Counting Regions in Hyperplane Arrangements

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Problems in combinatorics interest me due to the simplicity at which they can be stated and the elegance and depth of their solutions. In particular, I am drawn to geometric problems, such as ones on hyperplane arrangements, polytopes and simplicial complexes. In the following pages, we will state and prove an important theorem in geometric combinatorics, which I believe illustrates this characteristic of combinatorics– a theorem by Zaslavsky on how to count bounded and unbounded regions in a hyperplane arrangement. We begin my recalling a few definitions:

Definition 1. An **affine hyperplane** is a set of points in $\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R}$ satisfying equations of the form $a_1x_2 + a_2x_2 + \cdots + a_nx_n = b$.

In other words, it is an affine subspace of codimension 1 in an affine space. Hyperplanes in a real affine space divide the space into two: $a_1x_2 + a_2x_2 + \cdots + a_nx_n < b$ and $a_1x_2 + a_2x_2 + \cdots + a_nx_n > b$.

Definition 2. A collection of affine hyperplanes is called a **hyperplane arrangement**.

Such an arrangement is said to be in *general position* if you can move any of the hyperplanes slightly without changing the number of regions. For instance, the arrangement on the left hand side is not in general position while the one on the right is:

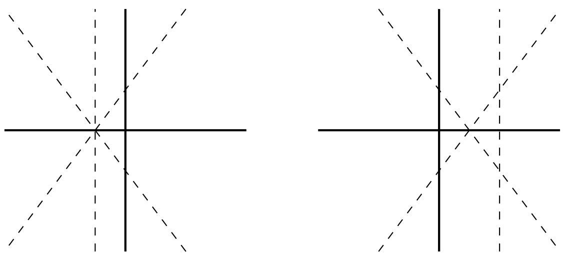

Figure 1: Examples of Hyperplane Arrangements

Definition 3. A **region** of a hyperplane arrangement, $\mathcal{A}$, is a connected component of the complement, $\mathbb{R}^n - \cup_{H\in\mathcal{A}} H$, where H denotes a hyperplane in $\mathcal{A}$.

Regions can be bounded or unbounded. We denote the total number of regions by $r(\mathcal{A})$ and the number of bounded regions by $b(\mathcal{A})$. We now arrive at the central question of this feature. *How do we count the number of bounded and total number of regions in a given hyperplane arrangement?*

One way is to draw the hyperplane arrangement; but, this process can get tedious. Luckily, there is a more rigorous way of counting such regions: a theorem by Zaslavsky. Before we go on to present this solution, we note a few important constructions in the theory of hyperplane arrangements.

Definition 4. Let H be a hyperplane in $\mathcal{A}$. We name $\mathcal{A}' = \mathcal{A} \backslash H$ the *deleted arrangement* and $\mathcal{A}'' = \{K \cap H : K \in \mathcal{A}'\}$ the *restricted arrangement*. The triple, $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$, is (creatively) named a triple of arrangements.

For instance, if we choose the red hyperplane the one we choose to create a triple of arrangements, we get the following result:

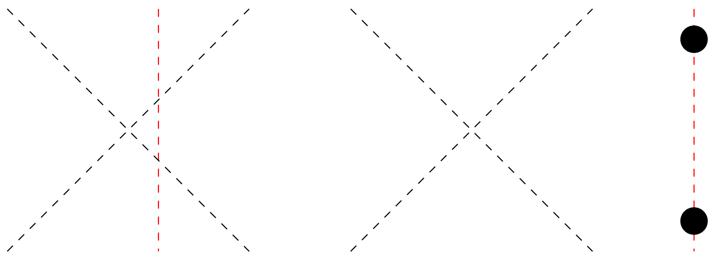

Figure 2: An Example of a Triple or Arrangements, with $\mathcal{A}, \mathcal{A}'$, and $\mathcal{A}''$, Respectively

Another theorem by Zaslavksy says that $r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'')$, which for now we will assume to be true. (In deed, for the above example, we get $7 = 4 + 3$.)

We can use this relationship to note a recursive relationship that we can use to count the number of regions when our hyperplanes are in general position. The procedure, for an arrangement in $\mathbb{R}^2$ goes as follows: take the hyperplane arrangement $\mathcal{A}$ with k lines that are in general position. Choose a particular hyperplane, H. Then, H meets $\mathcal{A}'$ in $k-1$ points, which divided H into k regions. Therefore, $r(\mathcal{A}'') = k$ and so $r(\mathcal{A}) = r(\mathcal{A}') + k$, where the arrangement $\mathcal{A}'$ contains $k-1$ hyperplanes. Repeat this process with the remaining lines to get,

$$r(\mathcal{A}) = r(\emptyset) + 1 + 2 + \cdots + k = 1 + k + \binom{k}{2}$$

We can use this recurrence and induction to note that if we have k hyperplanes in general position in an n-dimensional space, then

$$r(\mathcal{A}) = 1 + k + \binom{k}{2} + \binom{k}{3} + \cdots + \binom{k}{n}$$

Unfortunately, we cannot use this if the hyperplanes are not in general position, so we need to introduce more tools.

