\delta}} && \text{by Lemma 3.19} \end{aligned}$$

for a large enough constant C''_δ. Similarly,

$$\begin{aligned} \mathsf{DPLL}(\alpha_{count}(G,k) &\geq \frac{1}{n}\left(\mathsf{DPLL}(\alpha_{count}(G,n/3)) - 2n^2\right) && \text{by Lemma 3.2} \\ &\geq \frac{1}{n}\left(\frac{1}{2}\mathsf{DPLL}(\alpha_{block}(G,n/3))^{1/\log_2 6} - 2n^2\right) && \text{by Lemma 3.5} \\ &\geq 2^{C'''_\delta n/\Delta^{3+\delta}} && \text{by Lemma 3.19} \end{aligned}$$

for a large enough constant C_δ''''.

The bounds for the block encoding follow immediately from Lemmas 3.4 and 3.19. Finally, for the bound on the proof size in Chvátal's system,

$$\begin{aligned} Chv(G,k) &\geq Chv(G,n/3) - 1 && \text{by Proposition 3.3} \\ &\geq \frac{1}{4n}\mathsf{RES}(\alpha_{block}(G,n/3)) - 1 && \text{by Lemma 3.7} \\ &\geq 2^{C_\delta''''n/\Delta^{6+2\delta}} && \text{by Lemma 3.19} \end{aligned}$$

for a large enough constant c_δ''''. □

Corollary 3.3 (Vertex Cover Lower Bounds). *For each $\delta > 0$ there are constants $\widetilde{C}_\delta, C_\delta, C'_\delta > 0$ such that the following holds. Let $\Delta = np$, $t \geq 2n/3$, $(n-t)\,|\,n$, and $G \sim \mathbb{G}(n,p)$. With probability $1 - o(1)$ in n,*

$$\begin{aligned} \mathsf{RES}(VC_{count}(G,t)) &\geq 2^{\widetilde{C}_\delta n/\Delta^{6+2\delta}}, \\ \mathsf{RES}(VC_{block}(G,t)) &\geq 2^{C_\delta n/\Delta^{6+2\delta})}, \quad \textit{and} \\ \mathsf{DPLL}(VC_{block}(G,t)) &\geq 2^{C'_\delta n/\Delta^{3+\delta})}. \end{aligned}$$

The bounds for the block encoding require $(n-t)\,|\,(n/3)$.

Proof. Let C_δ, C'_δ, and C''_δ be the constants from Theorem 3.2 and let $\widetilde{C}_\delta$ be any constant less than C''_δ. For the counting encoding bound, apply Theorem 3.2 with k set to $n-t$ and use Lemma 3.9 to translate the results to the encoding of vertex cover. For the block encoding bounds, apply Theorem 3.2 in conjunction with Lemma 3.10. □

3.9 Hardness of Approximation

Instead of considering the decision problem of whether a given graph G has an independent set of a given size k, one may consider the related *optimization problem*: given G, find an independent set in it of the largest possible size. We call this optimization problem the *maximum independent set* problem. One may similarly define the optimization problem *minimum vertex cover* problem.

Since the decision versions of these problems are NP-complete, the optimization versions are NP-hard and do not have any known polynomial time solutions. From the perspective of algorithm design, it is then natural to ask whether there is an efficient algorithm that finds an independent set of size "close" to the largest possible or a vertex cover of size close to the smallest possible. That is, is there an efficient algorithm that finds an "approximate" solution to the optimization problem? In this section, we rule out the existence of any such efficient "resolution-based" algorithm for the independent set and vertex cover problems.

Remark 3.2. The results we prove in this section contrast well with the known approximation hardness results for the two problems which are both based on the PCP (probabilistically checkable proofs) characterization of NP [9, 8]. Håstad [62] showed that unless P = NP, there is no polynomial time $n^{1-\epsilon}$-approximation algorithm for the clique (and hence the independent set) problem for any $\epsilon > 0$. For graphs with maximum degree Δ_{max}, Trevisan [108] improved this to a factor of $\Delta_{max}/2^{O(\sqrt{\log \Delta_{max}})}$. More recently, Dinur and Safra [47] proved that unless P = NP, there is no polynomial time $10\sqrt{5} - 21 \approx 1.36$ factor approximation algorithm for the vertex cover problem. Our results, on the other hand, hold irrespective of the relationship between P and NP but apply only to the class of resolution-based algorithms defined shortly.

3.9.1 Maximum Independent Set Approximation

We begin by making several of the above notions precise. Let A be an algorithm for finding a maximum independent set in a given graph.

