$\{0,2\}$-Degree Free Spanning Forests in Graphs [*†]

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Abstract

Let G be a graph and S be a set of non-negative integers. By an S-degree free spanning forest of G we mean a spanning forest of G with no vertex degree in S. In this paper we study the existence of $\{0,2\}$-degree free spanning forests in graphs. We show that if G is a graph with minimum degree at least 4, then there exists an $\{0,2\}$-degree free spanning forest. Moreover, we show that every 2-connected graph with maximum degree at least 5 admits a $\{0,2\}$-degree free spanning forest.

1. Introduction

Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The *order* and *size* of G denote the number of vertices and the number of edges of G, respectively. We denote the minimum and maximum degree of the vertices of G by $\delta(G)$ and $\Delta(G)$, respectively. Also, the degree of v in graph G is denoted by $d_G(v)$. For every subset $U \subset V(G)$, we denote the induced subgraph on U by $\langle U \rangle$. Let $N_G(v)$ denote the neighbour set of v in G. The complete bipartite graph with two part sizes m

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and n is denoted by $K_{m,n}$. The complete graph and the cycle of order n, are denoted by K_n and C_n, respectively. A *factor* in a graph G is a spanning subgraph of G. A graph whose each vertex has even(odd) degree is called an *even(odd)* graph. A tree is called a *rooted tree* if one vertex has been designated the root. The *height* of a vertex in a rooted tree is the distance between that vertex and the root.

A graph is said to be *homeomorphically irreducible* if it contains no vertices of degree 2 [2]. A homeomorphically irreducible spanning tree is called HIST [2]. By an *S-degree free* spanning forest of G we mean a spanning forest of G with no vertex degree in S. Clearly, a $\{0, 2\}$-degree free spanning forest is a homeomorphically irreducible spanning forest, for abbreviation it is called HISF. In [2] the existence of HIST in a graph was investigated. In this paper we present some families of graphs which admit an HISF.

2. Results

We start by the following interesting lemma due to [1](Lemma 16.4).

Lemma 1. *If a graph is connected then it has an odd factor if and only if the number of its vertices is even.*

Corollary 1. *Every connected graph of even order admits an odd* HISF.

Proof. By Lemma 1, G has an odd factor, say H. If H is a forest, then we are done. Otherwise, we may remove all edges of a cycle from H and continue this procedure until we obtain a forest. Clearly, the resultant graph is an odd HISF and the proof is complete. □

Theorem 1. *For every graph G with $\delta(G) \geq 4$, there exists an* HISF *for G. Moreover, this* HISF *has at most one vertex of even degree.*

Proof. Clearly, we can assume that G is a connected graph. If G has even order, then by Corollary 1, there is an HISF with the desired property. Hence assume that G has odd order. For every $I \subset V(G)$, let $R(I)$ be a connected component of $\langle G \setminus I \rangle$ which has the maximum order. Assume that $U \subset V(G)$ and

$$|R(U)| = \max_{X \subset V(G), |X|=5} |R(X)|,$$

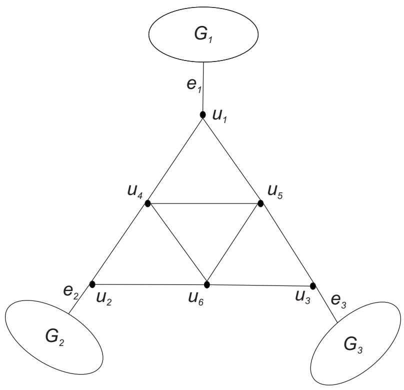
where the maximum takes over all vertex subsets X such that $|X| = 5$ and $\Delta(\langle X \rangle) = 4$. Suppose that $U = \{v, u_1, u_2, u_3, u_4\}$, $C = R(U)$ and $vu_i \in E(G)$, for $i = 1, \ldots, 4$. Let $T = G \setminus (U \cup V(C))$ and $V(T) = \{t_1, \ldots, t_{|V(T)|}\}$.

We claim that if $x \in U$ and $N(x) \cap V(C) \neq \varnothing$, then for every t_i, $1 \leq i \leq |V(T)|$, $xt_i \in E(G)$. By contradiction suppose that $xt_j \notin E(G)$, for some j. Then t_j has 4 neighbors other than x, say Q. Then we have $|V(R(\{t_j\} \cup Q))| > |V(R(U))|$, which contradicts the choice of U.

