BOOK EMBEDDINGS OF POSETS

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Abstract. We introduce a special type of graph embedding called book embedding and apply it to posets. A book embedding scheme for bipartite graphs is given, and is used to extend the embeddings to general k-partite graphs. Finally, we view the Hasse diagram of a poset as a directed k-partite graph and use this scheme to derive a book embedding for arbitrary posets. Using this book embedding scheme, we also find a bound on the book thickness of posets, which is a measure of the quality of a book embedding.

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1. Introduction

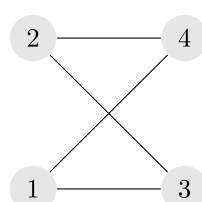
Graph embedding is an important tool in topological graph theory where, given a graph G and a surface S, we ask if G may be drawn on S such that no two edges intersect outside of the vertices.

Formally, an embedding is defined as follows:

Definition 1.1 (Graph Embedding)**.** An embedding of a graph G on a surface S is an injective function $\phi\colon G \to S$ that sends vertices of G to points of S and edges of G to simple arcs on S.

Note that the injectivity of ϕ ensures that arcs on the new surface may not intersect each other or the image of any vertex of G.

Example 1.2. Take the graph G,



As drawn above, it is not evident that G can be embedded in $\mathbf{R}^2$, but we can see this below:

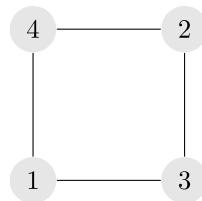


FIGURE 1. Planar drawing of G

Given a graph G, we are often interested in the types of spaces into which G may be embedded. Graphs that may be embedded in the plane, such as the example above, are called *planar graphs*. Various applications call for embeddings in different spaces, and in this paper we investigate embeddings in a class of spaces called *books*. Book embeddings are useful because they generalize to embedding in $\mathbf{R}^3$ and have applications in very-large-scale integrated (VLSI) circuit design. The number of layers required to build such a circuit can be determined by embedding the graph representation of the circuit in a book.

Before we discuss how to embed graphs in books, we must describe what this surface looks like. In a book embedding, each vertex lies on a line ℓ called the *spine* of the book. Each edge either lies in the spine or on exactly one half-plane bounded by ℓ (i.e. a plane to one side of ℓ) called a *page*. Edges on the same page do not intersect except at the vertices of the graph. An n-book is a book with n pages, and the resulting space is one we can visualize in $\mathbf{R}^3$ as the skeleton of an actual book. By convention, we allow ℓ to coincide with the z-axis, and we traverse the spine in the "up" and "down" directions. However, in the figures below, right will correspond to "up" and left will correspond to "down".

Example 1.3 (Embedding K_5 in a 3-book)**.** Figure 2 gives an example book embedding of K_5. If two edges are either both above or below the vertices and have the same color, then they are on the same page. Note that the edges along the spine may be contained in any page.

An n-book embedding of a graph G is a topological embedding that maps the vertices of G into the spine ℓ, and sends each edge of G into the interior of at most one page.

We give formal definitions of the above terms.

Definition 1.4 (Book)**.** A **book** in $\mathbf{R}^3$ is the union of a single line ℓ called the *spine*, and zero or more half-planes whose common boundary is ℓ. A book with n pages is called an n-book.

A graph admits a book embedding on k pages if its edges may be placed in k pages such that no two edges on the same page intersect. In general, a given graph has many possible book embeddings, so we are interested in comparing the "quality" or size of different embeddings. Therefore, we must define some properties that allow us to compare different embeddings.

Definition 1.5 (Book embedding)**.** An n-**book embedding** of a graph G is a topological embedding β of G into a book consisting of a spine ℓ and a set of pages $\mathcal{P}$, such that $\beta(\mathcal{V}(G)) \subset \ell$ and for each edge $e \in \mathcal{E}(G)$ there is a page $P \in \mathcal{P}$ such that $\beta(e) \subset P$.