Definition 3.11. Let $\gamma \geq 1$. A is a γ-*approximation algorithm* for the maximum independent set problem if on input G with maximum independent set size $\hat{k}$, A produces an independent set of size at least $\hat{k}/\gamma$.

In other words, if A produces an independent set of size $\bar{k}$ on input G, it proves that G does not have one of size $\bar{k}\gamma + 1$. This reasoning allows us to use our lower bounds from the previous section to prove that even approximating a maximum independent set is exponentially hard for certain resolution-based algorithms.

Definition 3.12. A γ-approximation algorithm A for the maximum independent set problem is *resolution-based* if it has the following property: if A outputs an independent set of size $\bar{k}$ on input G, then its computation history along with a proof of correctness within a factor of γ yields a resolution proof of $\alpha_{map}(G, k)$, $\alpha_{count}(G, k)$, or $\alpha_{block}(G, k)$ for $k \leq \bar{k}\gamma + 1, k \mid n$. (For the block encoding, we further require $k \mid (n/3)$.)

The manner in which the computation history and the proof of correctness are translated into a resolution refutation of an appropriate encoding depends on specific details and varies with the context. We will see a concrete example of this for the vertex cover problem when discussing Proposition 3.7 later in this section.

Let $\mathcal{A}_{\gamma}^{RES-ind}$ denote the class of all resolution-based γ-approximation algorithms for the maximum independent set problem. We show that while there is a trivial algorithm in this class for $\gamma \geq \Delta + 1$, there isn't an efficient one for $\gamma \leq \Delta/(6 \log \Delta)$.

Proposition 3.6. *For $\gamma \geq \Delta + 1$, there is a polynomial-time algorithm in $\mathcal{A}_{\gamma}^{RES-ind}$.*

Proof. Let A be the polynomial-time algorithm that underlies the bound in Turan's theorem (Proposition 3.1), that is, on a graph G with n nodes and average degree Δ as input, A produces an independent set of size $\bar{k} \geq n/(\Delta + 1)$. Since the size of a maximum independent set in G is at most n, A is a $(\Delta+1)$-approximation. We will argue that A is also resolution-based.

To be a resolution-based, the computation history of A on G along with a proof of correctness within a factor of $(\Delta+1)$ must yield a resolution proof of a suitable encoding $\alpha(G,k)$ for some $k \leq k^* = \bar{k}(\Delta+1)+1, k \,|\, n$. When G has no edges, $\Delta = 0$ and A produces an independent set of size $\bar{k} = n$. In this case, there is nothing to prove. When G has at least one edge (u,v), $k^* \geq n+1$ and we can choose $k = n$. In this case, A indeed yields a straightforward resolution proof of $\alpha(G,k)$ for both the mapping and the counting encodings by utilizing the edge clause(s) corresponding to (u,v). Therefore, A is resolution-based as a $(\Delta+1)$-approximation algorithm. □

While Proposition 3.2 guarantees that there is almost never an independent set of size larger than $(2n/\Delta)\log\Delta$, Theorem 3.2 shows that there is no efficient way to prove this fact using resolution. Indeed, there exist efficient resolution proofs only for the non-existence of independent sets of size larger than $n/3$. We use this reasoning to prove the following hardness of approximation result.

Theorem 3.3 (Independent Set Approximation). *There is a constant c such that the following holds. Let $\delta > 0$, $\Delta = np$, $\Delta \geq c$, $\gamma \leq \Delta/(6\log\Delta)$, and $G \sim \mathbb{G}(n,p)$. With probability $1-o(1)$ in n, every algorithm $A \in \mathcal{A}_\gamma^{RES-ind}$ takes time exponential in $n/\Delta^{6+2\delta}$.*

Proof. Recall the definitions of $k_{+\epsilon}$ and C_ϵ from Proposition 3.2. Fix $\epsilon > 0$ such that $k_{+\epsilon} < (2n/\Delta)\log\Delta$ and let $c \geq C_\epsilon$. The claimed bound holds trivially for $\Delta \geq n^{1/6}$. We will assume for the rest of the proof that $C_\epsilon \leq \Delta \leq n/\log^2 n$.