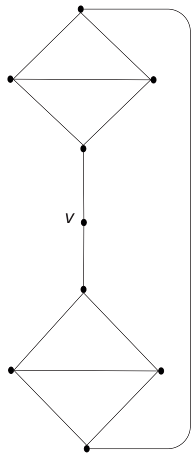
Now, two cases can be considered:

Case 1. C has even order. By Corollary 1, C has an odd HISF, say H_1. Let $x \in U$ and $N(x) \cap V(C) \neq \varnothing$. Now, define the graph H_2 with vertex set $V(T) \cup U$ and the edge set $\{vu_1, vu_2, vu_3, vu_4\} \cup \{xt_i | 1 \leq i \leq |V(T)|\}$. Clearly, $H_1 \cup H_2$ is an HISF for G. Also since C has even order and G has odd order, $|V(T)|$ is even and if $x \neq v$, then $d_{H_1 \cup H_2}(x)$ is odd. Hence the degree of each vertex of G other than v is odd.

Case 2. C has odd order. Let $x \in U$ and $N(x) \cap V(C) \neq \varnothing$. In this case, by Corollary 1, $\langle V(C) \cup \{x\} \rangle$ has an odd HISF, say H_1. Now, define the graph H_2 with the vertex set $U \cup V(T)$ and the edge set $\{vu_1, vu_2, vu_3, vu_4\} \cup \{xt_i | 1 \leq i \leq |V(T)|\}$. Obviously, $H_1 \cup H_2$ is an HISF for G. In this case, $|V(T)|$ and $d_{H_1}(x)$ are odd. Hence if $x \neq v$, then $d_{H_1 \cup H_2}(x)$ is odd and $d_{H_1 \cup H_2}(v) = d_{H_2}(v)$ is even, and hence the degree of each vertex of G in $H_1 \cup H_2$ other than v is odd. On the other hand, if $x = v$, then the degree of each vertex of G in $H_1 \cup H_2$ other than v is odd. The proof is complete. $\square$



(a) Graph G which admits no HISF.



(b) Graph H

Figure 1

Remark 1.
There exists a connected graph G with $\delta(G) = 3$ which has no HISF. To see this, first we introduce a graph G shown in Figure 1a in which G_1, G_2 and G_3 are isomorphic to the graph shown in Figure 1b. Call this graph by H. Three vertices u_1, u_2 and u_3 are joined to the vertex corresponding to v in G_1, G_2 and G_3, respectively. We claim that G has no HISF. By contradiction, suppose that G admits an HISF, say F. Since H has odd order and $\Delta(H) \leq 3$, all of e_1, e_2 and e_3 are contained in F. Moreover, since G has odd order, one vertex of u_4, u_5 and u_6 has degree 4 in F. With no loss of generality let $d_F(u_4) = 4$. On the other hand, since $d_F(u_1) \neq 2$, we conclude $u_1u_5 \in E(F)$ and there exists a triangle in F, a contradiction. Hence G has no HISF.

Before proving the next result we need two lemmas.

Lemma 2. *For every rooted tree T with root v, there exists a spanning forest F such that for each $u \in V(F) \setminus \{v\}$, $d_F(u) \notin \{0, 2\}$.*

Proof. We construct the forest F as follows: First let $E(F) = \varnothing$. In each step we consider all vertices with maximum height whose degree are 0 or 2 in F. Then we add to $E(F)$ those edges in T, which join them to their parents. Clearly, these vertices find degree 1 or 3. We continue this procedure until the vertices of height 1. Thus, the desired forest is obtained. □

Lemma 3. *Let T be a rooted tree with root v and $d_T(v) = 1$. If e is the edge incident with v and there exists a spanning subforest H of T such that $d_H(u) \notin \{0, 2\}$, for every $u \in V(T) \setminus \{v\}$ and $d_H(z) \geq 4$ for some vertex z, then there exists a spanning subforest H' of T such that $e \in (E(H) \setminus E(H')) \cup (E(H') \setminus E(H))$ and $d_{H'}(u) \notin \{0, 2\}$ for every $u \in V(T) \setminus \{v\}$.*

Proof. Let z be a vertex of T such that $d_H(z) \geq 4$ and $d(z, v)$ has the minimum possible value. Construct the spanning subforest H' as follows: Consider the path between z and v and call it by P. Let $E(H') = (E(H) \setminus E(P)) \cup (E(P) \setminus E(H))$. We claim that H' has the desired property. If $u \in V(T) \setminus V(P)$, then $d_H(u) = d_{H'}(u)$. Also if $u \in V(P) \setminus \{v, z\}$, then $d_H(u) \equiv d_{H'}(u) \pmod 2$. Since $d_H(u) \leq 3$ then $d_{H'}(u) \notin \{0, 2\}$. Moreover, $d_H(z) \geq 4$ and so $d_{H'}(z) \geq 3$. □

In the next result we show that every graph with no cut vertex whose maximum degree is large enough admits an HISF.

