Algorithmic Bounds on Hypergraph Coloring and Covering

A Thesis
by
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Under the supervision of

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Submitted in partial fulfillment of the requirements
for award of the degree of

Bachelor of Technology
in
Computer Science and Engineering

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING,
INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR

Certificate

This is to certify that this thesis entitled **"Algorithmic Bounds on Hypergraph Coloring and Covering"**, submitted by **Praveen Kumar**, Undergraduate Student, in the *Department of Computer Science and Engineering*, *Indian Institute of Technology*, *Kharagpur*, *India*, in partial fulfillment of the requirements for the degree of **Bachelor of Technology (Hons.)**, is a record of an original research work carried out by him under my supervision and guidance.

Dated : May 9, 2011
IIT Kharagpur **Prof. S. P. Pal**

Acknowledgments

With great pleasure and a deep sense of gratitude, I express my indebtedness to Prof. Sudebkumar Prasant Pal for his invaluable guidance and constant encouragement at each and every step of my project work. He exposed me to the intricacies of relevant topics through proper counseling and discussions and always showed great interest in providing timely support and suitable suggestions.

Praveen Kumar

Abstract

Consider the coloring of a vertex-labelled r-uniform hypergraph $G(V, E)$, where V is the vertex set of n labelled vertices, and E is the set of hyperedges. In case of proper bicoloring, given two colors, we need to assign each vertex with one of the colors so that none of the hyperedges is monochromatic. This may not always be possible. In such cases, we use multiple bicolorings to ensure that each hyperedge is properly colored in at least one of the colorings. This is called the bicolor cover of the hypergraph. We establish the following result: for r-uniform hypergraphs with hyperedge set E defined on n vertices, the size of bicolor cover is upper-bounded by $O(\log |E|)$. We also extend this result for tricoloring.

Consider again the coloring of vertices of a vertex-labelled hypergraph $G(V, E)$ using a given set of c distinct colors. In this work, we try to establish bounds on the number of hyperedges that will ensure the existence of a proper c-coloring, given $|e_i| \geq r$. We define the discrepancy in case of tricoloring (c = 3) as a measure of the uniformity of a particular coloring and then try to establish upper bounds on it. Further, we generalise the definition of discrepancy and proof for bounds on discrepancy for c-coloring where $c \geq 3$.

Contents

1 Introduction **2**

1.1 Definitions . . . 2

1.2 Overview of the work . . . 4

1.2.1 Bicolor cover . . . 4

1.2.2 Tricoloring of hypergraphs . . . 5

2 Covering hypergraphs using colorings **6**

2.1 Special case with one white and remaining black vertices in an edge . . . 6

2.2 Case with general proper bicoloring . . . 8

2.3 Lower bound on the size of bicolor cover . . . 10

2.4 Tricolor cover of a hypergraph . . . 10

3 Hypergraph c-coloring **13**

3.1 Existence of proper bicoloring . . . 13

3.2 Combinatorial discrepancy for bicoloring . . . 14

3.3 Existence of proper tricoloring . . . 15

3.4 Existence of proper c-coloring . . . 16

3.5 Bounded discrepancy tricoloring . . . 17

3.6 Lower bound on discrepancy for tricoloring . . . 21

3.7 Combinatorial discrepancy for c-coloring . . . 21

4 Conclusion and Future Work **24**

BIBLIOGRAPHY **25**

List of Figures

1.1 A hypergraph $G(V, E)$ with 9 vertices and 5 hyperedges. 2
1.2 A proper bicoloring of a hypergraph $G(V, E)$ using colors red and blue. . 3
1.3 A proper tricoloring of a hypergraph $G(V, E)$ using colors red, green and blue. 4

2.1 Bicolor cover of a non-bicolorable hypergraph. 7

Chapter 1

Introduction

1.1 Definitions

Definition 1. *A **hypergraph** G is a pair $G = (V, E)$ where V is a set of elements, called nodes or vertices, and E is a set of non-empty subsets of V called hyperedges or links. Therefore, E is a subset of $P(V) \setminus \phi$, where $P(V)$ is the power set of V.*

