Cake Cutting: Equitable Simple Allocations of Heterogeneous Goods

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Contents

Abstract

Cake cutting, a subfield of fair division, refers to the allocation of a heterogeneous, divisible, and continuous good among n players who have heterogeneous preferences. The challenge is to allocate the cake in such a way that is fair by using a procedure that encourages the players to be truthful about their preferences. There are several different notions of fairness studied in the literature; in this essay, we consider equitability – a fairness criteria where each player gets the same proportion of the cake as she values it. We give a comprehensive survey of prominent results for equitable allocations using $n-1$ cuts. We also focus on the relationship between equitability and other fairness criterion and note when it is possible to have procedures that are simultaneously equitable and proportional or envy-free.

1 Introduction

The problem of fair division of goods is as old as mankind. It can be traced back to the Hebrew Bible. When Abraham and Lot decide to separate in Genesis, they are presented with the challenge of dividing up the land. Abraham suggests to mark the place on which they stand as the cutting point and asks Lot to pick a side. Lot chooses the side that is well-watered and Abraham goes in the opposite direction [2].

Another instance is when two women come before King Solomon with a baby, each claiming that they are the true mother. King Solomon suggests "cut the living child in two and give half to one and half to the other." The true mother protests and offers to give the baby to the second woman, which reveals to King Solomon that she must not be the impostor. Fortunately, we now differentiate between divisible and indivisible goods.

Over the years, the field of fair division has gained popularity due to the increase in the number of resources to be allocated such as advertising spaces on websites and computational clouds as well as the decline in availability of some resources such as fossil fuels and land.

Fair division is categorized into different subfields using properties of the resources being allocated. The main distinctions are whether we are dividing,

- one good or multiple goods, such as a piece of land vs. a box of jewelry,
- divisible or indivisible goods, such as land vs. a piano,
- homogeneous or heterogeneous goods, such as money vs. a large cake,
- desirable or undesirable goods, such as inheritance vs. household chores.

Cake cutting refers to the allocation of a single, divisible, heterogeneous, and desirable good (such as a cake). The cake is heterogeneous since it is assumed to be made up of different ingredients and has different toppings. Therefore, pieces of cake of the same size might not be equally tasty to each player. The problem at hand is to divide up the cake among n players such that some fairness criteria is satisfied.

This cake may be continuous or discrete. Continuous goods can be divided up into arbitrarily small parts and at any point while discrete ones cannot. One of the earliest papers on this topic by Alon [3] uses a necklace made out of different beads to symbolize such a divisible good. A discrete model is likewise used in the first comprehensive book on cake cutting, which was by Brams and Taylor [7]. We will assume continuity, which is in line with the research done in recent decades.

The most popular example in cake cutting is that of a mother who must divide up a cake among her two children. Assume the children have different preferences: the oldest might want the biggest piece while the youngest insists on getting as much of the flower in the center as possible. Such preferences are captured by individual valuation functions that assign each portion of the cake a numerical value. This measures the players' level of entitlement over different parts of the cake. We assume that these functions are private and subjective.

A cake cutting procedure is a list of actions to be performed by the players that will result in an allocation; a valid procedure results in a fair allocation of the cake among the players if they act rationally according to their valuation functions. A challenge posed is to devise a procedure that encourages the players to report their true valuation functions.

One solution to the above problem is to ask one of the children to cut the cake and the second one to choose a piece. This is commonly known as the 'cut-and-choose' procedure. The first child has an incentive to cut the cake such that she is indifferent between the two resulting pieces and the second will pick the piece that she prefers.

Figure 1: Cake Representing Divisible Good [15]

This division scheme is fair in the sense that both children feel like they have received at least half of the cake and neither child strictly prefers the other piece over the one she is allocated. The former is known as *proportionality*. In the general setting with n players, a proportional allocation is one where each player receives what she feels is at least $1/n$ of the cake. The latter is known as *envy-freeness*, where none of the players benefit by trading with another player. There are various other criterion for fairness including equitability, efficiency, and exactness. In the next section, we will explicitly discuss the properties that characterize each of these and present the focus of this essay.

While the cake cutting problem has been around for quite some time, a rigorous mathematical treatment of it was only initiated at the end of World War II. At the time Steinhaus, who is regarded as the father of cake cutting, observed that the cut-and-choose procedure can be extended to three players and asked whether it can be done for any n. This was resolved affirmatively by Steinaus, Banach, and Knaster and started a snowball effect where similar ideas were applied to envy-freeness. The formality and mathematical-rigor of the models used have been evolving over the decades, allowing for more precision in the results. A summary of the classical results is given by Brams and Taylor [6] and there is an up-to-date discussion by Procaccia [17].

Not all cake cutting problems have as simple of a solution as the cut-and-choose procedure. Moreover, not all fairness criterion are easy to satisfy. An infamous example is an envy-free allocation for $n > 4$, for which there is no known discrete procedure with a bounded number of cuts. Therefore, recent research in cake cutting focuses on giving improved procedures, coming up with new ones that simultaneously satisfy various fairness criteria, and applications to real-world problems. Cake cutting is thus an active area of research in mathematics, economics, operations research, and more recently computer science. Results are applied in various domains including political science, sociology, and economics.

In this essay, we will focus on a specific fairness criteria known as equitability. An equitable allocation is one where each player feels that she has received equal proportions of the cake according to her individual preferences. Therefore, each player is not concerned about the piece other players get and receives a fixed proportion p of the cake. Such an allocation need not be envy-free or even proportional.

One instance in which equitability is natural is if several research groups are using the same laboratory equipment. The resource being allocated here is time with the equipment. Each group may have their own preferences depending on what time of the day it is. Moreover, they might need it for a specific proportion p of the day, after which point the equipment is of no use. In such a scenario, we would like to allocate the equipment such that each group gets it for this proportion of the day. We are interested in proving the existence of such procedures as well as giving an explicit step-by-step instruction for finding them.

Equitability is not as well understood as envy-freeness and proportionality. Its relationship with other fairness criterion is also not as clear. Therefore, we give a thorough survey of the scattered work involving equitability. We also investigate the simultaneous satisfaction of equitability with other fairness criterion. This paper is written with a mathematical audience in mind. Thus, concepts from economics and computer science will be defined explicitly and mathematically whenever possible. We pay particular attention to the mathematical framework and rigor underlying the results. The results will be analytical and combinatorial in flavor. We also highlight some holes in the research done on equitability and suggest further directions.

2 Preliminaries

The goal of this section is to establish a mathematically rigorous framework under which we present the results and discussions. While there is a fair amount of overlap across the literature, there is some inconsistency for the model and definitions. We thus explicitly define all tools and properties as they will be used in this essay. The set up presented here largely follows that given by Chen et al. [12] and Procaccia [17].

2.1 The Cake

We first assume that there are n individuals that desire the cake, which we call the *players*. We denote them using the set $\{1, 2, \cdots, n\} = [n]$.

A heterogeneous, divisible and continuous good – the cake – is represented by the interval $[0, 1]$. A *piece of cake* is a finite union of non-overlapping (interior disjoint) subintervals of $[0, 1]$. The length of a piece of cake is its Lebesgue measure. We will only use this when A is a finite union of intervals.

Each individual may have different preferences for two subintervals with the same length. Moreover, two players may have different preferences over the same subinterval.

Definition 1. A **(simple) allocation** of the interval $[0, 1]$ among n players is a partition of it into n non-overlapping closed intervals A_i.

Equivalently, a simple allocation is a pair (C, π), where C is the set of *cut-points* $(0 = c_0 \leq c_1 \leq \cdots \leq c_n = 1)$ and π is a permutation of $[n]$, which we call the *player order*. For such an allocation, the player $j = \pi(i)$ is assigned the piece $[c_{i-1}, c_i]$. We can give an $n-1$-tuple for the set of cut-points as c_0 and c_1 are fixed at 0 and 1, respectively.

Some of the literature assumes that we can throw away a piece of cake. In this case, we partition $[0, 1]$ into $n + 1$ pieces, where A_{n+1} corresponds to the piece being thrown away. In this essay, we assume that no piece will be thrown away (i.e., the allocation is total). Thus, the definition above suffices. This is because once the cake has been allocated, players can choose to throw away pieces of their cake.