Theorem 2. *Every 2-connected graph of maximum degree at least 5 admits an HISF.*

Proof. First one may assume that the graph is of odd order. Otherwise, by Corollary 1, there exists an HISF for G. Let $v \in V(G)$ and $d_G(v) \geq 5$. Assume that $N_G(v) = \{v_1, \ldots, v_k\}$. We claim that if $G \setminus \{v_1, v_i\}$ is not connected, for every i, $2 \leq i \leq k$, then G contains an HISF. Let G_i be that connected component of

$G \setminus \{v_1, v_i\}$ which contains v. Define $H_i = G \setminus (\{v_1\} \cup V(G_i))$. We claim that H_i has three properties:

(i) H_i is connected, since $G \setminus \{v_1\}$ is connected.

(ii) $V(H_i) \cap V(H_j) = \varnothing$, because $vv_i \in E(G_j)$ and $vv_j \in E(G_i)$ and these imply that $V(H_i) \subset V(G_j)$ and $V(H_j) \subset V(G_i)$.

(iii) $N_G(v_1) \cap (V(H_i) \setminus \{v_i\}) \neq \varnothing$, for every i, $2 \leq i \leq k$, since $G \setminus \{v_i\}$ is connected.

We claim that there exists a family of subgraphs of G, say $\{H'_i \mid i = 2, \ldots, k\}$ with the following properties : (*)
H'_i is connected, $V(H_i) \subset V(H'_i)$ and $\{H'_i \mid i = 2, \ldots, k\}$ is a partition of $V(G) \setminus \{v, v_1\}$.

To see this, for every $u \in V(G) \setminus (\{v, v_1\} \cup \bigcup_{i=2}^{k} V(H_i))$, let P_u be one of the shortest paths between u and $\bigcup_{i=2}^{k} V(H_i)$ in $G \setminus \{v, v_1\}$ (Note that since $G \setminus \{v_1\}$ is connected P_u exists.) . Let the other end point of P_u be $w_u \in V(H_t)$ and t be the minimum possible number. Now, define $H'_i = H_i \cup \{u | w_u \in V(H_i)\}$ for $i = 2, \ldots, k$. Obviously, the set $\{H'_i \mid i = 2, \ldots, k\}$ has the mentioned properties and the claim is proved.
Among sets with (*) properties, let $\{H'_2, \ldots, H'_k\}$ be a set with the minimum value of $\sum_{2 \leq i,j \leq k, i \neq j} \left| \{uv \mid u \in V(H'_i) \setminus \{v_i\}, v \in V(H'_j) \setminus \{v_j\}, uv \in E(G)\} \right|$. Call this minimum value by m.

We claim that $m = 0$. By contradiction suppose that there exists a vertex $u \in V(H'_i) \setminus \{v_i\}$ and $w \in V(H'_j) \setminus \{v_j\}$, where $uw \in E(G)$. Let Γ be the set of vertices which are joined to w by a path in $H'_j \setminus V(H_j)$. One can remove Γ from H'_j and add it to H'_i to obtain a value less than m, a contradiction. Hence we are done.

Let T_i be a spanning rooted tree of H'_i with root v_i, for $i = 2, \ldots, k$. Let T be a spanning tree of $G \setminus \{v_1\}$ with the edge set $\bigcup_{i=2}^{k} E(T_i) \cup \{vv_2, \ldots, vv_k\}$. By Lemma 2, T has a spanning forest F in which the degree of each vertex $x \in V(T) \setminus \{v\}$ is neither 0 nor 2.