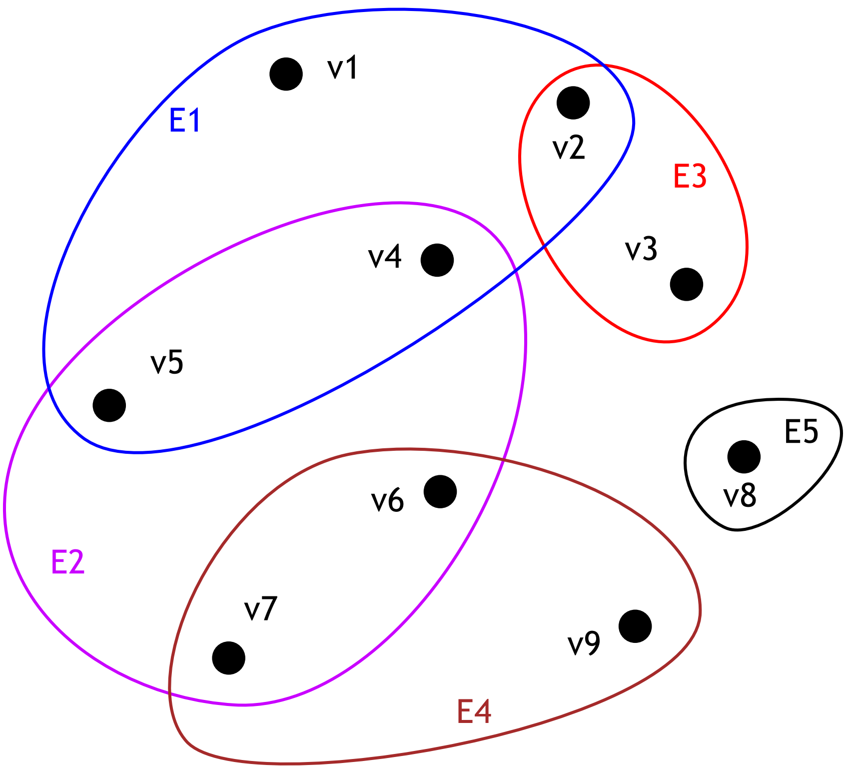

Figure 1.1: A hypergraph $G(V, E)$ with 9 vertices and 5 hyperedges.

So, a hypergraph is similar to a graph except that in case of a hypergraph, a hyperedge may connect any number of vertices while an edge in a graph can connect only two vertices.

Definition 2. *A* ***r-uniform hypergraph*** *is a hypergraph such that all its hyperedges have size r.*

Definition 3. *A* ***bicoloring*** *of a hypergraph G is a coloring of the vertices of G with two colors. Each of the vertices is assigned one of the two colors.*

A *proper bicoloring* refers to a bicoloring of the vertices of the hypergraph in such a way that no hyperedge is monochromatic, i.e., each hyperedge has atleast one vertex of each color.