Definition 5. A **partially ordered set** (poset) is a set, P, together with a binary relation $\leq$ such that for all $x, y, z \in P$,

- $x \leq x$,
- If $x \leq y$ and $y \leq z$, then $x \leq z$,
- If $x \leq y$ and $y \leq x$, then $x = y$.

We can represent a poset with what is called a *Hasse diagram*. For instance, the following Hasse diagram is of the set $\{a, b, c, d\}$, with relations $a \leq b$, $a \leq c$, $b \leq d$

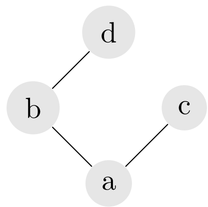

Figure 3: A Hasse Diagram of a Poset with Four Elements

A *closed interval*, $[x, y]$, of a poset is the set of all points between x and y, including x and y. Using this, we can define the Mobius function, $\mu(x, y)$, as follows:

Definition 6. The **Mobius function**, $\mu(x, y)$ is defined recursively on the interval $[x, y]$ by the following two properties:

- $\mu(x, x) = 1$, for all $x \in P$
- If $x < y$, then $\sum_{z \in [x,y]} \mu(x, z) = 0$.

If we have a poset with a minimal element, which we will denote by $\hat{0}$, then we will write that $\mu(x) = \mu(\hat{0}, x)$,

The *intersection poset*, denoted by $L(\mathcal{A})$, has as its elements all the intersections of all the hyperplanes in the arrangement. The relation here is inverse inclusion, where $A \subseteq B$ if and only if $A \geq B$. The minimal element is the ambient space $\mathbb{R}^n$.

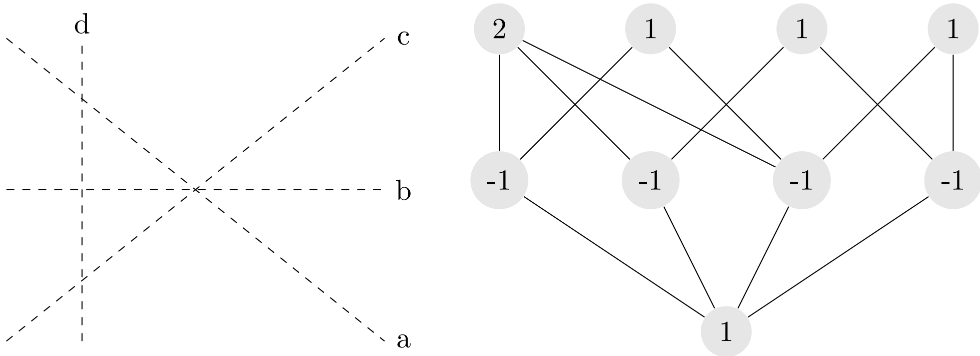

Figure 4: An Arrangement in $\mathbb{R}^2$ with its Poset

Definition 7. The **characteristic polynomial** associated with an arrangement $\mathcal{A}$ is defined as

$$\chi(\mathcal{A}, q) := \sum_{x \in L(\mathcal{A})} \mu(\hat{0}, x) q^{dim(x)}$$

For instance, the arrangements in Figure 4 and Figure 3 have the following characteristic polynomials $q^2 - 4q + 5$ and $q^2 - 3q + 3$, respectively.

We are finally ready to state Zaslavsky's theorem!

Theorem 1 (Main Theorem)**.** *Let $\mathcal{A}$ be an arrangement in an n-dimensional real vector space. Then,*

$$r(\mathcal{A}) = |\chi(\mathcal{A}, -1)| \quad (1)$$

$$b(\mathcal{A}) = |\chi(\mathcal{A}, 1)| \quad (2)$$

In deed, this gives us that there are 10 regions (2 of which are bounded) for the arrangement in Figure 4, and 7 regions (only 1 of which is bounded) for the arrangement in Figure 3. The proof of the theorem will make use of the following two claims:

Lemma 1. *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of real arrangements. Denote the chosen hyperplane for this triple of arrangements with H_0. Then, we have that*

- $r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'')$
- $b(\mathcal{A}) = b(\mathcal{A}') + b(\mathcal{A}'')$ *if* $\text{rank}(\mathcal{A}) = \text{rank}(\mathcal{A}')$*, and* 0 *otherwise*

Proof. The proof for the two equations is analogous, so we will only show the first one. It is easy to see that $r(\mathcal{A})$ is equal to $r(\mathcal{A}')$ plus the number of regions $\mathcal{A}'$ is cut into two regions by the chosen hyperplane H_0. We will prove the equality by showing a bijection between regions of $\mathcal{A}'$ that are cut into two by the hyperplane H_0 and the number of regions of $\mathcal{A}''$.