From Proposition 3.2, with probability $1-o(1)$ in n, a maximum independent set in G is of size $k_{max} \leq k_{+\epsilon} < (2n/\Delta)\log\Delta$. If A approximates this within a factor of γ, then, in particular, it proves that G does not have an independent set of size $k = k_{max}\gamma + 1 \leq n/3$. Convert the transcript of the computation of A on G along with an argument of its correctness within a factor of γ into a resolution proof π of an appropriate encoding $\alpha(G,k)$. From Theorem 3.2, $size(\pi)$ must be exponential in $n/\Delta^{6+2\delta}$. □

3.9.2 Minimum Vertex Cover Approximation

A similar reasoning can be applied to approximation algorithms for finding a minimum vertex cover.

Definition 3.13. Let $\gamma \geq 1$. A is a *γ-approximation algorithm* for the minimum vertex cover problem if on input G with minimum vertex cover size $\hat{t}$, A produces a vertex cover of size at most $\hat{t}\gamma$.

Definition 3.14. A γ-approximation algorithm A for the minimum vertex cover problem is *resolution-based* if it has the following property: if A outputs a vertex cover of size $\bar{t}$ on input G, then its computation history along with a proof of correctness within a factor of γ yields a resolution proof of $VC_{count}(G,t)$ or $VC_{block}(G,t)$ for $t \geq \bar{t}/\gamma - 1, (n-t) \,|\, n$. (For the block encoding, we further require $(n-t) \,|\, (n/3)$.)

Let $\mathcal{A}_{\gamma}^{RES-VC}$ denote the class of all resolution-based γ-approximation algorithms for the minimum vertex cover problem.

As the following proposition shows, the usual greedy 2-approximation algorithm for vertex cover, for instance, is in $\mathcal{A}_{2}^{RES-VC}$. It works by choosing an arbitrary edge, say (u,v), including both u and v in the vertex cover, throwing away all edges incident on u and v, and repeating this process until all edges have been removed from the graph. This gives a 2-approximation because any optimal vertex cover will also have to choose at least one of u and v. For concreteness, we describe this algorithm below as Algorithm 3.1 and denote it by `VC-greedy`. We use $E(G)$ for the set of edges in G and $E(w)$ for the set of edges incident on a vertex w.

```
Input  : An undirected graph $G$ with minimum vertex cover size $t+1$
Output : A vertex cover for $G$ of size at most $2(t+1)$
begin
    $cover \leftarrow \phi$
    while $E(G) \neq \phi$ do
        Choose an edge $(u,v) \in E(G)$ arbitrarily
        $cover \leftarrow cover \cup \{u,v\}$
        $E(G) \leftarrow E(G) \setminus (E(u) \cup E(v))$
    Output $cover$
end
```

Algorithm 3.1: `VC-greedy`, a greedy 2-approximation algorithm for the minimum vertex cover problem

Proposition 3.7. *Let $t = \bar{t}/2 - 1$ and $(n-t) \,|\, n$. If* `VC-greedy` *outputs a vertex cover of size $\bar{t}$ on input G, then* $\mathsf{RES}(VC_{count}(G,t)) \leq 8t^2$.

Proof. Consider a run of `VC-greedy` on G that produces a vertex cover of size $\bar{t} = 2(t+1)$. This yields a sequence of $\bar{t}/2 = t+1$ *vertex disjoint* edges of G that are processed sequentially till G has no edges left. Without loss of generality, assume that these $t+1$ edges are $(v_1,v_2),(v_3,v_4),\ldots,(v_{2t+1},v_{2t+2})$. Extend this ordering of the vertices of G to the remaining $n-2t-2$ vertices. Under this ordering, we will construct a refutation of $\alpha_{count}(G,t)$ of size at most $8t^2$. Note that $\alpha_{count}(G,t)$ includes $(x_{2p-1} \vee x_{2p}), 1 \leq p \leq t+1$, among its edge clauses.

In order to construct this derivation, it will be helpful to keep in mind that one can resolve any clause $(y_{q,i} \vee B), 1 \leq q < n, 1 \leq i \leq t$, with the initial clause $(y_{q+1,i} \vee \neg y_{q,i} \vee x_{q+1})$ to derive $(y_{q+1,i} \vee x_{q+1} \vee B)$. For convenience, we will refer to this as a Z_1 derivation. Similarly, for $i < t$, $(y_{q,i} \vee B)$ can be resolved with the initial clause $(y_{q+1,i+1} \vee \neg y_{q,i} \vee \neg x_{q+1})$ to derive $(y_{q+1,i+1} \vee \neg x_{q+1} \vee B)$. We will refer to this as a Z_2 derivation.