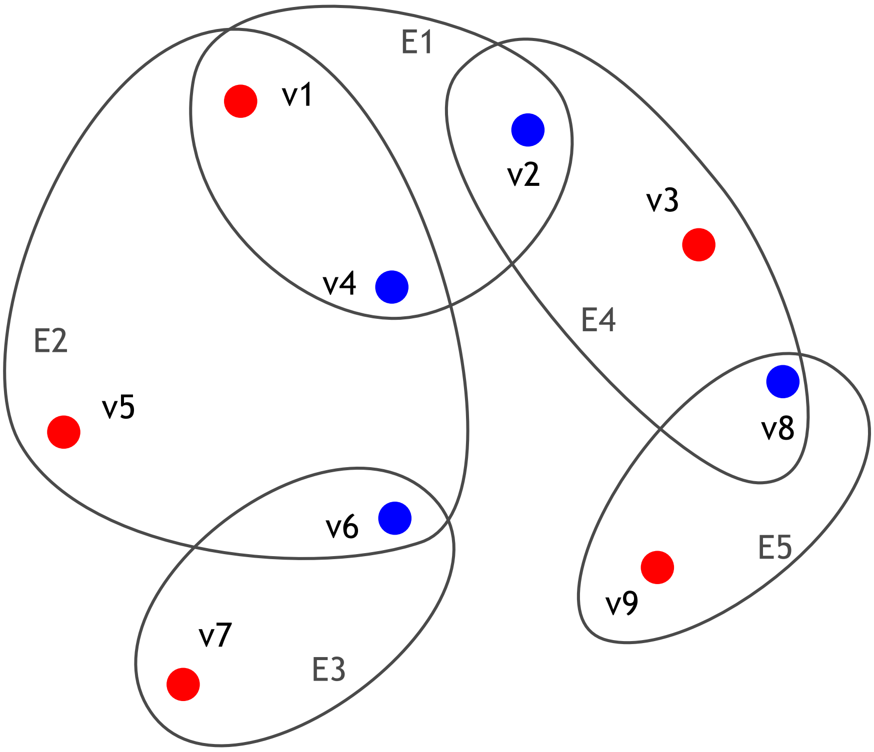

Figure 1.2: A proper bicoloring of a hypergraph $G(V, E)$ using colors red and blue. mydef

Definition 4. *A* ***tricoloring*** *of a hypergraph G is a coloring of the vertices of G with three colors.*

A *proper tricoloring* refers to a tricoloring of the vertices of the hypergraph in such a way that every hyperedge has atleast one vertex of each of the three colors.

Definition 5. *The chromatic number* $\chi(G)$ *of a graph is the minimum number of colors required to color the vertices of a graph such that no two adjacent vertices receive the same color.*

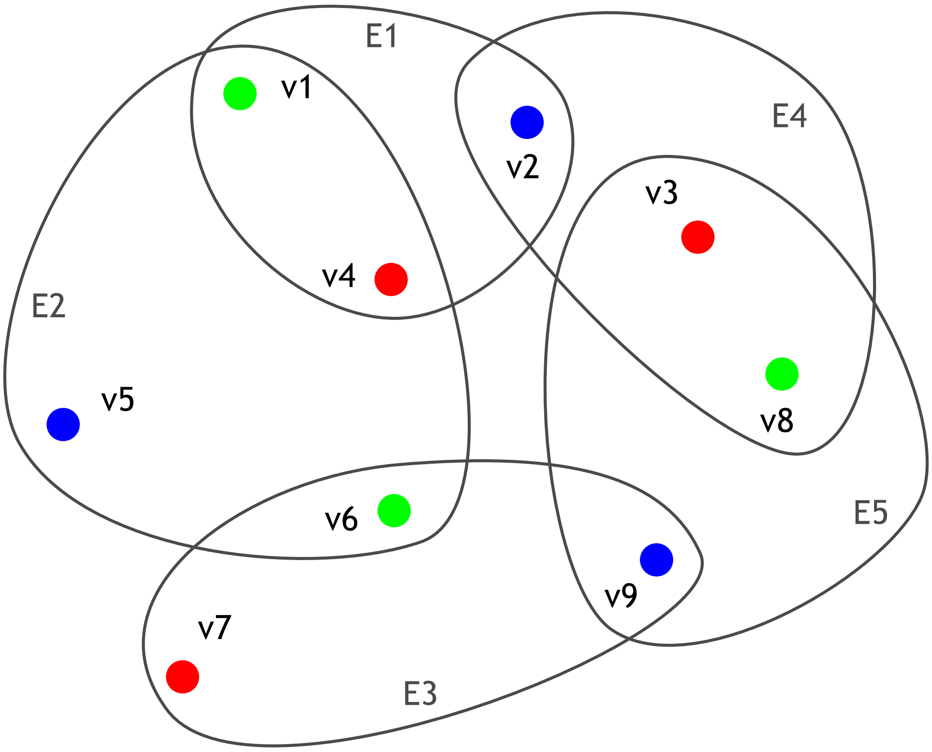

Figure 1.3: A proper tricoloring of a hypergraph $G(V, E)$ using colors red, green and blue.