We must emphasize the fact that edges may occupy at most one page and therefore are not allowed to cross the spine in a book embedding. If we omit this condition, then we find that all graphs can be embedded in a 3-book.

Definition 1.6 (Page-number)**.** The **page-number** of a book embedding of a graph G into a book B is the number of pages in B. Thus, an n-book has a page-number of n.

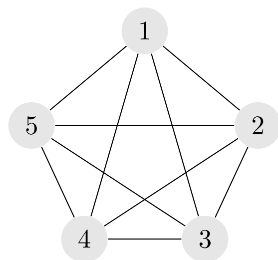


FIGURE 2. The complete graph K_5 is not planar

As we can see above, there are many graphs that cannot be embedded in the plane, much less in a one-page book. In these cases, more pages must be added to circumvent the edge intersections. For example, the complete graph K_5 shown in figure 2 is a well-known example that is not planar.

Definition 1.7 (Book thickness)**.** The **book thickness** of a graph G is the minimum number $n \in \mathbf{N}$ such that G may be embedded in an n-book. Alternatively, it is the minimum page-number of all book embeddings of G. We write $bt(G)$ for the book thickness of G.

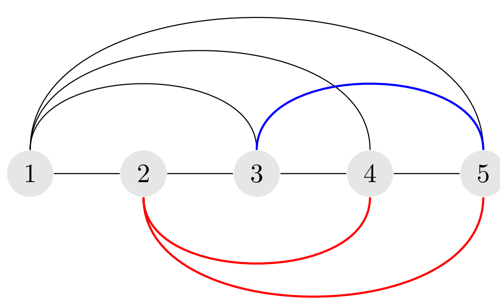


FIGURE 3. K_5 embedded in 3 pages

Figure 3 shows a 3-page embedding of K_5, thus we find $bt(K_5) \leq 3$. Soon it will become clear that the book thickness of this graph is, in fact, 3.

We begin by studying graphs with small book thickness. Before we begin discussing graphs with the smallest book thickness, we give a few definitions.

Definition 1.8 (Path)**.** A graph G with $n \geq 2$ vertices is called a **path** if G has exactly two vertices with degree 1 and exactly $n-2$ vertices with degree 2.

Definition 1.9 (Outerplanar graph)**.** A graph G is **outerplanar** if the graph G' constructed from G by adding a single new vertex and the edges connecting it to all other vertices is planar. Alternatively, G is outerplanar if it may be drawn in the plane such that no vertex is completely surrounded by edges.

Definition 1.10 (Hamiltonian graph)**.** A **Hamiltonian cycle** is a graph cycle that visits each vertex of the graph exactly once. A graph containing a Hamiltonian cycle is called a **Hamiltonian graph**.

The following proposition is due to [3].

Proposition 1.11. *Let G be a connected graph. Then,*

- $bt(G) = 0$ *if and only if G is a path.*
- $bt(G) \leq 1$ *if and only if G is outerplanar.*
- $bt(G) \leq 2$ *if and only if G is the subgraph of a Hamiltonian planar graph.*

Definition 1.12 (Width)**.** The **width** of a page P of a book B is the maximum number of edges intersecting any half-line in P perpendicular to the spine of B.

Definition 1.13 (Page-width)**.** The **page-width** of a book embedding is the maximum width of all pages of the book.

In the above example book embedding of K_5, the page of edges incident with vertex 1 has width 3, the page containing the edges incident with vertex 2 has width 2, and the page containing the edge between vertices 3 and 5 has width 1. Thus, the page-width of this embedding of K_5 is 3. Note that the embedding sending each edge to its own page has page-width 1, so every graph admits a book embedding with page-width 1.

Definition 1.14 (Printing cycle)**.** The **printing cycle** of a book embedding is the sequence of vertices ordered from bottom to top along the spine of the book.

The printing cycle of an embedding is cyclic in the sense that the vertices may be rotated or reversed while preserving the page-number of the previous embedding.