Several results in cake cutting are concerned with non-simple allocations, where players may be given multiple disjoint intervals. This further complicates the problem and is also impractical in real-life settings. For instance, consider the above case of several research groups that have to share a single equipment. It is unrealistic to have members of the lab groups using the equipment in six five-minute intervals instead of a continuous half-hour interval. Henceforth, we consider only simple allocations.

2.2 Preferences

We encode the players' preferences over the cake using valuation functions. These are borrowed from the notion of utility functions in economics, but permit applications of more concepts from analysis and probability theory.

Each player $i \in [n]$ has a private *valuation function* V_i, which maps a given piece of cake to the value player i assigns to them. Thus, given two pieces of cake A and B, if $V_i(A) > V_i(B)$, then player i prefers A over B. For subintervals $[x, y]$, we write $V_i(x, y)$ instead of $V_i([x, y])$ for simplicity. We formalize this further.

Definition 2. Each player i has a **value density function** $v_i : [0, 1] \to [0, \infty\}$, which is piecewise continuous, non-negative, and integrable. It encodes the player's preference over the interval $[0, 1]$.

Valuation functions, defined as follows, assign a value to each piece of cake.

$$V_i(A) = \int_A v_i(x)dx = \sum_{I \in A} \int_I v_i(x)dx. \tag{1}$$

where the sum in the third term is over all subintervals in A. These functions have four properties that we make repeated use of:

1. Valuation functions are *countably additive*. In particular, for two disjoint pieces of cake A and B,
$$V_i(A \cup B) = V_i(A) + V_i(B).$$
2. They are *divisible*, i.e., for every subinterval I and $0 \leq \lambda \leq 1$, there exists a subinterval $J \subseteq I$ such that $V_i(J) = \lambda V_i(I)$.
3. They are *non-atomic*. That is,
$$V_i(x, x) = 0, \forall x \in [0, 1].$$
This is especially useful for cut-points, which define the border of a piece. These cuts have length 0 and are thus of no value to any of the players. Therefore, we do not have to worry about boundaries during allocations as a subinterval $[x, y] \subseteq [0, 1]$ has the same value as (x, y) for each player. It is for this reason that we treat interior-disjoint subintervals as disjoint and we sometimes use closed intervals instead of open ones. We also assume that each player is indifferent between receiving a piece that has value 0 and receiving no piece.
4. They are *non-negative* and *non-decreasing*, which follows from Equation (1). Thus, $V_i(I) \geq 0$ for every subinterval $I \subseteq [0, 1]$.

We normalize the valuation functions such that,

$$V_i(0, 1) = \int_0^1 v_i(x)dx = 1.$$

We assume that the players have pairwise distinct valuation functions. While this is not always necessary to assume, it will allow us to avoid some corner cases that arise in Section 4 without this assumption.

We can treat the V_i as probability measures and define the *distribution functions* $F_i(x) = V_i(0, x)$ such that the measure on an interval $[a, b]$ is $F_i(b) - F_i(a)$. The relationship between valuation and distribution function is,

$$F_i(b) - F_i(a) = V_i(a, b) = \int_a^b v_i(x)dx.$$

The F_i are non-negative, non-decreasing, and continuous on the whole interval $[0, 1]$. We note, $F_i(0) = 0$ and $F_i(1) = 1$.

2.3 Fairness

There are several notions of fairness that are used in cake cutting. The two most popular ones are proportionality and envy-freeness.

- An allocation is said to be **proportional** if $V_i(A_i) \geq 1/n, \forall i \in [n]$. Therefore, $\sum_{i\in[n]} V_i(A_i) \geq 1$.

- An allocation is **envy-free** if $V_i(A_j) \leq V_i(A_i), \forall i, j \in [n]$.

An envy-free allocation is also proportional since $\sum_{j\in[n]} V_i(A_j) = 1, \forall i \in [n]$. Thus, there must exist $j \in [n]$ such that $V_i(A_j) \geq 1/n$. Since the allocation is envy-free, we have that $V_i(A_i) \geq V_i(A_j)$ giving us $V_i(A_i) \geq 1/n$. For two players, proportionality and envy-freeness are equivalent since $V_i(A_i) \geq 0.5$ and $V_i(0, 1) = 1, \forall i \in \{1, 2\}$. This does not necessarily hold for $n \geq 3$.

There are other fairness criterion that are more elusive.

- An allocation is said to be **equitable** if $V_i(A_i) = V_j(A_j), \forall i, j \in [n]$.

- An allocation is **exact** if $V_i(A_i) = V_i(A_j), \forall i, j \in [n]$.

An equitable allocation need not be envy-free or proportional. An exact allocation encompasses all the other notions of fairness.

Lemma 3. *An exact simple allocation is also proportional, envy-free, and equitable.*

Proof. We note $V_i(A_j) = V_i(A_i)$, thus $V_i(A_j) \leq V_i(A_i)$, which gives envy-freeness. Envy-freeness implies proportionality for all values of n.

Finally, $V_i(A_j) = V_i(A_i), \forall i, j \in [n]$ and $V_i(0, 1) = 1$ implies that $V_i(A_j) = 1/n, \forall i, j \in [n]$ and in particular when $i = j$. □

Example 4. Take two players with value density functions,

$$v_1(x) = 1, \forall x \in [0, 1]$$
$$v_2(x) = 2x, \forall x \in [0, 1].$$

Assume Player 1 is the cutter. Then, she cuts the cake at 0.5, while Player 2 picks the subinterval $[0.5, 1]$. Therefore, $V_1(A_1) = 0.5$ and $V_2(A_2) = 0.75$. This is a simple allocation with cut-point 0.5 and player order $(1, 2)$.

If Player 2 was the cutter instead, then she would cut the cake at $\sqrt{0.5}$ and Player 1 would pick the subinterval $[0, \sqrt{0.5}]$. Thus, $V_1(A_1) = \sqrt{0.5}$ while $V_2(A_2) = 0.5$. Both of these allocations are envy-free (and proportional) but not equitable.

It is easy to see that if the cutter is risk-averse, which we will discuss in the truthfulness section, the cut-and-choose procedure restricts her to having a piece of cake that she values to be exactly one half.

Due to Lemma 4, results in exactness are preferred. Needless to say, they are also harder to obtain. Equitability is far less demanding, but has not been explored as much as envy-freeness or proportionality. Moreover, it is does not imply envy-freeness or proportionality in the same way that envy-freeness implies proportionality.

Note that an allocation that assigns each player a piece with value 0 (or no piece) is equitable, but far from a desirable allocation. Therefore, we are also concerned with the simultaneous satisfaction of equitability with other fairness criterion.

In this essay, we focus on equitable allocations. Equitable allocations were first studied by Dubins and Spanier [13] and Alon [3], who prove the existence of equitable allocations where the pieces could be any members of a σ algebra or use as many as $n(n-1)$ cuts, respectively. For reasons discussed earlier, we only consider simple allocations.

2.4 Common Techniques

A procedure is a list of actions to be performed by the players in accordance to their valuation functions that will result in an allocation. In fair division, we assume that the players are rational and will act in a way that maximizes the value of their allocated piece using the information available to them. We note that each player has no information about the other players' valuation functions, unless otherwise specified.

The first cake cutting procedures were given informal descriptions like the cut-and-choose procedure. With the introduction of a more mathematical framework to the problem, more formalized procedures emerged. These can be roughly be divided into two: *discrete* (e.g. protocols and algorithms) and *continuous* (e.g. moving-knife techniques). Such procedures might prove the *existence* of an allocation satisfying certain properties or they might go beyond and give explicit instructions for finding it.

These techniques may require a *referee* or might be possible to carry out without one. The referee is assumed to be an indifferent mediator whose purpose is to ensure that the players are adhering to the instructions of the procedure and carry out the allocation.

2.4.1 Discrete Techniques

Protocols are one of the earliest techniques to be applied to the cake cutting problem. The notion that we use is that introduced by Even and Paz [14]:

> "By a protocol, we understand a computer programmable interactive procedure. It may issue queries to the participants whose answers may affect its future decisions. It may issue instructions to the participants such as: 'Cut a piece of cake according to the following specifications: $\cdots$' or 'The piece labeled X is alloted to participant A', etc. The protocol has no information on the measures of the various pieces as measured by the different participants. It is assumed also that if the participants obey the protocol, then each participant will end up with his piece of cake after finitely many steps."