Using the above derivations as building blocks, we show that for $0 \leq p < t, 0 \leq i < t-1$, and any clause $(y_{2p,i} \vee A)$, we can derive the clause $(y_{2p+2,i+1} \vee y_{2p+2,i+2} \vee A)$ in 8 resolution steps. First, apply a Z_1 derivation to $(y_{2p,i} \vee A)$ to obtain $(y_{2p+1,i} \vee x_{2p+1} \vee A)$. Apply a Z_2 derivation to this to get $(y_{2p+2,i+1} \vee x_{2p+1} \vee \neg x_{2p+2} \vee A)$. Resolve this with the edge clause $(x_{2p+1} \vee x_{2p+2})$ to finally obtain the clause $C_1 = (y_{2p+2,i+1} \vee x_{2p+1} \vee A)$. Starting again from $(y_{2p,i} \vee A)$, apply a Z_2 derivation to obtain $(y_{2p+1,i+1} \vee \neg x_{2p+1} \vee A)$. Apply Z_1 and Z_2 derivations separately to this clause and resolve the results together on the variable x_{2p+2} to obtain the clause $C_2 = (y_{2p+2,i+1} \vee y_{2p+2,i+2} \vee \neg x_{2p+1} \vee A)$. Resolving clauses C_1 and C_2 on the variable x_{2p+1} finishes the 8 step derivation of $(y_{2p+2,i+1} \vee y_{2p+2,i+2} \vee A)$. We will refer to this derivation as Z_3.

A similar argument shows that for the boundary case $i = t-1$, one can derive from $(y_{2p,t-1} \vee A)$ in at most 8 steps the clause $(y_{2p+2,t} \vee A)$.

We are ready to describe the overall construction of the refutation. Starting from the initial clause $y_{0,0}$, apply Z_3 to derive $(y_{2,1} \vee y_{2,2})$. Now apply Z_3 successively to the two literals of this clause to obtain $(y_{4,2} \vee y_{4,3} \vee y_{4,4})$. Applying Z_3 repeatedly to the literals of the clause obtained in this manner results in the derivation of $(y_{2p,p} \vee y_{2p,p+1} \vee \ldots \vee y_{2p,r})$ in at most $8p^2$ steps, where $1 \leq p \leq t$ and $r = \min(2p, t)$. For $p = t$, this gives a derivation of $y_{2t,t}$ in a total of $8t^2$ steps.

Resolving $y_{2t,t}$ with the initial clauses $(\neg y_{2t,t} \vee \neg x_{2t+1})$ and $(\neg y_{2t,t} \vee \neg x_{2t+2})$, and resolving the two resulting clauses with the edge clause $(x_{2t+1} \vee x_{2t+2})$ derives the empty clause Λ and finishes the refutation. $\square$

Theorem 3.4 (Vertex Cover Approximation). *There is a constant c such that the following holds. Let $\delta > 0$, $\Delta = np$, $\Delta \geq c$, $\gamma < 3/2$, and $G \sim \mathbb{G}(n,p)$. With probability $1 - o(1)$ in n, every algorithm $A \in \mathcal{A}_{\gamma}^{RES-VC}$ takes time exponential in $n/\Delta^{6+2\delta}$.*

Proof. This proof is very similar to that of Theorem 3.3. Recall the definitions of $k_{+\epsilon}$ and C_ϵ from Proposition 3.2. Fix $\epsilon > 0$ such that $k_{+\epsilon} < (2n/\Delta)\log\Delta$ and let $c \geq C_\epsilon$. The claimed bound holds trivially for $\Delta \geq n^{1/6}$. We will assume for the rest of the proof that $C_\epsilon \leq \Delta \leq n/\log^2 n$.

From Proposition 3.2 and the relation between independent sets and vertex covers, with probability $1-o(1)$ in n, a minimum vertex cover in G is of size $t_{min} \geq n - k_{+\epsilon} > n - (2n/\Delta)\log\Delta$. If A approximates this within a factor of γ, then, in particular, it proves that G does not have a vertex cover of size $t = t_{min}/\gamma - 1 \geq 2n/3$. Convert the transcript of A's computation on G along with an argument of its correctness within a factor of γ into a resolution proof π of an appropriate encoding $VC(G,t)$. From Corollary 3.3, $size(\pi)$ must be exponential in $n/\Delta^{6+2\delta}$. $\square$

3.10 *Stronger Lower Bounds for Exhaustive Backtracking Algorithms and* DPLL

We conclude this chapter with a stronger lower bound for a natural class of backtracking algorithms for the independent set and vertex cover problems, namely the class of exhaustive backtracking search algorithms. The key difference between the algorithms captured by resolution that we have considered so far and the ones in this class is that the latter do *not* reuse computation performed for previous branches; instead, they systematically rule out all potential independent sets or vertex covers of the desired size by a possibly smart but nonetheless exhaustive search. As an illustration, we will give an example of a non-trivial exhaustive backtracking algorithm for the independent set problem shortly.