Book embeddings may also be realized as a sort of circular embedding. First, assign each page a distinct color, and give each edge the color of its page. Then, project all of the edges into a single page, so that no two edges of the same color cross. Finally, curve the spine of the book, identifying the two points at infinity to form a circle.

We may construct a circle embedding that corresponds to a book embedding by first, placing all of the vertices on a circle. Then we draw each edge as a chord in the interior of the circle. Finally, we color each edge such that no two edges of the same color intersect. Each color in this circular embedding corresponds to a single page in the book embedding. Figure 3 shows the circular version of the embedding of K_5 shown in figure 2.

The following results are clear when one considers the circular version of book embeddings.

Lemma 1.15. *Suppose G is a graph with an n-book embedding with printing cycle $\sigma = v_1, \ldots, v_p$. Then,*

- *the graph G also has an n-book embedding with printing cycle $v_2, \ldots, v_p, v_1$.*

- *the graph G also has an n-book embedding with the reverse printing cycle $v_p, \dots, v_1$.*

Proof. Suppose an n-book embedding β of G with printing cycle σ has been constructed. Convert this into the circular version of the embedding γ by identifying a point on the spine ℓ below v_1 with a point on ℓ above v_p such that the printing cycle can be read off the resulting circle clockwise, and then coloring each page of edges differently and finally projecting the edges into the chords of the circle γ. Both claims follow from simple symmetries of the circle. The first claim follows from rotating the vertices of G clockwise by $2\pi/p$, putting v_p in v_1's previous position and sending v_i to v_{i+1}'s previous position for $1 \le i < p$. Since all edges are also rotated by the same amount, no new edge intersections are created, and the new printing cycle is, in fact, $v_p, v_1, \dots, v_{p-1}$. The second claim follows by observing that no new edge crossings are created when reflecting γ about the line between the center of γ and v_1. The resulting printing cycle is $v_1, v_p, v_{p-1}, \dots, v_2$, and rotating the vertices counterclockwise by 1 gives the required printing cycle. $\square$

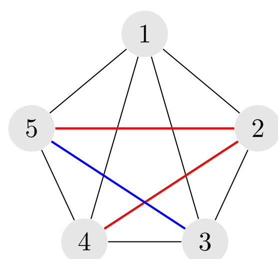


FIGURE 4. Circular book embedding of K_5

2. Posets and Hasse Diagrams

Definition 2.1 (Poset)**.** A **poset**, or **partially-ordered set** is a set E with a relation $\le$ such that for all $x, y, z \in E$, we have $x \le x$ (reflexive), $x \le y$ and $y \le z$ implies $x \le z$ (transitive), and $x \le y$ and $y \le x$ implies $x = y$ (antisymmetric).

Definition 2.2 (Hasse Diagram)**.** A **Hasse diagram** H is a directed graph representation of a poset E whose vertices represent elements of E and whose directed edges signify the $\le$ relation. That is, an edge $v \to w$ means $v \le w$.

In other words, the Hasse diagram H of a poset E is defined

$$\mathcal{V}(H) = E$$

$$\mathcal{E}(H) = \left\{ v \to w \;\middle|\; v,\, w \in E,\, v \le w,\, \nexists t \in E \text{ such that } v \le t \le w \right\}.$$

The second condition is important because, as we will see, it gives us a well-defined notion of the "level" of a vertex in the Hasse diagram. We do not allow "shortcuts" between two vertices, meaning that all directed paths between two comparable elements must have the same length. Furthermore, this condition essentially allows Hasse diagrams to encode the transitive property of posets. It is also clear that Hasse diagrams may not contain any cycles due to the antisymmetric property of $\le$.

Definition 2.3 (Indegree and outdegree)**.** Given a directed graph G, the **indegree** of a vertex $v \in \mathcal{V}(G)$ is the number of vertices $w \in \mathcal{V}(G)$ such that the directed edge $w \to v$ is in $\mathcal{E}(G)$. Similarly, the **outdegree** of v is the number of vertices $w \in \mathcal{V}(G)$ such that the directed edge $v \to w$ is in $\mathcal{E}(G)$.