A protocol thus gives the players instructions, which they must then follow. The instructions must be enforceable without knowing the players' true valuation functions. (For instance, 'cut the cake into two pieces' is enforceable whereas 'cut the cake into two parts you value equally' is not.) When called upon by the protocol, each player chooses a strategy. Strategies are defined as an adaptive sequence of moves consistent with the protocol. When presenting procedures, we differentiate the strategies by putting them in brackets.

This description of protocols has evolved to be more precise over the decades. In particular, the types of instructions (or queries) that the referee can issue to the players have been restricted to two:

1. Cut(i, y) returns the minimum value of x such that $V_i(0, x) = y$ and,
2. Eval(i, x) returns $y = V_i(0, x)$ for $y \in [0, 1]$ and Player i.

The former is known as the *cutting query* and the latter is known as the *evaluation query*. For instance, the cut-and-choose protocol can be described as,

Protocol 1 Cut-and-Choose

1: $x := \text{Cut}(1, 0.5)$. (Player 1 does this such that the resulting pieces A_1 and A_2 satisfy $V_1(A_1) = V_1(A_2)$.)

2: If $\text{Eval}(2, x) \geq 0.5$, then assign $[0, x]$ to Player 2 and $[x, 1]$ to Player 1. Otherwise, assign $[0, x]$ to Player 1 and $[x, 1]$ to Player 2.

A protocol that uses only finitely many of the above queries is known as a *finite protocol*. The *complexity* of such a protocol is the worst case number of Cut and Eval queries required to complete the desired allocation.

In recent years, algorithms have gained popularity and attracted more computer scientists to the problem. Algorithms take as *inputs* the value density functions of the players and *output* the allocation. Unlike protocols, the task of performing computations using the value density functions is done by a computational center instead of the players themselves. A *finite algorithm* is one that can be performed using Cut and Eval.

We want to design protocols and algorithms that, if a player uses a strategy that does not attempt to misrepresent her valuation function, then she is guaranteed a piece of cake that satisfies the specified fairness criteria regardless of what other players do. When presenting protocols and algorithms, we will make them as precise as possible without losing readability.

2.4.2 Moving-Knife Procedures

Moving-knife techniques are borne out of a literal interpretation of the cake cutting problem. They were introduced by Stromquist [23]. A simple example is,

Moving-Knife 1. A referee moves a knife from 0 to 1. Let $c(t) \in [0, 1]$ be the location of the knife at time t. Each player i computes $V_i(0, c(t))$ and calls out 'cut' when $V_i(0, c(t)) = 1/n$. The first player who called out 'cut' at time t_1 is given the piece $[0, c(t_1)]$. If multiple players call out 'cut' at the same time, then one of them is chosen at random and given this piece.

For each stage j, let the position of the knife be denoted by $c(t_j)$. The referee repeats this step by moving her knife from $c(t_{j-1})$ to 1 and a player i calls out 'cut' when $V_i\left(c(t_{j-1}), c(t_j)\right) = 1/n$. The procedure is repeated $n - 1$ times and the last piece is given to the player who has not been allocated a piece.

Note that each player is guaranteed a piece that they value to be $1/n$, with one player potentially getting more. This procedure need not be envy-free or equitable. The discrete form of this is presented by Even and Paz [14].

Generally, moving-knife techniques proceed as follows: a referee moves a knife from 0 to 1 on the interval $[0, 1]$. Each player has her own knife that she might also be moving over the cake in accordance to some rule specified by the procedure. Each of the knives are assumed to be perpendicular to the cake and hovering over it. The players are instructed to say 'cut' when the position of the knives satisfies certain criteria according to their valuation functions. At this point, all knives stop moving and a subset of the knives make a cut at their position. Some pieces of cake are allocated. This procedure can be repeated a finite number of times until the entire cake is allocated.

Due to their continuous nature, moving-knife techniques are difficult to work with. As discussed by Robertson and Webb [20], they are not finite or even countable since each player must make a computation at each point the referee's knife is passing over. This is ignoring the corner case where a player says cut at $c_0 = 0$. Thus, we discretize them whenever possible.

2.5 Truthfulness

Each player's valuation function is private information. Thus, players might choose to misrepresent their valuation functions to increase the value of their allocated piece. We would like procedures that encourage players to be truthful. Just as in the case of fairness, there are various notions of truthfulness. There is perhaps the most amount of confusion in terminology regarding truthfulness. We adopt a mixture given by Brams et al. [5] and Chen et al. [12].

The definition as presented here follows that by Brams et al. [5]. It relies on the assumption that players are risk-averse.

Definition 5. A piece of cake is called a **maximin** piece if it is the $\max(\min A_i)$ that a player i can be assigned by adopting a certain strategy.

In a weak-truthful procedure, the goal of each player is to maximize the $V_i(\max(\min A_i))$ that she is guaranteed regardless of what the other players do. In other words, there *exists* some possible valuation function of the other players such that truth-telling results in a piece that she values at least as much as that she receives by lying.

Example 6. Take the players in Example 4. Assume that Player 1 cuts the cake at point $x < 0.5$, hoping to get the piece $[x, 1]$. She is reporting a false valuation function where $V_1'(0, x) = 0.5 > V_1(0, x)$.

There is a valuation function such that she can end up with a smaller piece. Namely, if Player 2 has value density function $v_2(x) = 2x$, then Player 2 would pick the piece $[x, 1]$ and $V_1(0, x) = x < 0.5$. Hence, as a risk-averse player, Player 1 would not choose this strategy.

In a procedure that is not weak-truthful, at least one player is guaranteed a bigger maximin piece by misrepresenting her valuation function.

The above definition is weak compared to that used in the social choice literature. A weak-truthful procedure is one where a player is not guaranteed a bigger piece by lying. This is not as helpful in real-world settings where the player might have an idea of the other players' valuation functions or is not 100% risk-averse. Indeed, in cut-and-choose, if Player 1 suspects that Player 2 likes subintervals closer to 1, then she can obtain a bigger piece by using this information.

Another notion, which we call *string-truthfulness*, is one where a player can never benefit by lying for all possible valuation functions of the other players. As described by Chen et al. [12] "truth-telling is the dominant strategy." In this case, a player cannot benefit by lying even if she knows the other players' valuation functions. This criteria of truthfulness is more common in computer science. It is useful as it also encompasses weak-truthfulness.

Example 7. Take the players in Example 4 and assume that Player 1 is taken to be the cutter. If Player 1 had a guess about Player 2's valuation function then she can choose $x \in (0.5, \sqrt{0.5})$ and Player 2 would still choose the piece $[0, x]$. Such a strategy ensures Player 1 a piece with value $x > 0.5$.

Therefore, cut-and-choose is a weak-truthful but not string-truthful procedure.

3 Existence

Existence results in fair division are more tractable than step-by-step algorithms. The first existence result for equitable allocations was given by Alon [3]. In this paper, he studied discrete goods (namely a necklace made out of different beads) and proved that an exact allocation for n players always exists. This allocation can have as many as $(n-1)n$ cut-points and is thus not simple.

Progress made towards equitable allocations was at a standstill until recent results by Brams et al. [5] and Cechlárová et al. [9][10][11]. We will present most of the results in this essay or discuss them in Section 6. In particular, the results in this section are by Brams et al. [5] and Cechlárová et al. [9][10].

3.1 Existence Results for Two Players

We start with the simple case of 2 players. We desire to find an equitable cut-point c such that $V_1(0,c) = V_2(c,1)$.

Theorem 8 ([10])**.** *Given two players, there always exists a cut-point $c \in [0,1]$ such that $V_1(0,c) = V_2(c,1)$. Moreover, all such points satisfying this property result in allocations with pieces of the same value for a given player order.*

Proof. Define $f : (0,1) \to \mathbb{R}$ such that $f(x) = V_1(0,x) - V_2(x,1)$. Since $V_1(0,x)$ is continuous and non-decreasing in x and $V_2(x,1) = 1 - V_2(0,x)$ is continuous and non-increasing in x, the function $f(x)$ is continuous and non-decreasing. Since $f(0) = -1$ and $f(1) = 1$, using the Intermediate Value Theorem, there exists a point $c \in (0,1)$ such that $f(c) = 0$.