The argument for our lower bound is based on the density of independent sets and vertex covers in random graphs and is quite straightforward in the light of Lemma 3.1. We derive as a consequence a tighter lower bound for the DPLL complexity of the mapping and counting encodings of the two problems that allows the edge density in the underlying graph to be much higher than in Theorem 3.2 and Corollary 3.3.

Returning to the class of exhaustive backtracking algorithms, recall that the approach we used for our upper bounds (cf. Section 3.5) was to systematically rule out all potential independent sets of a certain size $k' = k_{min}$. This is the simplest algorithm in the class. Of course, instead of simply considering all $\binom{n}{k'}$ subsets of vertices of size k' as we did, one can imagine more complex techniques for exhaustive search. For instance, an idea similar to the one used by Beame et al. [14] for the graph coloring problem would be to consider all subsets of size $u < k'$ in the first stage. For a random graph, most of these subsets are very likely to already contain an edge and need not be processed further. For any remaining subset S, one can recursively refute the existence of an independent set of size $k' - u$ in the residual graph with $|n - k - N(S)|$ vertices, where $N(S)$ denotes all neighbors of S outside S. This is also an exhaustive backtracking algorithm.

Such algorithms may require a more complex analysis than we gave in our upper bound proofs and could potentially be more efficient. However, as the following result shows, any technique that systematically rules out all possible k'-independent sets by an exhaustive backtracking search cannot improve the relatively simple upper bounds in Theorem 3.1 and Corollary 3.1 by more than a constant factor in the exponent.

Let $\mathcal{A}^{ind}_{exhaustive}$ (or $\mathcal{A}^{VC}_{exhaustive}$) denote the class of backtracking algorithms for proving non-existence of independent sets (vertex covers, resp.) of a given size in a given graph, that work by recursively subdividing the problem based on whether or not a set of vertices is included in the independent set (vertex cover, resp.) and that do not reuse computation performed in previous branches. For example, our approach in Section 3.5 as well as the algorithm based on [14] sketched above, both belong to $\mathcal{A}^{ind}_{exhaustive}$.

Theorem 3.5 (Exhaustive Backtracking Algorithms). *There are constants C and c such that the following holds. Let $\Delta = np$, $c \leq \Delta \leq n/\log^2 n$, and $G \sim \mathbb{G}(n,p)$. With probability $1-o(1)$ in n, every algorithm $A \in \mathcal{A}^{ind}_{exhaustive}$ (or $\mathcal{A}^{VC}_{exhaustive}$) running on input (G,k) must branch at least $2^{C(n/\Delta)\log^2\Delta}$ times when G does not have an independent set (vertex cover, resp.) of size k.*

Proof. Let C be the constant from Lemma 3.1. Recall the definitions of $k_{+\epsilon}$ and C_ϵ from Proposition 3.2. Fix $\epsilon > 0$ such that $k_{+\epsilon} + 1 > (2n/\Delta)\log\Delta$ and let $c \geq C_\epsilon$. With probability $1-o(1)$ in n, algorithm $A \in \mathcal{A}^{ind}_{exhaustive}$ succeeds in proving the non-existence of a k-independent set in G only when $k \geq k_{+\epsilon} + 1$. However, Lemma 3.1 says that G almost surely contains at least $2^{C(n/\Delta)\log^2\Delta}$ independent sets of size $k^* = \lfloor (n/\Delta)\log\Delta \rfloor$, which is less than $(k_{+\epsilon}+1)/2$. Hence, while recursively subdividing the problem based on whether or not to include a vertex in the k-independent set, A must explore at least $2^{C(n/\Delta)\log^2\Delta}$ distinct k^*-independent sets before finding a contradictory edge for each and backtracking.