Definition 2.4 (Minimal and maximal vertices)**.** Let H be a Hasse diagram. A vertex $v \in \mathcal{V}(H)$ is **minimal** if it has indegree 0. Similarly, v is **maximal** if it has outdegree 0.

Definition 2.5 (Level)**.** The **level** of a vertex $v \in \mathcal{V}(H)$, denoted $\ell(v)$, is one more than the maximum length of a path from a minimal vertex to v.

Observe that if v is minimal, then $\ell(v) = 1$, and for every vertex v on level m, there is a vertex w on level $m-1$ such that $w \to v \in \mathcal{E}(H)$. Thus every connected Hasse diagram is a directed k-partite graph whose vertices are partitioned by level.

The goal of this paper is to construct book embeddings β of Hasse diagrams that preserve levels, that is, if $v, w \in \mathcal{V}(H)$, then $\ell(v) < \ell(w)$ implies $\beta(v) < \beta(w)$ in terms of the ordering of vertices on the spine. We shall accomplish this by first examining book embeddings of bipartite graphs, extending the results to k-partite graphs, and finally apply this by giving an algorithm constructing book embeddings for arbitrary posets.

3. Bipartite and k-partite Graphs

We begin by studying the book embedding of bipartite graphs. Bipartite graphs are graphs whose vertices can be partitioned into two disjoint sets such that every edge connects vertices. This definition extends to k-partite graphs where we partition the vertices into k disjoint sets.

Definition 3.1 (Complete bipartite graph)**.** The **complete bipartite graph** $K_{m,n}$ is the graph consisting of $m+n$ vertices, which are partitioned into two disjoint sets, M containing the m vertices, and N containing the other n vertices. The edge set of $K_{m,n}$ is the set

$$\mathcal{E}(K_{m,n}) = \{(v, w) \mid v \in M,\, w \in N\}$$

of all edges between elements of M and elements of N.

Generally, the notation $K_{n_1,\ldots,n_k}$ denotes the **complete k-partite graph**, which is the graph on $n_1 + \cdots + n_k$ vertices which are partitioned into k pairwise disjoint sets $N_1, \ldots, N_k$ such that $|N_i| = n_i$, with edge set

$$\mathcal{E}(K_{n_1,\ldots,n_k}) = \{(v, w) \mid v \in N_i,\, w \in N_{i+1},\, 1 \le i \le k-1\}.$$

We are now ready to state some results about the book thickness of complete bipartite graphs.

Theorem 3.2. *For $m \le n$ with $n \ge m^2 - m + 1$, we have $bt(K_{m,n}) = m$.*

This theorem gives a bound when the sizes of the two sets of vertices are significantly different. When the two are the same, we have a slightly better bound.

Theorem 3.3. *For $m \ge 4$, we have $bt(K_{m,m}) \le m - 1$.*

We state a simple corollary that will suffice for our construction of an embedding of $K_{m,n}$.

Corollary 3.4. *For all $m, n \in \mathbf{N}$, we have $bt(K_{m,n}) \le \min(m, n)$*

We now construct a book embedding for the complete bipartite graph $K_{m,n}$. This construction is not designed to be minimal, but will be used for embedding k-partite graphs and posets.

Construction 3.5 (Embedding $K_{m,n}$ in n pages)**.** Label the m vertices "1" and the n vertices "2". Assign an ordering σ to the $m+n$ vertices on the spine such that if v is labelled 1 and w is labelled 2 then $v <_\sigma w$. Unless otherwise specified, this will be the printing cycle for all remaining book embeddings in this paper. Finally, for each vertex w with label 2, create a new page P_w, and send each edge incident with w to P_w. We can see that drawing all of the edges as nested arcs sharing a single vertex will never produce any edge crossings, which is why each vertex gets its own page. Figure 5 shows an example of this configuration.