First we claim that one can assume that exactly one H'_i has odd order, for $i \in \{2, \dots, k\}$. Since G has odd order, then H'_i has odd order, for at least one $i \in \{2, \dots, k\}$. Furthermore, if there exist two indexes i, j, such that H'_i and H'_j have odd order, then either $vv_j, vv_i \in E(F)$ which by adding vv_1 to F we obtain an HISF for G or by Lemma 3, there exists a subforest F' of T such that for every $u \in V(T) \setminus \{v\}$, $d_{F'}(u) \notin \{0, 2\}$ and $vv_j, vv_i \in E(F')$. Hence by adding vv_1 to F' we obtain the desired HISF for G and the claim is proved.

Now, we claim that there exists $w_i \neq v_i \in V(H'_i)$ such that $H'_i \setminus \{w_i\}$ is connected, and moreover $w_i \in N_G(\{v_1, \dots, v_k\} \setminus \{v_i\})$. To see this, we note that, by the property (iii), $S_i = N_G(\{v_1, \dots, v_k\} \setminus \{v_i\}) \cap V(H'_i) \neq \varnothing$. Now, let $x \in S_i$. If $H'_i \setminus \{x\}$ is connected, then we are done. Otherwise, let C be a component of $H'_i \setminus \{x\}$ not containing v_i. Since $G \setminus \{x\}$ is connected, $S_i \cap V(C) \neq \varnothing$. Continue this procedure with one vertex of $S_i \cap V(C)$, say x'. Similarly, if $H'_i \setminus \{x'\}$ is not connected, then let C' be a component of $H'_i \setminus \{x'\}$ not containing v_i. Clearly, $|V(C')| < |V(C)|$ and $S_i \cap V(C') \neq \varnothing$. By continuing this procedure we obtain w_i.

If there are two indexes i, p such that H'_i, H'_p have even order and $v_p w_i \in E(G)$, then we claim there exists an HISF for G. To see this, let T' be the spanning tree of $G \setminus \{v_1, w_i\}$ and F' be the spanning subforest of T' obtained from Lemma 2. If $vv_i \notin E(F')$ or $vv_p \in E(F')$, since $|V(H'_i) \setminus \{w_i\}|$ is odd and $|V(H'_p)|$ is even, then we can use Lemma 3. Hence there exists a subforest F'' of T' such that $vv_i \in E(F''), vv_p \notin E(F'')$ and $d_{F''}(u) \notin \{0, 2\}$, for every $u \in V(T') \setminus \{v\}$. Now, if we add $v_p w_i, vv_p$ and vv_1 to F'', then the desired HISF in obtained and the claim is proved. Hence there exists two indexes $2 \leq i, j \leq k$ such that H'_i, H'_j have even order and either $v_1 w_i, v_1 w_j \in E(G)$ or $v_p w_i, v_p w_j \in E(G)$ in which H'_p has odd order. Let T' be the spanning tree of $G \setminus \{v_1, w_i, w_j\}$ and F' be the spanning subforest of T' obtained from Lemma 2. If $vv_i \notin E(F')$ or $vv_j \notin E(F')$, since $|V(H'_i) \setminus \{w_i\}|$ and $|V(H'_j) \setminus \{w_j\}|$ are odd, then we can use Lemma 3. Hence there exists a subforest F''

of T' such that $vv_i, vv_j \in E(F'')$ and $d_{F''}(u) \notin \{0,2\}$, for every $u \in V(T') \setminus \{v\}$. In the first case that $v_1w_i, v_1w_j \in E(G)$, by adding v_1w_j, v_1w_i and vv_1 to F'' we obtain an HISF for G. In the other case, by adding v_pw_i, v_pw_j and vv_1 to F'' the desired HISF is obtained and we are done.

Therefore one may assume that $G \setminus \{v_1, v_2\}$ is connected. First we claim that if $G \setminus \{v_1, v_2, v_i\}$ is connected for some $i = 3, \ldots, k$, then G contains an HISF. To see this, since $G \setminus \{v_1, v_2, v_i\}$ has an even order, by Corollary 1, it has an HISF F. Now, by adding vv_1, vv_2 and vv_i to F we obtain the desired HISF for G. Hence suppose that $G \setminus \{v_1, v_2, v_i\}$ is not connected, for $i = 3, \ldots, k$. Now, define G'_i and L_i as follows:

The graph G'_i is the connected component of $G \setminus \{v_1, v_2, v_i\}$ containing v and $L_i = G \setminus (\{v_1, v_2\} \cup V(G'_i))$. We claim that L_i has three following properties:

(i) L_i is connected, because $G \setminus \{v_1, v_2\}$ is connected.