To note the second assertion, let c_1 and c_2 be cut-points such that $c_1 \leq c_2$. Let $c \in [c_1, c_2]$. Then,

$$V_1(0,c_1) \leq V_1(0,c) \leq V_1(0,c_2) = V_2(c_2,1) \leq V_2(c_1,1) = V_1(0,c_1).$$

Thus, $V_1(0,c_1) = V_1(0,c_2) = V_2(c_1,1) = V_2(c_2,1)$. □

It is easy to note that if such an equitable point exists, then switching the players will also result in an equitable allocation. Since $V_i(0,c) = 1 - V_i(c,1)$, at least one of these results in a proportional allocation. Therefore, for two players, there always exists a proportional, envy-free, and equitable allocation for some player order. There exists an equitable allocation for any player order, but it does not need to be proportional and envy-free.

Example 9. Refer back to Example 4. The cut-point $\frac{\sqrt{5}-1}{2}$ and player order $(2,1)$ gives an equitable allocation where both players get pieces that they value to be $\frac{3-\sqrt{5}}{2} \approx 0.382$. This is neither an envy-free nor a proportional allocation. However, the order $(2,1)$ gives a proportional, envy-free, and equitable allocation with value approximately $0.618, \forall i \in \{1,2\}$.

An equitable allocation can be made into an envy-free (and proportional one) for the two person case by adding an extra step at the end of the procedure where the players are allowed to trade. The players will do so if and only if $V_i(A_i) < 0.5$. As this is a simple allocation, $1 - V_1(A_2) = 1 - V_2(A_1) > 0.5$.

3.2 Equitable Allocations for $n \geq 3$

The challenge is to achieve similar results for 3 or more players or prove they are not always possible. Before doing so, we set up the necessary terminology.

Recall that simple allocations can be specified by their cut-points and player order. Thus, an allocation in this section consists of an $n+1$-tuple,

$$(0 = c_0, c_1, \cdots, c_{n-1}, c_n = 1),$$

and a permutation π of $[n]$, which assigns each player to exactly one subinterval determined by the cut-points. That is, if $\pi(i) = j$, then player j is given the piece $[c_{i-1}, c_i]$.

For instance, if π is the identity permutation, then an equitable allocation is one where the following system of equalities is satisfied:

$$\begin{aligned} F_1(c_1) =& F_2(c_2) - F_2(c_2) \\ =& F_3(c_3) - F_3(c_2) \\ &\vdots \\ =& F_{n-1}(c_{n-1}) - F_{n-1}(c_{n-2}) \\ =& 1 - F_n(c_{n-1}). \end{aligned}$$

We start with 3 players. First, assume that the players have everywhere positive density functions. Thus, their distribution functions are strictly increasing. Let c_2 be an arbitrary point such that Player 3 is given the piece $[c_2, 1]$. By using the previous subsection, we note that there exists $c_1 \in [0, c_2]$ such that $F_1(c_1) + F_2(c_1) = F_2(c_2)$, i.e. Players 1 and 2 get an equitable allocation of the piece $[0, c_2]$. Then, the function $F_1 + F_2$ has inverse,

$$c_1 = (F_1 + F_2)^{-1}(F_2(c_2)).$$

The complication is that in general the density functions are not assumed to be everywhere positive. Therefore, the distribution functions are only weakly increasing. Note the following example to illustrate the challenge:

Example 10. Assume we have three players with value density functions,

$$\begin{aligned} v_1(x) &= \begin{cases} 2, x \in [0, 1/2] \\ 0, x \in (1/2, 1] \end{cases} \\ v_2(x) &= \begin{cases} 0, x \in [0, 1/2] \\ 2, x \in (1/2, 1] \end{cases} \\ v_3(x) &= 1, \forall x \in [0, 1]. \end{aligned}$$

Then, for player order $(2, 1, 3)$, the equitable cut-points must satisfy,

$$c_1 = \begin{cases} c_2, & \text{if } c_2 \in [0, 1/4] \\ \in [1/4, c_2], & \text{if } c_2 \in (1/4, 3/4] \\ \in [1/4, 3/4], & \text{if } c_2 \in (3/4, 1]. \end{cases}$$

Therefore, c_1 is not always uniquely determined by c_2.

The other challenge is that c_2 cannot be chosen arbitrarily; it must be chosen such that the resulting allocation is equitable for all three players. i.e.,

$$F_1(c_1) = F_2(c_1) = F_3(c_3) - F_3(c_2).$$

This calls for a generalized notion of inverses, conveniently called generalized inverses (sometimes quantile functions) from probability theory and statistics. We define them in a particular form and point out some useful properties below. A full exposition including proofs of the properties is given by Pfeiffer [16] and Resnick [19].

Definition 11. Let $h : [0,1] \to [0,2]$ be a non-decreasing, continuous function such that $h(1) \geq 1$. The **generalized inverse** of h, which we will denote by h^-, is the function $h^- : [0,1] \to [0,1]$ such that,

$$h^-(x) = \inf\{z \in [0,1] : h(z) \geq x\}.$$

Note this is the regular inverse for strictly increasing functions. This function has the following properties:

1. $h^-(0) = 0$ and $h^-(1) \leq 1$,
2. h^- is non-decreasing,
3. $h^-(x) = \min\{z \in [0,1] : h(z) = x\}, \forall x \in [0,1]$. Thus, $h(h^-(x)) = x$.

   For the remainder of the properties, let

$$h^-(x^-) = \lim_{t \to x^-} h^-(t) \tag{2}$$
$$h^-(x^+) = \lim_{t \to x^+} h^-(t). \tag{3}$$

4. Both one-sided limits exist $\forall x \in (0,1)$ and $h^-(x^-) \leq h^-(x) \leq h^-(x^+)$,
5. $h(h^-(x^-)) = x, \forall x \in (0,1]$ and $h(h^-(x^+)) = x, \forall x \in [0,1)$,
6. both the left and right hand side limits exist for each $x \in (0,1)$ and $h^-(x^-) \leq h^-(x) \leq h^-(x^+)$,
7. the generalized inverse is left-continuous in each $x \in (0,1]$.

We can now use generalized inverses in the above setup where we fix c_2 and try to find c_1. In particular, we have one possible choice,

$$c_1 = (F_1 + F_2)^-(F_2(c_2)).$$

Of course, this value of c_1 is dependent on c_2 in a non-decreasing manner. We would like to find a value of c_2 and c_1 such that,

$$F_3(c_2) = F_1(c_1) + F_2(c_1),$$

without assuming that the density functions are everywhere positive.

The crux of the argument is to define continuous functions on which we can apply the Intermediate Value Theorem. Using the definition of generalized inverses we note the following result.

Lemma 12 ([9])**.** *Let f and g be continuous, non-decreasing functions on $[0,1] \to [0,1]$ such that $f(0) = g(0) = 0$ and $f(1) = 1$. Then, the function $g \circ (f+g)^-$ is continuous. Moreover, $\forall y \in [0,1]$, there exists $x \in [0,y]$ such that $g(x) + f(x) = f(y)$.*

Proof. By Properties 3 and 5 above,

$$g((f+g)^-(x)) + f((f+g)^-(x)) = (f+g)((f+g)^-(x)) = x \tag{4}$$
$$g((f+g)^-(x^+)) + f((f+g)^-(x^+)) = (f+g)((f+g)^-(x^+)) = x. \tag{5}$$

Next, $(f+g)^-(x) \leq (f+g)^-(x^+)$ by Property 3 and both f and g are non-decreasing functions. Therefore,

$$f((f+g)^-(x)) \leq f((f+g)^-(x^+) \tag{6}$$
$$g((f+g)^-(x)) \leq g((f+g)^-(x^+). \tag{7}$$

When adding inequalities (6) and (7) together, we get $x \leq x$ by Equations (4) and (5) above. This forces equality in (6) and (7). Equality in (7) shows that $(g \circ (f+g)^-)$ is right continuous. Left continuity is given by Property 7.

The second part is a consequence of the Intermediate Value Theorem. Clearly, $f+g$ is continuous on $[0,1]$ (and so on $[0,y], \forall y \in [0,1]$). Also, $f(0)+g(0) \leq f(y)$ and $f(y)+g(y) \geq f(y)$, since f and g are continuous and non-decreasing functions with $f(0)=g(0)=0$. Thus, by the Intermediate Value Theorem, there exists $x \in [0,y]$ such that $g(x)+f(x)=f(y)$. □

We can translate this back to the distribution functions in order to apply it to the cake cutting problem. Namely, let $G_j : [0,1] \to [0,1]$ be defined as,

$$\begin{aligned}
&G_1 = F_1 \\
&G_2 = G_1 \circ (F_1+F_2)^- \circ F_2 \\
&\quad\vdots \\
&G_j = G_{j-1} \circ (F_{j-1}+F_j)^- \circ F_j, \forall j \in \{2,3,\cdots,n-1\}.
\end{aligned}$$

We are now able to prove a key result in this section.