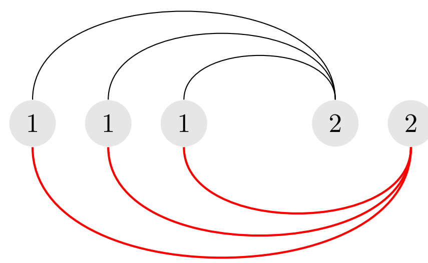


Figure 5. Example of a 2 page embedding of $K_{3,2}$

The main idea of this construction is to identify a set of vertices that are collectively incident with every edge in the graph, and then assign a single page to each of these vertices. Since all edges incident with a single vertex can be drawn on a single page, and since all edges are incident with at least one of these vertices, this is a valid book embedding. Finally, this strategy works given any printing cycle of the vertices. Denote the book thickness of a graph G with a fixed printing cycle σ by $bt(G, \sigma)$.

Definition 3.6. The **point-line covering number** of a graph G, denoted $\alpha(G)$, is the smallest number of vertices incident with every edge in G.

Lemma 3.7. *If G is a graph and σ is any listing of the vertices, then $bt(G, \sigma) \leq \alpha(G)$.*

Proof. Let $A \subset \mathcal{V}(G)$ be a subset of vertices such that each edge in $\mathcal{E}(G)$ is incident with at least one element of A. Create one page P_v for each element $v \in A$ and place all edges incident with v in P_v. Repeat this for each element of A and for each edge that has not already been embedded. This embeds every edge in $\alpha(G)$ pages since A covers $\mathcal{E}(G)$, thus $bt(G, \sigma) \leq \alpha(G)$ as desired. □

In the following lemma, we are particularly interested in the case of 4-partite graphs because they arise in the book embedding of posets, but this result generalizes to any k-partite graph with even k.

Lemma 3.8. *Given a 4-partite graph $bt(K_{m,n,p,q}) \leq \max(n, q) + p - 1$*

Proof. Due to lemma 1.11, it suffices to show this for $n \leq q$. Let $K_{m,n} = M \cup N$ and $K_{p,q} = P \cup Q$ be the partitions of the vertex sets. Construct separate embeddings of $K_{m,n}$ and $K_{p,q}$ using n and q pages respectively, as in construction 2.4. Merge the two embeddings on the spine such that all vertices of $K_{m,n}$ are below the vertices of $K_{p,q}$ on the spine. Since no edge of $K_{m,n}$ crosses an edge of $K_{p,q}$ given this printing cycle, we may reduce the number of pages from $n+q$ to q by sending the edges of $K_{m,n}$ to the q pages of $K_{p,q}$. We must now connect the vertices of N to the vertices of P. Using the page of the bottom-most vertex in N, connect each of the vertices of N to the bottom-most vertex of P. This can be done because the edges in this page are all "below" N, so each vertex in N is accessible from M. Finally, create $p-1$ new for each of the remaining vertices in P and connect them to the vertices in N as in 2.4. The resulting book has book thickness $q+p-1$, thus $bt(K_{m,n,p,q}) \leq \max(n,q) + p - 1$. □

4. Embedding Hasse Diagrams

Since Hasse diagrams are directed k-partite graphs, we can use a similar strategy to embed them in books as the 4-partite example in the previous section. In particular, we allow for an arbitrary number of levels, partition the level sets into "even" and "odd" levels, then embed each disjoint adjacent even and odd level as a bipartite graph as before. Finally, we fill in the gaps using the same strategy as before.