Definition 6. *A **bicolor cover** is a set of bicolorings such that each bicoloring individually bicolors a set of hyperedeges properly and the union of all the hyperedges properly bicolored using the bicolorings is the set of all the hyperedges in the hypergraph.*

1.2 Overview of the work

We have worked on two problems pertaining to hypergraphs :

1.2.1 Bicolor cover

There is an already existing work [2] on the bound on the maximum number of bicolorings required to cover a hypergraph. However, it deals with only a special type of bicoloring in which only one vertex of a hyperedge is colored with one color(say white) and all the remaining vertices of the hyperedge are colored with the other color(say black). In our work (presented in chapter 2), we have proved a bound on the maximum number of bicolorings required to cover a hypergraph using general bicolorings. A lower bound on the size of bicolor cover for r-regular hypergraphs is also provided [2]. We have also proved the upper bound on the number of tricolorings required to cover a hypergrah.

1.2.2 Tricoloring of hypergraphs

There are bounds on the size of hyperedges and number of hyperedges that ensure the existence of a proper bicoloring[3]. The Chernoff bound exists for the discrepancy of hypergraph bicoloring. There also exists a Las Vegas algorithm for finding a bicoloring for a hypergraph with bounded discrepancy. In our work (presented in chapter 3), we have proved a bound on the number of hyperedges which ensures the existence of proper tricoloring. We also provide a scheme to establish an upper bound on the number of hyperedges which ensures the existence of proper c-coloring for $c > 3$. Further, we have defined discrepancy for coloring with more than 2 colors and established an upper bound on the discrepancy in tricoloring of a hypergraph and also demonstrated how to find an upper bound on discrepancy for c-coloring for $c > 3$.

Chapter 2

Covering hypergraphs using colorings

Given a hypergraph H, we wish to find the minimum number of bicolorings required that can cover all the hyperedges. In case a hypergraph is properly bicolorable, this number is 1 because a single bicoloring covers all the hyperedges. If a hypergraph is not properly bicolorable, then a certain bicoloring will properly bicolor a set of hyperedges and not all the hyperedges. For the remaining hyperedges, we need more bicoloring scheme(s). And so, the bicolor cover will have size greater than 1.

Fig. 2.1(a) shows a hypergraph which is not bicolorable. The bicoloring in Fig.2.1(b) can properly bicolor hyperedges $E1$ and $E2$ only while that in Fig.2.1(c) can properly bicolor hyperedges $E1$ and $E3$ only. However, the union of the hyperedges properly colored by either of the bicolorings contains all the hyperedges in the hypergraph and hence, the bicolorings in Fig.2.1(b) and (c) together cover the hypergraph.

In this chapter, we discuss the already existing bound for a special type of bicoloring[2] and then move on to provide a proof for the number of bicolorings required in the general case.

2.1 Special case with one white and remaining black vertices in an edge

In this section, we consider a special case of bicoloring(using colors say black and white) in which the bicoloring is said to be proper if there exists only vertex in an hyperedge which is colored white and all the remaining vertices in the hyperedge are colored black[2].

Theorem 2.1. *The number of such bicolorings reqiured to cover a hyperedge with C hyperedges is upper bounded by $O(\log C)$.*