Denote the regions of $\mathcal{A}'$ that are cut into two by H_0 by R'. Then, $R' \cap H_0$ is a region of $\mathcal{A}''$. Likewise, if R'' is a region in $\mathcal{A}''$, then the points near R'' on either side of H_0 have to belong to the same region R', which is a region in $\mathcal{A}'$. This is because any H, which is a region in $\mathcal{A}'$ that separates them would also intersect R'. Therefore, R' is cut into two by H_0. This creates the desired bijection. □

Definition 8. A subarrangement, $\mathcal{B}$, is said to be **central** if $\cap_{H \in \mathcal{B}} H \neq \emptyset$.

For instance, the example in Figure 4 has the following central arrangements: in the example below the central arrangements are:

$$\{\emptyset\}, \{a\}, \{b\}, \{c\}, \{d\}, \{ab\}, \{ac\}, \{ad\}, \{bc\}, \{bd\}, \{cd\}, \{abc\}$$

Theorem 2 (Whitney's theorem)**.** *Let $\mathcal{A}$ be an arrangement in an n-dimensional vector space. Then,*

$$\chi(\mathcal{A}, t) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{\#\mathcal{B}} q^{n - \text{rank}(\mathcal{B})}$$

We will not prove this theorem, but working out the case for the example in Figure 4 gives us $\chi(\mathcal{A}, q) = q^2 - 4q + (6 - 1)$, which agrees with the result above.

Lemma 2. *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple real arrangement. Then,*

$$\chi(\mathcal{A}, q) = \chi(\mathcal{A}', q) - \chi(\mathcal{A}'', q)$$

Proof. Let H_0 be the chosen hyperplane that defines the triple of arrangements. Using Whitney's theorem

$$\chi(\mathcal{A}, q) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{\#\mathcal{B}} q^{n-\mathrm{rank}(\mathcal{B})}$$

If we split this sum into two, depending on whether H_0 is in the given central arrangement or not, we get the following result.

- The sum in the case where $H_0 \notin \mathcal{B}$ is $\chi(\mathcal{A}', q)$,
- If $H_0 \in \mathcal{B}$, then set $\mathcal{B}_1 = (\mathcal{B} - \{H_0\})_{H_0}$, which is a sub-arrangement of $\mathcal{A}''$ that is central in H_0. Clearly, $\#\mathcal{B}_1 = \#\mathcal{B} - 1$ and $\mathrm{rank}(\mathcal{B}_1) = \mathrm{rank}(\mathcal{B}) - 1$. Thus, the sum in this case will be,

$$\sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{\#\mathcal{B}} q^{n-\mathrm{rank}(\mathcal{B})} = \sum_{\mathcal{B}_1 \in \mathcal{A}''} (-1)^{\#\mathcal{B}_1+1} q^{n-1-\mathrm{rank}(\mathcal{B}_1)} = -\chi(\mathcal{A}'', q)$$

Thus, $\chi(\mathcal{A}, q) = \chi(\mathcal{A}', q) - \chi(\mathcal{A}'', q)$, as desired.

□

Proof of Main Theorem. The arrangement $\mathcal{A} = \emptyset$ has characteristic polynomial q^n, and clearly satisfies equation 1. If we write $r(\mathcal{A}) = (-1)^n \chi(\mathcal{A}, -1)$, using Lemma 1 and Lemma 2 above, we can see that both sides of the equation satisfy the same recurrence relation, which completes the proof.

Likewise, equation (2) holds for $\mathcal{A} = \emptyset$.[1] Let $d(\mathcal{A}) = (-1)^{\mathrm{rank}(\mathcal{B})} \chi(\mathcal{A}, 1)$. If $\mathrm{rank}(\mathcal{A}) = \mathrm{rank}(\mathcal{A}')$, which in turn equals $\mathrm{rank}(\mathcal{A}'') + 1$, then $d(\mathcal{A}) = d(\mathcal{A}') + d(\mathcal{A}'')$. But, if $\mathrm{rank}(\mathcal{A}) = \mathrm{rank}(\mathcal{A}) + 1$, then $b(\mathcal{A}) = 0$ and $L(\mathcal{A}') \cong L(\mathcal{A}'')$. Using Lemma 2, $d(\mathcal{A}) = 0$. Therefore, $b(\mathcal{A})$ and $d(\mathcal{A})$ satisfy the same recurrence relation and agree on $\mathcal{A} = \emptyset$, as desired.

□

References

[1] R. Stanley: *An Introduction to Hyperplane Arrangements*. IAS/Park City Mathematics Series, Volume 14, 2004.

[1] Note that this counts the number of relatively bounded regions, and in this case the entire ambient space $\mathbb{R}^n$ is relatively bounded.