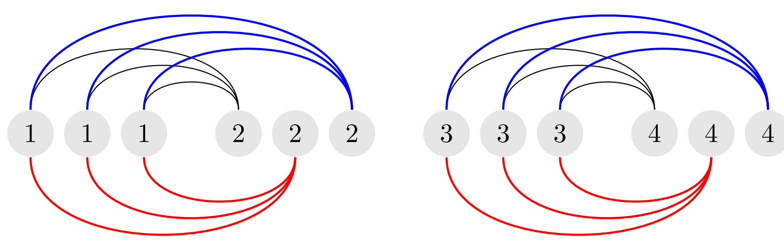


Figure 6. Step 1 – Connect disjoint pairs of adjacent even and odd levels

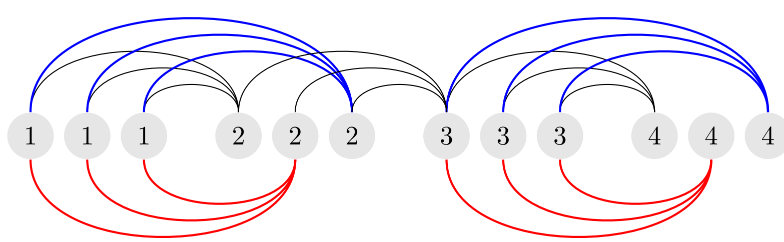


Figure 7. Step 2 – Connect the bottommost vertex in level 3 to all vertices in level 2 without creating a new page

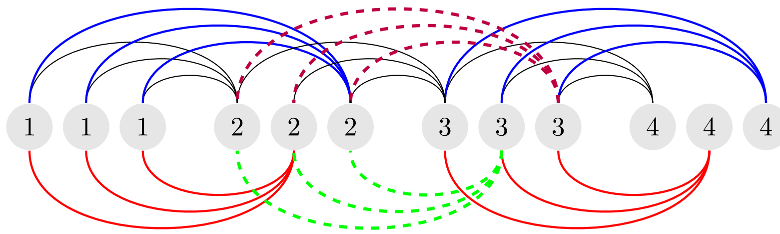


FIGURE 8. Step 3 – add $p-1$ pages to complete the embedding

Theorem 4.1 (Main Theorem)**.** *Let H be a Hasse diagram. Let $\mathcal{O}$ be the set of subsets of $\mathcal{V}(H)$ corresponding to the odd levels of H and let $\mathcal{E}$ be the set of subsets of $\mathcal{V}(H)$ corresponding to the even levels of H.*

Then, we have,

$$bt(H) \leq \max_{U \in \mathcal{O}} |U| + \max_{V \in \mathcal{E}} |V| - 1.$$

Proof. Let H be a Hasse diagram with N levels. Let $\mathcal{L} = \{K \subset \mathcal{V}(H) \,\big|\, v, w \in K \iff \ell(v) = \ell(w)\}$, that is, the set of vertices of H grouped by level. Let $E \in \mathcal{L}$ be the level of H with the most number of vertices. Suppose $E \in \mathcal{E}$ (otherwise, rotate the vertices of level 1 to the top of the printing cycle by lemma 1.11), and create $|E|$ new pages. Assign each vertex $v \in E$ its own page, P_v, and add each edge incident with v and some vertex on level $l(v)-1$ to P_v. Repeat this process of connecting all of the vertices of the remaining even levels to their neighbors on the level below. All edges may be assigned to the $|E|$ existing pages. All that remains is to connect each odd level to its even neighbor below. Let $F \in \mathcal{O}$ be the odd level with the most vertices. Connect the lowest vertex of F in the printing cycle, $v \in F$, to its neighbors below using the page of the lowest vertex in level $l(v) - 1$. Create $|F| - 1$ new pages, assign one to each of the remaining elements of F, and connect the vertices to their neighbors below using these pages. Repeat this process for the remaining odd levels without making any new levels. Finally, we have a book embedding with $|E| + |F| - 1$ pages. $\square$

It is interesting to observe that the book thickness of a Hasse diagram depends mainly on the size of its largest levels, and vertices and edges may be added to smaller or even new levels without increasing the book thickness of the resulting Hasse diagram, provided that the size of the largest even or odd level does not increase in the process.

This construction does not consider any particular ordering of vertices within their levels. It may be possible to achieve lower page-number embeddings by finding a nice printing cycle. For instance, a "beat point" of a Hasse diagram has either indegree or outdegree 1, so this single edge may be embedded as a spine arc if it is placed at the top or bottom of the level, depending on if the point is upbeat or downbeat.

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