Lemma 13 ([9]). *If the distribution functions of the players are non-decreasing and continuous, then all of the G_k are non-decreasing and continuous with $G_k(0)=0$.*

Proof. We prove this by induction on j. It is clear that the G_j are non-decreasing and satisfy $G_j(0)=0$. To prove continuity, define inductively $H_1(x)=x$ and $H_j = G_{j-1} \circ (F_{j-1}+F_j)^-$. Then, $H_j \circ F_j = G_j$ so it remains to prove that H_j is continuous. We note,

$$H_j = H_{j-1} \circ F_{j-1} \circ (F_{j-1}+F_j)^-, \forall j > 1.$$

By induction, H_{j-1} is continuous and $F_{j-1} \circ (F_{j-1}+F_j)^-$ is continuous by the lemma, and we are done. □

We use these G_j to construct an equitable allocation. First, define $\{c_n, c_{n-1}, \cdots, c_0\}$ where $c_0=0$, $c_n=1$ and inductively,

$$c_{j-1} = ((G_{j-1}+F_j)^- \circ F_j)(c_j), \forall j < n.$$

Then, $c_{j-1} < c_j$. So, $(0 = c_0 \leq c_1 \leq \cdots \leq c_{n-1} \leq c_n = 1)$.

Lemma 14 ([9]). *For distribution functions F_n and G_n defined as above*

$$G_j(c_j) = G_{j-1}(c_{j-1}), \forall j \in \{2,3,\cdots,n-1\}.$$

Proof. By definition, $G_{j-1}(c_{j-1}) = G_{j-1}((G_{j-1} + F_j)^{-} \circ F_j)(c_j) = G_j(c_j)$. $\square$

Now, $c_{j-1} = ((G_{j-1} + F_j)^{-} \circ F_j)(c_j)$, so applying $G_{j-1} + F_j$ to both sides gives us,

$$F_j(c_j) - F_j(c_{j-1}) = G_{j-1}(c_{j-1}) = G_j(c_j) = G_1(c_1) = F_1(c_1).$$

So, $0 = c_0, c_2, \cdots, c_{n-1}, c_n = 1$ gives an equitable allocation, as required.

Theorem 15 (Existence of Equitable Allocations)**.** *There exists an equitable allocation for any number of players and any player order. Moreover, if the density function of each player is everywhere positive, then this allocation is unique for each player order.*

Proof. We have proved that there exists an equitable allocation for the identity permutation. For any other permutation, we can simply relabel the players to have the order $(1, 2, \cdots, n)$ and apply the same argument above.

The second part can be noted using the additional observation that Lemma 13 can be strengthened further by saying that if f is strictly increasing, then $\forall y \in [0, 1]$ we can find a unique point x such that $g(x) + f(x) = f(y)$. $\square$

3.3 Equitable and Proportional Allocations

Consider the following example to demonstrate the need for this section:

Example 16. Assume players 1, 2, and 3 have valuation functions,

$$V_1(0, 1/3) = V_2(2/3, 1) = V_3(0, 1) = 1,$$

where the valuation is uniformly distributed.

In the previous subsection, we have noted that an equitable allocation exists for each player order. Since the valuation functions are not absolutely continuous (as measures) with respect to one another, the cut-points may not be unique.

Take the player order $(3, 1, 2)$. Then, $c_1 = 1/4$ and $c_2 = 11/12$ gives us an equitable allocation where each player gets a piece that they value to be $1/4$.

For the player order $(2, 1, 3)$, since we must assign a piece to Player 2 first, it is easy to see that the only equitable allocation is one that assigns each player a piece that they value to be 0. This is obtained by setting $c_1 \leq 2/3$ and $c_2 > 2/3$.

This last allocation satisfies equitability, but is not ideal. For instance, if we set $c_1 = 2/3$ and $c_2 = 3/4$, we have an allocation that is neither proportional nor envy-free. Indeed, $V_i(A_i) = 0$ for each player i, Player 2 gets the piece $[0, 2/3]$. Player 1 values Player 2's piece to be 1 and Player 3 values this piece to be $2/3$.

We thus desire allocations that simultaneously satisfy another fairness criteria. This subsection focuses on allocations that are simultaneously proportional and equitable. We first note the following result by Robertson and Webb [20] that we will not prove but is useful for the main theorem.

Lemma 17 (Robertson-Webb Model)**.** *For any number of players n, there exists a permutation π for which a proportional allocation exists.*

In the previous subsection, we have shown that there exists an equitable allocation for any player order. Therefore, if we take this permutation π, we desire an allocation that is simultaneously proportional and equitable. We note the following technical but essential lemma.

Lemma 18 ([11]). *Let A_i' be an allocation for player order π, where Player i is assigned the piece A_i'. Then, an equitable allocation A_i with the same player order satisfies,*

$$\min\{V_j(A_j'), j \in [n]\} \leq V_j(A_j) \leq \max\{V_j(A_j'), j \in [n]\}.$$

Proof. We will only prove the inequality on the left hand side as the second inequality can be shown analogously.

As before, without loss of generality, we can assume that π is the identity permutation. Let the A_i' be given by cut-points $\{c_0' = 0 \leq c_1' \leq \cdots \leq c_{n-1}' \leq c_n' = 1\}$ and the A_i be given by cut-points $\{c_0 = 0 \leq c_1 \leq \cdots \leq c_{n-1} \leq c_n = 1\}$.

Since the A_i give equitable allocations, $V_i(A_i) = V_j(A_j), \forall i, j, \in [n]$. Thus, proving the result for some i resolves the case for all other players. There are three cases:

1. $c_1 \geq c_1'$, thus $[0, c_1'] \subset [0, c_1]$. Thus, $V_1(A_1) \geq V_1(A_1')$.
2. $c_j < c_j', \forall j \in [n-1]$. Then, $[c_{n-1}', 1] \subset [c_{n-1}, 1]$. Therefore, $V_n(A_n) \geq V_n(A_n')$.
3. There exists $k < n$ such that $c_j \leq c_j', \forall j \in [k-1]$ and $c_k \geq c_k'$. Then, $[c_{k-1}', c_k'] \subset [c_{k-1}, c_k]$. Thus, $V_k(A_k) \geq V_k(A_k')$, as desired.

□

The following theorem is a consequence of taking the player order given by the Robertson-Webb model and Lemma 18.

Theorem 19 (Equitable and Proportional Allocation). *For any number of players, there exists a player order that admits an allocation that is simultaneously proportional and equitable.*

Another consequence of Lemma 18 is that if there two (or more) equitable allocations for the same player order say A_i and A_i', then $V_i(A_i) = V_i(A_i')$.

3.4 Equitable and Envy-Free Allocations

Envy-freeness and proportionality are equivalent for $n = 2$. Therefore, there exists a proportional, equitable, and envy-free allocation for two players. What about three or more players?

For $n \geq 4$, there are no known envy-free allocations that use a bounded number of cuts, and certainly not simple ones that only use $n - 1$ cut-points. The known procedures make heavy use of 'trimming', which is discussed by Even and Paz [14]. So, we focus on the case for $n = 3$. As it turns out, it is not always possible to have an allocation that is simultaneously envy-free and equitable and we give an explicit counter-example. This example is inspired by that given by Brams et al. [5].

Example 20. Assume $n = 3$ with value density functions,

$$v_1(x) = \begin{cases} -2x + 3/2, x \in [0, 1/2] \\ 2x - 1/2, x \in (1/2, 1] \end{cases}$$

$$v_2(x) = \begin{cases} -x + 5/4, x \in [0, 1/2] \\ x + 1/4, x \in (1/2, 1] \end{cases}$$

$$v_3(x) = 1$$

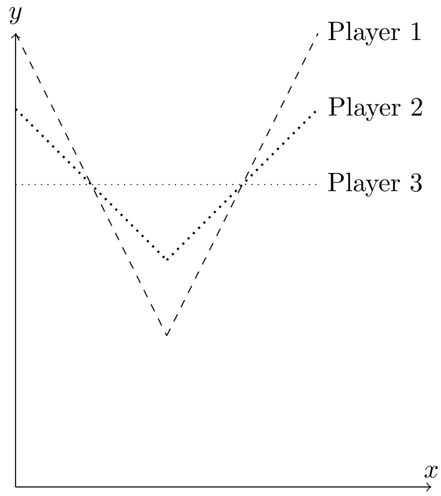

Figure 2: Counterexample for Envy-Free and Equitable Allocation

We seek cut-points c_1 and c_2 such that for some player order, the corresponding allocation is simultaneously envy-free and equitable. First, we note that $c_1 < 1/2$. If $c_1 \geq 1/2$, then the player receiving the piece $[0, c_1]$ would value her piece to be at least $1/2$ and the entire remaining piece would be valued to be at less than $1/2$ by the remaining players, rendering it inequitable.