(ii) $V(L_i) \cap V(L_j) = \varnothing$, for every i, j, $3 \leq i < j \leq k$, because $V(L_j) \subset V(G'_i)$ and $V(L_i) \subset V(G'_j)$.

(iii) $(N_G(v_1) \cup N_G(v_2)) \cap (V(L_i) \setminus \{v_i\}) \neq \varnothing$, for every $i = 3, \ldots, k$, because $G \setminus \{v_i\}$ is connected.

Let $\{L'_i \mid i = 3, \ldots, k\}$ be the a family of subgraphs of G such that for $i = 3, \ldots, k$, L'_i is connected, $V(L_i) \subset V(L'_i)$ and $V(L'_i), i = 3, \ldots, k$ is a partition of $V(G) \setminus \{v, v_1, v_2\}$. Furthermore, assume that $\sum_{3 \leq i,j \leq k, i \neq j} |\{uv \mid u \in L'_i \setminus \{v_i\}, v \in L'_j \setminus \{v_j\}, uv \in E(G)\}| = 0$. The proof of existence of L'_i is similar to that of H'_i. By the definition of L'_i, $N_G(V(L'_i) \setminus \{v_i\}) \subset V(L'_i) \cup \{v_1, \ldots, v_k\}$. Let T'_i be a spanning rooted tree of L'_i with root v_i, for $i = 3, \ldots, k$. Let T' be a spanning tree of $G \setminus \{v_1, v_2\}$ with the edge set $\bigcup_{i=3}^{k} E(T'_i) \cup \{vv_3, \ldots, vv_k\}$. By Lemma 2, T' has a spanning subforest F_1 in which the degree of each vertex $x \in V(T') \setminus \{v\}$ is neither 0 nor 2. Now, noting to F_1, two cases may be considered:

(i) $d_{F_1}(v) \neq 0$. In this case add vv_1 and vv_2 to F_1 to obtain an HISF for G.

(ii) $d_{F_1}(v) = 0$. First we claim that one can assume that L'_i has even order, for $i = 3, \dots, k$. If not, then there is some index j, such that L'_j has odd order. Thus, then either $vv_j \in E(F_1)$ which is Case (i) or there is $z \in V(L'_j)$ such $D_{F_1}(z) \geq 4$. Now, by Lemma 3, there exists a subforest F'_1 of T' such that for every $u \in V(T') \setminus \{v\}$, $d_{F'_1}(u) \notin \{0, 2\}$ and $vv_j \in E(F'_1)$. Hence Case (i) occurs and we are done.

Now, we claim that there exists $z_i \neq v_i \in V(L'_i)$ such that $L'_i \setminus \{z_i\}$ is connected, and moreover $z_i \in N_G(\{v_1, \dots, v_k\} \setminus \{v_i\})$. To see this, we note that, by the property (iii), $S_i = N_G(\{v_1, \dots, v_k\} \setminus \{v_i\}) \cap (V(L'_i) \setminus \{v_i\}) \neq \varnothing$. Now, let $x \in S_i$. If $L'_i \setminus \{x\}$ is connected, then we are done. Otherwise, let C be a component of $L'_i \setminus \{x\}$ not containing v_i. Since $G \setminus \{x\}$ is connected, $S_i \cap V(C) \neq \varnothing$. Continue this procedure with one vertex of $S_i \cap V(C)$, say x'. Similarly, if $L'_i \setminus \{x'\}$ is not connected, then let C' be a component of $L'_i \setminus \{x'\}$ not containing v_i. Clearly, $|V(C')| < |V(C)|$ and $S_i \cap V(C') \neq \varnothing$. By continuing this procedure we obtain z_i.
If one z_i is adjacent to a vertex, say v_p in $\{v_3, \dots, v_k\} \setminus \{v_i\}$, then we construct an HISF as follows:

Consider the spanning tree T''_1 of $G \setminus \{v_1, v_2, z_i\}$ with root v. Also, let F_2 be the spanning subforest of T''_1 obtained from Lemma 2. If $vv_i \notin E(F_2)$ or $vv_p \in E(F_2)$, since $|V(L'_i) \setminus \{z_i\}|$ is odd and $|V(L'_p)|$ is even, then we can use Lemma 3. Hence, there exists a subforest F'_2 of T''_1 such that $vv_i \in E(F'_2)$, $vv_p \notin E(F'_2)$ and $d_{F'_2}(u) \notin \{0, 2\}$, for every $u \in V(T''_1) \setminus \{v\}$. Now, by adding vv_1, vv_2, vv_p and $z_i v_p$ to F'_2 we obtain an HISF for G.
So, assume that for every i, $N_G(z_i) \cap \{v_1, v_2\} \neq \varnothing$. Thus, with no loss of generality suppose that v_1 is adjacent to z_3 and z_4, because $k \geq 5$. Now, consider the spanning subtree T''_2 of $G \setminus \{v_1, v_2, z_3, z_4\}$. Now, consider the spanning forest F_3 of T''_2. if vv_3 or vv_4 are not contained in $E(F_3)$, since $|V(L'_3) \setminus \{z_3\}|$ and $|V(L'_4) \setminus \{z_4\}|$ are odd, we can use Lemma 3. Hence, there is a subforest F'_3 of T''_2 such that $vv_3, vv_4 \in E(F'_3)$ and $d_{F'_3}(u) \notin \{0, 2\}$, for every $u \in V(T''_2) \setminus \{v\}$. Now, by adding $vv_1, vv_2, v_1 z_3$ and $v_1 z_4$ to F'_3 we obtain an HISF for G. The proof is complete. $\square$

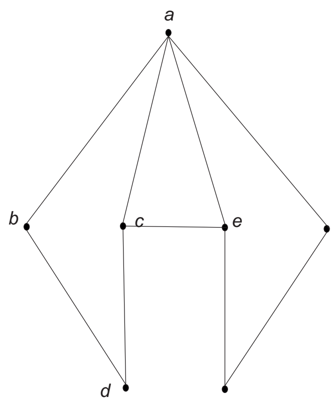


Figure 2: 2-connected graph G with no HISF

Remark 2. There exists a 2-connected graph of maximum degree 4 which has no HISF. We prove that the graph G shown in Figure 2 admits no HISF. By contradiction suppose that G admits an HISF, say F. Since G have odd order and $d_G(u) \leq 3$, for every $u \in V(G) \setminus \{a\}$, then $d_F(a) = 4$. Moreover, since $d_F(b) \neq 2$, bd is not contained in F. On the other hand, ce, cd are not contained in F, since there is no cycle in F and $d_F(c) \neq 2$. This implies $d_F(d) = 0$, a contradiction. Hence G has no HISF.

Remark 3. Every connected graph G of order n with maximum degree at least $\lceil \frac{n}{2} \rceil + 1$ admits an HISF. To see this, let v be a vertex of maximum degree in G. Let T be a spanning rooted tree of G with root v such that $d_T(v) \geq \lceil \frac{n}{2} \rceil + 1$. Let S be the set of pendant edges incident with v. By Lemma 2, T admits a spanning forest F such that for each $u \in V(F) \setminus \{v\}$, $d_F(u) \notin \{0, 2\}$. Since $d_T(v) \geq \lceil \frac{n}{2} \rceil + 1$, we have $|S| \geq 3$. Since all edges in S have to be used in F, we have $d_F(v) \geq |S| \geq 3$. Hence F is an HISF.
Note that this lower bound is sharp. To show this, let T' be a tree with one vertex of degree $\lceil \frac{n}{2} \rceil$ adjacent to two vertices of degree 1 and $\lceil \frac{n}{2} \rceil - 2$ vertices of degree 2. Clearly, T' has no HISF.

Theorem 3. *Every* 3*-connected graph admits an* HISF.

Proof. First one may assume that the graph is of odd order. Otherwise, by Corollary 1, there exists an HISF for G. If all vertices in G have odd degree, then G is of even order, a contradiction. Hence there exists a vertex $v \in V(G)$ such that $d_G(v) \geq 4$. Assume that $N_G(v) = \{v_1, \ldots, v_k\}$ with $k = d_G(v) \geq 4$. Since G is 3-connected, $G \setminus \{v_1, v_2\}$ is connected. So $G \setminus \{v, v_1, v_2\}$ can be partitioned into $G_3, G_4, \ldots, G_k$ such that for each i with $3 \leq i \leq k$, G_i is connected and contains v_i. For simplifying the argument, for $i = 1, 2$, we let G_i be the graph consisting of only v_i. Note that v has no neighbors in $G_i \setminus \{v_i\}$ for each $1 \leq i \leq k$. By symmetry, we may assume that $|V(G_1)|, \ldots, |V(G_p)|$ is odd and $|V(G_{p+1})|, \ldots, |V(G_k)|$ is even for some p with $1 \leq p \leq k$. Since $|V(G_1)| = |V(G_2)| = 1$, we have $p \geq 2$. Note that $\sum_{i=3}^{k} |V(G_i)| = |V(G)| - 3$ is even, and hence p is even. We take such a partition $G_3, \ldots, G_k$ so that