For the vertex cover case, note that the algorithms in $\mathcal{A}^{VC}_{exhaustive}$ are the duals of the algorithms in $\mathcal{A}^{ind}_{exhaustive}$; including a vertex in a vertex cover to create a smaller subproblem is equivalent to not including it in an independent set. Further, the number of vertex covers of size $n-k$ in G is exactly the same as the number of independent sets of size k in G. Hence, the above lower bound applies to the algorithms in $\mathcal{A}^{VC}_{exhaustive}$ as well. $\square$

Theorem 3.6 (Stronger DPLL Lower Bounds). *There are constants C and c such that the following holds. Let $\Delta = np$, $c \leq \Delta \leq n/(2\log^2 n)$, and $G \sim \mathbb{G}(n,p)$. With probability $1-o(1)$ in n,*

$$\begin{aligned}
\mathsf{DPLL}(\alpha_{map}(G,k)) &\geq 2^{C(n/\Delta)\log^2\Delta}, \\
\mathsf{DPLL}(\alpha_{count}(G,k)) &\geq 2^{C(n/\Delta)\log^2\Delta}, \\
\mathsf{DPLL}(VC_{map}(G,t)) &\geq 2^{C(n/\Delta)\log^2\Delta}, \quad \textit{and} \\
\mathsf{DPLL}(VC_{count}(G,t)) &\geq 2^{C(n/\Delta)\log^2\Delta}.
\end{aligned}$$

Proof. The DPLL complexity of the encodings, by our convention, is ∞ if G does have an independent set of size k. If it does not, the tree T associated with any DPLL refutation of $\alpha_{map}(G,k)$ or $\alpha_{count}(G,k)$ can be viewed as the trace of an exhaustive backtracking algorithm $A \in \mathcal{A}^{ind}_{exhaustive}$ on input (G,k) as follows. An internal node in T with variable x_v as its secondary label corresponds to the decision of A to branch based on whether or not to include vertex v in the independent set it is creating. Nodes in T with counting variables as secondary labels represent the counting process of A.

Given this correspondence, Theorem 3.5 immediately implies the desired lower bounds for the independent set problem. The results for the vertex cover problem can be derived in an analogous manner. Note that refuting the block encoding may be easier than ruling out all independent sets (vertex covers, resp.) of size k. Hence, Theorem 3.5 does not translate into a bound for this encoding. $\square$

3.11 Discussion

In this chapter, we used a combination of combinatorial and probabilistic arguments to obtain lower and upper bounds on the resolution complexity of several natural CNF encodings of the independent set, vertex cover, and clique problems. Our results hold almost surely when the underlying graph is chosen at random from the $\mathbb{G}(n, p)$ model. Consequently, they hold (deterministically) for nearly all graphs. A key step in the main lower bound arguments was to simplify the task by considering the induced block graph in place of the original graph. The expansion properties of the block graph then allowed us to relate refutation width with structural properties of the graph.

Our results imply exponential lower bounds on the running time of resolution-based backtracking algorithms for finding a maximum independent set (or, equivalently, a maximum clique or a minimum vertex cover) in a given graph. Such algorithms include some of the best known ones for these combinatorial problems [105, 106, 67, 100].

A noteworthy contribution of this work is the hardness of approximation result. We showed unconditionally that there is no polynomial time resolution-based approximation algorithm that guarantees a solution within a factor less than $\Delta/(6\log\Delta)$ for the maximum independent set problem or within a factor less than $3/2$ for the minimum vertex cover problem. This complements the hardness results conditioned on P $\neq$ NP that rule out efficient approximations within factors of $\Delta_{max}/2^{O(\sqrt{\log\Delta_{max}})}$ [108] and $10\sqrt{5} - 21 \approx 1.36$ [47] for the two problems, respectively. (Here Δ_{max} denotes the maximum degree of the underlying graph rather than the average degree.)

On the flip side, some algorithms, such as those of Robson [97], Beigel [21], Chen et al. [33], and Tomita and Seki [107], employ techniques that do not seem to be captured by resolution. The techniques they use, such as unrestricted without loss of generality arguments [97], vertex folding [33], creation of new vertices [21], and pruning of search space using approximate coloring [107], represent global properties of graphs or global changes therein that appear hard to argue locally using a bounded number of resolution inferences. For instance, the algorithm of Robson [97] involves the reasoning that if an independent set contains only one element of $N(v)$, then without loss of generality, that element can be taken to be the vertex v itself. It is unclear how to model this behavior efficiently in resolution.

Restricted versions of these general properties, however, can indeed be simulated by resolution. This applies when one restricts, for instance, to vertices of small, bounded degree, as is done in many case-by-case algorithms cited at the beginning of this chapter [105, 106, 67, 100].

Finally, as we mentioned in the introduction, the spectral algorithm of Coja-Oghlan [37] achieves an $O(\sqrt{\Delta}/\log\Delta)$ approximation and, in the light of our lower bounds, cannot be simulated by resolution.