The player order must be of the form $(1, 3, 2)$ and $(2, 3, 1)$. Player 3 cannot subsume the first or last position. Assume, for the sake of contradiction, that she is in the first position. Then, $V_3(0, c_1) = c_1$, and we would like that $V_1(0, c_1) \leq c_1$ and $V_2(0, c_1) \leq c_1$, by envy-freeness. But, a simple calculation shows that this is only satisfied for $c_1 \geq 1/2$, which is not allowed. A similar argument can also be applied for Player 3 being in the last position.

For player orders $(1, 3, 2)$ and $(2, 3, 1)$, the following equality must be satisfied in order for the allocation to be equitable: $c_1 = 1 - c_2$. Else, one of Players 1 or 2 would envy the other player's piece. But, if this equality is satisfied, then we cannot have an equitable allocation since Player 1 will receive a piece she values more than Player 2 values her piece.

Envy-freeness is a harder criteria to satisfy than proportionality and this example highlights some of the difficulties underlying the simultaneous satisfaction of certain fair division properties.

4 Equitable Procedures

In this section, we present results which not only establish the existence of allocations satisfying certain properties but also give explicit ways of finding them. The results are by Brams et al. [5] and Cechlárová et al. [11].

We assume that the players have valuation functions that are absolutely continuous with respect to each other as measures. Therefore, any piece of cake that has positive value for at least one player cannot have zero value for any other player.

4.1 Equitability for Two Players

In the previous section, we have shown that there exists an equitable allocation for two players for any player order. We have not, however, given an explicit procedure for finding it. In this section, we discuss procedures that satisfy certain notions of equitability. In particular, we present *proportional equitability* and discuss its advantages.

4.1.1 Proportional Equitability

An allocation is said to be proportionally equitable if,

1. it first ensures that each player gets a piece that she values to be exactly a half,
2. then allocates the remainder of the cake (the *surplus*) such that each player gets the same proportion of the surplus as she values it.

That is, the valuation of each player over the surplus is renormalized to be 1 and the surplus is allocated equitably with respect to the renormalized valuations. Hence, a player that values the surplus more will receive more of it. Step 1 ensures that this is an overall proportional allocation and Step 2 ensures that it is equitable on the surplus. The allocation need not necessarily be equitable on the whole cake. We give an informal description of an algorithm that takes the players value density functions as inputs and outputs a proportionally equitable allocation.

Algorithm 1 The Surplus Algorithm

1: Players 1 and 2 independently report their value density functions $v_1'(x)$ and $v_2'(x)$ over the entire cake.

2: Determine points a and b, where without loss of generality $a \leq b$, such that the $V_i(0, a) = 1/2$ and $V_j(b, 1) = 1/2$ for $i \neq j \in \{1, 2\}$.

3: If $a = b$, then $[0, a]$ and $(a, 1]$ are randomly assigned to the two players.

4: If $a < b$, assume without loss of generality that Player 1 is assigned the piece $[0, a]$ and Player 2 is assigned the piece $[b, 1]$. Determines a point c such that,

$$\frac{\int_a^c v_1'(x)dx}{\int_a^b v_1'(x)dx} = \frac{\int_c^b v_2'(x)dx}{\int_a^b v_2'(x)dx}. \tag{8}$$

5: Assign piece $[0, c]$ to Player 1 and $(c, 1]$ to Player 2.

This is not a finite algorithm. For instance, Step 4 cannot be computed using only Cut and Eval queries. Similarly, Step 2 issues a Cut query to Player 1 and a 'Reverse-Cut' query to Player 2. Reverse-Cut queries are those, for a value $y \in [0, 1]$, determine the largest value of x such that the value of the interval $[x, 1]$ is y. While this query is sometimes included in the list of allowable queries, it cannot be realized using a finite combination of Cut and Eval queries. Therefore, it is not truly finite.

As always, we do not assume that either player has any information about the other player's preference. We also do not assume that the players are being truthful. i.e., $v_i(x) = v_i'(x)$ where the v_i are the players true value density functions.

In Step 2, we call a and b the *midpoints* with respect to Players i and j, respectively. If the midpoints are the same, then the algorithm halts and issues an exact (and thus equitable and proportional) allocation. If not, then we reach the main contribution of this protocol, which is Step 4. The surplus (a, b) is being allocated such that each player gets the same proportion of the surplus as she values it. When both players have uniform densities over the surplus, then this results in an equitable allocation. If not, this step ensures that the player who values the surplus more gets more of the piece, resulting in an allocation that is not absolutely equitable.

Lemma 21 ([5]). *The Surplus Algorithm, which results in a proportionally equitable allocation, is weak-truthful.*

Proof. Using the definition of weak-truthfulness, it suffices to show that if either of the players is not truthful about her value density function, then she might end up with a piece that she values less. Without loss of generality, assume that Player 1 is not truthful about her value density function while Player 2 is truthful. There are two ways in which Player 1 can be dishonest. She can report a different midpoint a or she can misreport her value on the surplus (a, b). Both of these contradict the property that Player 1 is risk-averse, which we show below:

In the first case, Player 1 could report a different value density function such that the referee determines a different midpoint $a' \neq a$, where a is her actual midpoint. Player 1 desires to increase her initially allocated piece by moving a' to the right. This contradicts that she is risk-averse since she does not know the value of b. Indeed, if she moved a' such that $a' > b$, then her initial allocated piece will be $(a', 1)$, which she values to be less than a half. The surplus in this case will be (b, a') and she is assured some non-complete subset of this subinterval. Therefore, since $a < b$, she will value the overall assigned piece to be less than a half.

The second is if Player 1 is truthful about her midpoint a, but misreport her value on the surplus (a, b). That is, she wants to move c to the right to obtain a new cut-point c' and thus obtain more of the surplus. This can be done by either decreasing the numerator on the left hand side of (8) or increasing its denominator. In order to do this, Player 1 must know the value of b, which she does not. Therefore, she is also ignorant of the value of c before the procedure is completed. Therefore, Player 1 cannot assuredly reduce her value on the interval $[a, c]$ relative to $[a, b]$ in order to get more of the surplus. In fact, such an attempt might move c to the left which would reduce the value of the surplus that she is assigned. □

Therefore, if a player is truthful about her value density function, she can guarantee at least half of the share and generally more while being dishonest can result in a share that she values to be even less than a half.

4.1.2 Absolute Equitability

We consider a natural potential extension of the Surplus Algorithm. After both players report their value density functions, the algorithm could instead determine the point e such that the resulting allocation is overall equitable. This is achieved by setting the denominators in (8) to be 1.

We call c the proportionally equitable cut-point and e the (absolute) equitable cut-point. Using the previous section, we have noted that e is unique. It turns out that this modification comes at a high cost.

Lemma 22 ([5]). *The Absolute Equitable Algorithm, which uses e as a cut-point, is not weak-truthful.*

Proof. We prove this by showing that there is a way that Player 1 can misreport her value density function to assuredly increase her piece regardless of what Player 2 does.

The new value density function that she reports must have the same equitable point a since Player 1 is unaware whether e lies to the left or right of a on the interval. Thus, to increase her final allocated piece, she must decrease her value on the interval (a, e), without knowing the location of e.

This can be achieved by reporting a value density function that concentrates nearly half of its value near 0 and the other half near 1, while maintaining a as the midpoint of this new value density function. The remaining small portion is spread out over the remainder of the interval. This can be done by letting the value strictly between the midpoint and both ends of the cake approach 0 in the limit.

This decreases Player 1's value to both the left and the right side of the midpoint. Therefore, regardless of which side the equitable point lies on, Player 1 will be assigned more of the surplus by moving the cut-point closer to b, which is Player 2's midpoint. $\square$

4.2 Equitable Allocations for $n \geq 3$

In this section, we present a simple general algorithm that results in an equitable allocation for three or more players as presented by Brams et al. [5].