$$p \text{ is as large as possible.} \tag{1}$$

Suppose first that $p \geq 4$. Then $|V(G_i) \cup \{v\}|$ is even for each i with $1 \leq i \leq p$, and $|V(G_i)|$ is even for each i with $p+1 \leq i \leq k$. Hence by Corollary 1, $\langle V(G_i) \cup \{v\} \rangle$ has an odd HISF T_i for each i with $1 \leq i \leq p$, and G_i has an odd HISF T_i for each i with $p+1 \leq i \leq k$. Let T be a spanning subgraph of G with the edge set $\bigcup_{i=1}^{k} T_i$. Note that each vertex u in G with $u \neq v$ has odd degree, and the degree of v is at least $p \geq 4$. Hence T is an HISF of G.

Hence we may assume that $p = 2$, that is, $|V(G_i)|$ is even for all i with $3 \leq i \leq k$. Let $G_3^1, \ldots, G_3^t$ be the components of $G_3 \setminus \{v_3\}$. Since $\sum_{j=1}^{t} |V(G_3^j)| = |V(G_3)| - 1$ is odd, there exists an index j such that $|V(G_3^j)|$ is odd. Say $j = 1$ by symmetry. Note that $G_3 \setminus V(G_3^1)$ is connected. Since $G \setminus \{v_3\}$ is 2-connected, $V(G_3^1)$ has at least two neighbors u_3^1, u_3^2 in $G \setminus (V(G_3^1) \cup \{v_3\})$. By the choice of $G_3, G_4, \ldots, G_k$, we have $u_3^1, u_3^2 \neq v$, and by the choice of $G_3^1, \ldots G_3^t$, we have $u_3^1, u_3^2 \notin \bigcup_{j=2}^{t} V(G_3^j)$.

Suppose that $u_3^1 \in V(G_i)$ for some i with $4 \leq i \leq k$, say $i = 4$ by symmetry. Let $G_3' = G_3 \setminus V(G_3^1)$ and $G_4' = \langle V(G_4) \cup V(G_3^1) \rangle$. Then both G_3' and G_4' are connected, and both $|V(G_3')|$ and $|V(G_4')|$ are odd, which contradicts condition (1) for the choice of $G_3, G_4, \ldots, G_k$. Hence $u_3^1, u_3^2 \notin V(G_i)$ for any i with $4 \leq i \leq k$. This (and the symmetry) implies that $u_3^1 = v_1$ and $u_3^2 = v_2$.

By the same argument as above for G_4, there exists a component G_4^1 of $G_4 \setminus \{v_4\}$ such that $|V(G_4^1)|$ is odd, and G_4^1 has two neighbors u_4^1 and u_4^2 with $u_4^1 = v_1$ and $u_4^2 = v_2$.

Now, let $G_1' = \langle \{v_1\} \cup V(G_3^1) \cup V(G_4^1) \rangle$, $G_2' = G_2$, $G_3' = G_3 \setminus V(G_3^1)$, $G_4' = G_4 \setminus V(G_4^1)$ and $G_i' = G_i$ for each i with $5 \leq i \leq k$. Then G_i' is connected for each i with $1 \leq i \leq k$, all of $|V(G_1')|, |V(G_2')|, |V(G_3')|$ and $|V(G_4')|$ are odd, and $|V(G_i')|$ is even for each i with $5 \leq i \leq k$. Hence by Corollary 1, $\langle V(G_i') \cup \{v\} \rangle$ has an odd HISF T_i' for each i with $1 \leq i \leq 4$, and G_i' has an odd HISF T_i' for each i with $5 \leq i \leq k$. Let T' be a spanning subgraph of G with the edge set $\bigcup_{i=1}^{k} T_i'$. Then T' is an HISF of G. This proves Theorem 3. □

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