Algorithm 2 Equitable Algorithm

1: Players $1, 2, \cdots, n$ report their value density functions $v_1'(x), v_2'(x), \cdots, v_n'(x)$ over the whole interval.
2: For each possible permutation of the players, determine the $n-1$ equitable cut-points.
3: From the $n!$ possible permutations, chooses the assignment that maximizes the value of the pieces the players receive.

This, again, is not a finite algorithm. Step 3 ensures that it is, however, efficient in the sense that no player can get a better piece without another player being worse off.

This procedure is similar to the Absolute Equitable Algorithm in the previous section with two players. In that case, a player can assuredly get a piece that she values more by reporting a value density function that minimizes her value around her midpoint. One might wonder whether such a strategy can also be adapted to the Equitable Algorithm.

As it turns out, this procedure is in fact weak-truthful. The key here is that in the two person case, Player 1 is truthful about her midpoint a and is therefore indifferent between $[0, a]$ or $[a, 1]$. She is able to assuredly increase the amount that she receives from the surplus by decreasing her value around the midpoint. When n is greater than two, there is no specified surplus to be distributed.

Lemma 23 ([5]). *The Equitability Algorithm is weak-truthful.*

Proof. Without loss of generality, assume that Player 1 is not truthful about her value density function while the other players are. Player 1 desires to increase her allocated piece $[a, b]$ by decreasing a or increasing b. However, Player 1 does not know the reported value density functions of the other players. Therefore, she does not know the values of a and b. Therefore, she cannot adapt a strategy like that in the absolute equitability section to assuredly increase her assigned piece. $\square$

Note that in the third step, the algorithm picks the allocation that increases the common value from the $n!$ potentially distinct possibilities. At least one of these is a proportional allocation. In this next lemma, we assume that the players are truthful about their value density functions.

Lemma 24 ([5]). *The equitability procedure results in a proportional allocation.*

Proof. Assume that in the second step, the algorithm uses Moving-Knife 1. It is clear that this moving-knife procedure assigns all but one of the players pieces that they value to be $1/n$ and assigns the last player a piece that they value to be at least $1/n$. The last remaining piece can be divided into a piece the last player values to be $1/n$ and a surplus. As described by Shishido and Zeng [22], by moving these initial cut-points to the right, the referee can give each player an equal amount greater than $1/n$, thereby exhausting the surplus. □

Thus, if a player is truthful about her value density function, she is assured at least $1/n$ of the cake regardless of whether the other players are truthful. Moreover, if a player is not truthful, she may even get a piece that she values to be less than $1/n$. An example is presented by Brams et al. [5].

4.3 Discussion of Procedures

Throughout this section, we have discussed three main procedures – the Surplus Algorithm, Absolute Equitable Algorithm, and Equitable Algorithm – that may be used to achieve allocations that satisfy equitability in some sense. We also present the delayed discussion about the relative advantages and disadvantages to using these procedures.

The Surplus Algorithm outputs an allocation that satisfies equitability in the relative sense. This approach, introduced only in 2006 by Brams et al. [5], manages to be weak-truthful which the analogous procedure that uses equitable points instead of proportionally equitable points does not.

Both the Surplus and Absolute Equitable Algorithms are highly information-demanding in that they require the players to report their value density functions over the entire cake, which some of the previous procedures such as Moving-Knife 1 do not. This can be minimized if we presented this as a protocol that first asks the players to indicate their midpoints and then submit their value density function for the half of the cake which includes the midpoint of the other player. While this makes the procedure less information-demanding, it partially compromises the anonymity of the players' preferences.

The Equitable Algorithm is also information-demanding in that it asks each player to report her entire value density function over the entire cake. In the previous section, it has been show that envy-freeness and equitability may be incompatible for more than two players. Therefore, while it is absolutely equitable and proportional by construction, it may not be envy-free. The same counterexample can be used to see this. However, like the Surplus Algorithm, it can be carried out such that it is weak-truthful.

Note that the Equitable Algorithm is equivalent to the Absolute Equitable Algorithm for $n = 2$, except the former calculates the equitable points for both player orders $(1, 2)$ and $(2, 1)$. However, the Absolute Equitable Algorithm is not weak-truthful while the Equitable Algorithm is. In both cases, the players have an incentive to be truthful about their $1/n$ points. For $n = 2$, the player knows that she is guaranteed a piece that includes either the interval $[0, a]$ or $[a, 1]$, where a is her midpoint. For $n > 2$, the player has to report $n - 1$ points, which we denote by $a_1, \cdots, a_{n-1}$; and there are potential allocations that do not include any of the subintervals $[a_i, a_{i+1}]$. Indeed, Moving-Knife 1, which is used in the proof of Lemma 24 constantly evaluates the $1/n$ points after a player calls 'cut'. Thus, there are no points around which a player can minimize her value to assuredly increase her allocated piece.

5 Finite Algorithms

We focus on finite algorithms in this section. In particular, we are concerned with what can be accomplished using such algorithms. The results presented in the next subsection are mostly by Cechlárová et al. [11] and we give a discussion of these and related results in the last subsection.

5.1 Existence Results for Three Players

In Section 3, we proved that there exists a simple allocation for each player order for any number of players and that there exists a player order for which the allocation is simultaneously proportional and equitable. This result does not, however, give a finite step-by-step procedure for finding these allocations. As it turns out, such procedures are elusive. We will prove that no finite algorithm can give an allocation that is simultaneously equitable and proportional for more than two players. This result, which is presented fully by Cechlárová et al. [11], was inspired by Stromquist [23].

Definition 25. A **stiff measure system** for three players 1, 2, and 3, is a triple of value density functions $\{v_1, v_2, v_3\}$ such that each is everywhere positive and there exists four parameters $\{s, t, x, y\}$ such that the following holds:

	$[0, x]$	$[x, y]$	$[y, 1]$
V_1	t	s	s
V_2	s	t	s
V_3	s	s	t

where $0 < s < 1/6, 2s + t = 1$, and $0 < x < y < 1$.

Lemma 26 ([11]). *If Players* 1, 2, *and* 3, *have value density functions forming a stiff measure system with parameters defined above, then there exists a unique simultaneously proportional and equitable allocation with player order* $(1, 2, 3)$ *and cut-points* x *and* y.

Proof. We first show that a proportional allocation must have player order $(1, 2, 3)$. Assume otherwise. Then, we have three cases:

- Player 3 comes first. In this case, Player 3 must be given a subinterval $[0, z]$ for $z \geq y$, since $s < 1/6$ and the value of Player 3 on the interval $[0, y]$ is less than $1/3$. But, the remaining piece is a subinterval of $[y, 1]$, which has value less than $1/3$ for both the remaining players.
- Player 2 comes first. In this case, she must get a subinterval $[0, z]$ such that $z > x$; but, in that case Player 1 must get a subinterval of $[x, 1]$, which would have value less than a third, again because $s < 1/6$.
- We have player order $(1, 3, 2)$. In this case, Player 2 would have to get a piece of the form $[z, 1]$, where z is to the left of y. But, the piece $[0, z]$ has value less than $1/3$ for Player 2, leading to a contradiction.

Therefore, we conclude that the only possible player order is $(1, 2, 3)$. It is clear that this proportional allocation can be achieved by using cut-points x and y. Moreover, it is unique for this player order and hence overall since the valuation functions are everywhere positive. $\square$

The following lemma is useful for proving the main theorem of this section. Since it is overly technical without being insightful, we state it without proof. Its proof in the particular form that we use it can be found by Cechlárová et al. [11] and is heavily inspired by Stomquist [23] (Lemma 3).

Lemma 27 ([11])**.** *Suppose that $\{v_1, v_2, v_3\}$ form a stiff measure system with parameters denoted as usual. Let $\epsilon > 0$ and $\mu > 0$ be such that $2/3 + \mu < t < 1 - \mu$.*

Let $t' \in (2/3 + \mu, 1 - \mu)$ be sufficiently close to t depending on ϵ, but distinct from t. Then, there exists a stiff measure system $\{v_1', v_2', v_3'\}$ with parameters $\{s', t', x', y'\}$ such that,

1. *$v_1'(z) = v_1(z), \forall z \in [0, 1]$,*
2. *$v_j'(z) = v_j(z)$ for all $j \neq 1$ and all z that are outside the ϵ-neighborhood of x and y,*
3. *x' and y' are within ϵ-neighborhoods of x and y and distinct from x and y, respectively.*

The interpretation of this lemma is as follows: assume that one of the players, say Player 1, knows that the three probability densities form a stiff measure system. Player 1 knows her own density in full and might know the densities of the other players' outside the neighborhoods of suspected cut-points. However, the lemma tells us that she cannot determine the parameters of the stiff measure system by herself. This is used to prove the main theorem of this section.

Theorem 28 ([11])**.** *There is no finite algorithm for finding an allocation that is simultaneously proportional and equitable for 3 players.*

Proof. Assume otherwise, i.e., there exists a finite algorithm $\mathfrak{A}$ that gives the desired allocation. We will further assume that the value density functions $\{v_1, v_2, v_3\}$ form a stiff measure system with parameters $\{s, t, x, y\}$.

Assume that the algorithm has already performed $r \in \mathbb{N}$ steps. If, at this step, there have been no marks at x or y, then the algorithm must proceed to the next step. If, on the other hand, $\mathfrak{A}$ made a mark on at least one of the points (say x), then by the previous lemma all the marks made so far could have been obtained for another stiff measure system $\{v_1', v_2', v_3'\}$ with parameters $\{s', t', x', y'\}$. But then we can take ϵ such that no mark is within an ϵ-neighborhood of x and y. This implies that $\mathfrak{A}$ has not found the correct cut-point and must thus proceed to the next step, rendering the algorithm infinite. □

5.2 Extension of Existence Results

The results in the previous subsection hold in general for any n. To extend this for any number of players, we let $0 < s < \frac{1}{n(n-1)}$ and $t = 1 - (n-1)s$, and consider value density functions which form a stiff measure system in the sense of the following diagram. This reduces to the previous case for $n = 3$.

	$[0, x_1]$	$[x_1, x_2]$	$\cdots$	$[x_{n-1}, 1]$
V_1	t	s	$\cdots$	s
V_2	s	t	$\cdots$	s
$\vdots$	$\vdots$		$\ddots$	$\vdots$
V_n	s	s	$\cdots$	t

That is, the entries follow the rule,

$$V_i(x_j, x_{j+1}) = \begin{cases} s, & \text{if } j \neq i \\ t, & \text{otherwise.} \end{cases}$$

Using the same arguments, we can see that the only player order that admits a proportional allocation is $(1, 2, \cdots, n)$. In particular, we apply a proof by contradiction where if we assume that Player j, for $j > i$, were in position i, then her piece would end in a point $y > x_j$. But, this would not leave enough for the remaining players.

We get the desired allocation using the cut-points $(x_1, x_2, \cdots, x_{n-1})$ for player order $(1, 2, \cdots, n)$. Each player gets a piece that they value to be t, which satisfies $t > \frac{(n-1)}{n} > \frac{1}{n}$. Thus, this is the unique proportional and equitable allocation since the v_i are everywhere positive. Using the same argument as in the previous section, we remark that this cannot be realized using a finite algorithm.

5.3 Discussion of Finite Algorithms

The key property of finite algorithms is that, as implied by their name, they require a finite number of specified steps to complete. In previous sections, we have discussed certain popular methods, such as Moving-Knife 1, that require the referee to perform infinitely many computations. Finite algorithms, on the other hand, can be performed using finitely many queries.

In this section, we have shown that there exists no finite algorithm that can come up with simultaneously proportional and equitable allocations. Whether finite algorithms that can find equitable allocations exist remains an open problem.

A notable progress on this end is to give a 'finite' algorithm (one that allows the 'Reverse-Cut' query) that comes up with a simultaneously proportional and near-equitable allocation. Near-equitable allocations are those where the values the players have for their pieces do not differ by more than ϵ, for a fixed ϵ. Cechlárová et al. [11] were able to show this for any n. A truly finite algorithm that accomplishes a simultaneously proportional and equitable or near-equitable allocation is another open problem in this field.

6 Conclusion and Further Work

Problems in cake cutting are simple to state and motivate, but present a computational and mathematical challenge. Needless to say, they arise in various real-world settings including divorce settlements, inheritance allocations, airport traffic management, and of course birthday parties. As discussed by Procaccia [18], problems in fair division in general and cake cutting in particular are showing up in various contexts spanning from crowd-sourcing to political science. Due to the nature of the field, these problems also allow for conversations across various disciplines such as philosophy, economics, sociology, mathematics, etc.

In this essay, we focus on simple equitable allocations as they are natural to consider but not as well understood as proportionality and envy-freeness. We give a survey of existence results as well as explicit procedures. Relaxing one of these conditions, such as the number of cuts, would include a wider class of results than those presented in this essay. For instance, Robertson and Webb [20] give a finite near-equitable algorithm for any number of players, but this algorithm can assign various small pieces scattered across the whole cake to a player.

Some potential directions to pursue are finding truly finite algorithms that result in equitable or near-equitable allocations for any n. Progress made in this direction is the work of Cechlárová et al. [10][11], where they are able to achieve this by including the 'Reverse-Cut' query.

In line with the work by Chen et al. [12], it would also be interesting to consider restricted valuation functions such as piecewise uniform ones for equitable allocations.

Some other considerations are to model the cake not using the interval $[0, 1]$ but a circle and consider equitable allocations with geometric constraints, such as if each player demands that her piece is a square or a rectangle with a bounded length to width ratio as presented by Shishido and Zeng [21]. This is another natural question to consider as it might show up in, for instance, land allocation. There has also been a surge in interest in pie cutting, where radial cuts are made from the center of the disk. Pie cutting is even less understood and younger than cake cutting and might pose interesting challenges.

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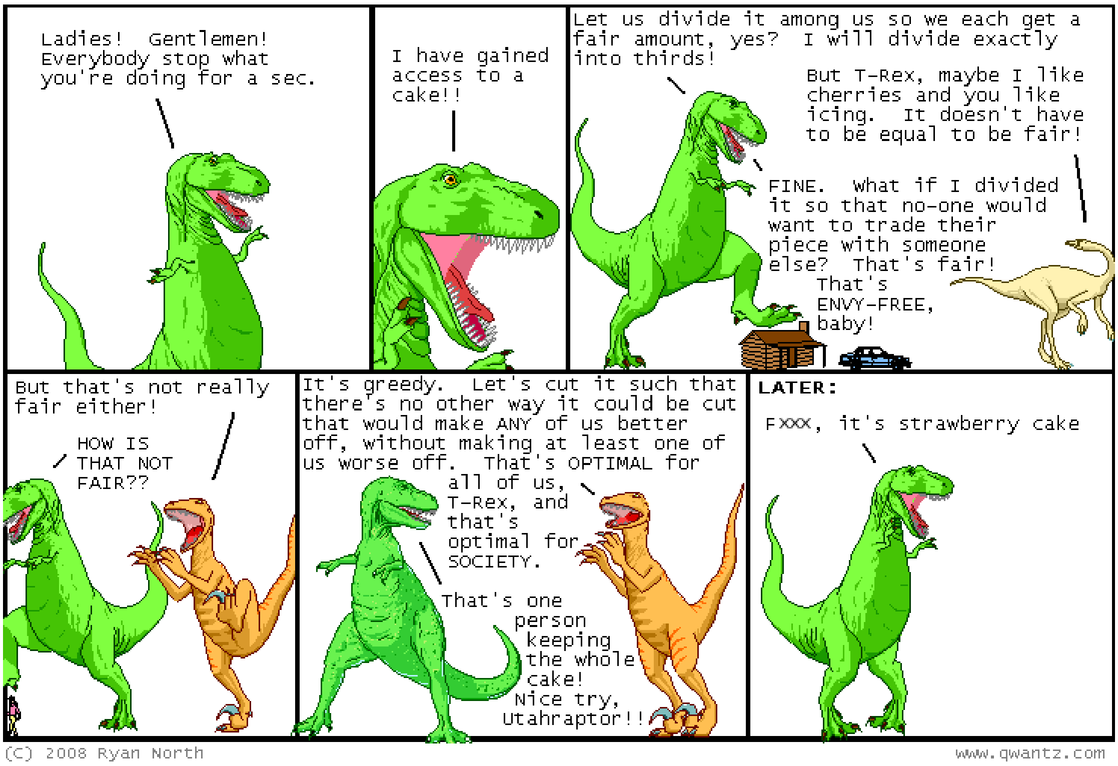

Figure 3: Comic 1345 of Dinosaur Comics [1]

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