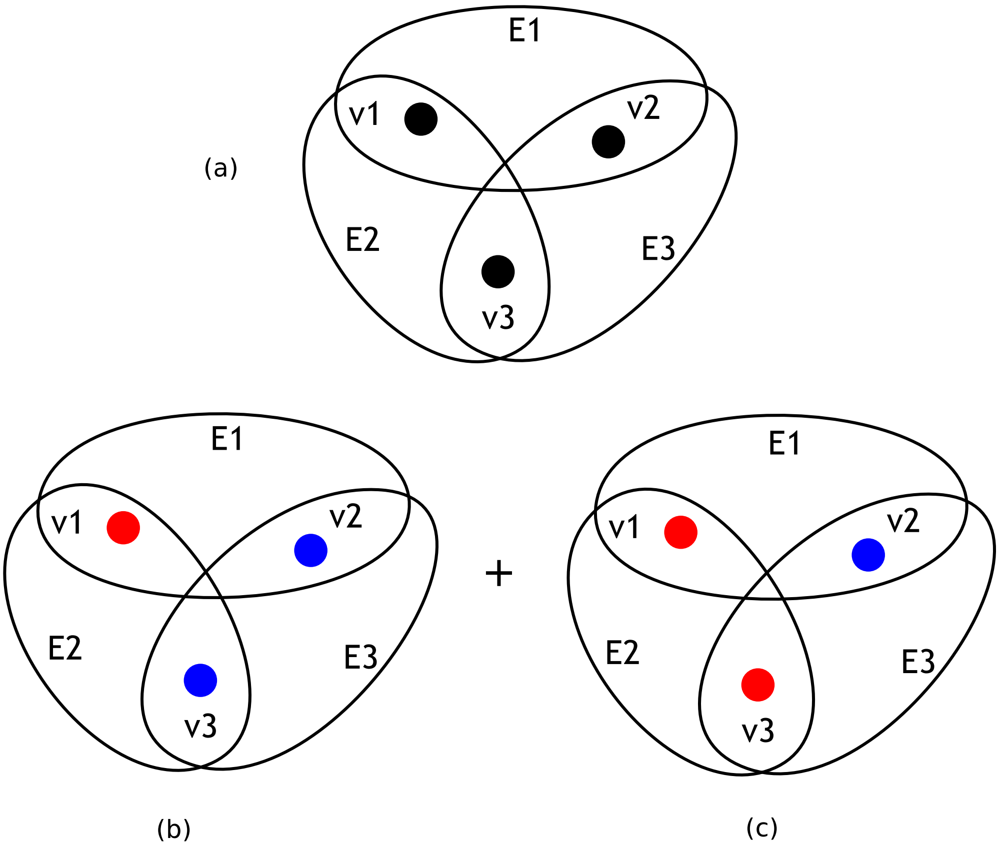

Figure 2.1: Bicolor cover of a non-bicolorable hypergraph.

Proof. Let us consider an r-uniform hypergraph. Let $P(A_i^1)$ denote the probability that the i^{th} hyperedge h_i is not properly bicolored by a random bicoloring. There are r choices for the white vertex in a hyperedge and for each choice, the probability of that vertex being white is p. The probability that the rest of the vertices are black is $(1-p)^{(r-1)}$.

$$P(\overline{A_i^1}) = rp(1-p)^{r-1} \tag{2.1}$$

Therefore, the probability that the strategy does not properly bicolor an edge h_i is

$$P(A_i^1) = 1 - rp(1-p)^{r-1} \tag{2.2}$$

Suppose we repeat the bicoloring x times. Then, the probability that none of the x strategies properly bicolors h_i is

$$P(A_i^x) = (1 - rp(1-p)^{r-1})^x \tag{2.3}$$

Let b_i denote the indicator variable which equals 1 if hyperedge h_i is not satisfied (by any of the x strategies in the proposed solution) and 0, otherwise.

Let $B = \sum_{1}^{C} b_i$. $B = 0$, if and only if the x randomly chosen strategies bicolor all the C hyperedges properly.

$$E(B) = E\left(\sum_{1}^{C} b_i\right) = \sum_{1}^{C} E\left(b_i\right) = C \times P\left(A_i^x\right) \tag{2.4}$$

Suppose x is such that $E(B) < 1$. Since $E(B) < 1$, the integral random variable B should take the value 0 for some random choice of x strategies. So, an integral value of x satisfying the strict inequality is the suffficient number of strategies that together satisfy H.

$$C \times P\left(A_i^x\right) < 1 \tag{2.5}$$

So, we have

$$C \times (1 - rp\,(1-p)^{r-1})^x < 1 \tag{2.6}$$

We now find x satisfying the above inequality as,

$$\begin{aligned}
&(1 - rp\,(1-p)^{r-1})^x < 1/C \\
&\Rightarrow x \log(1 - rp\,(1-p)^{r-1}) < -\log C \\
&\Rightarrow x > \frac{-\log C}{\log\left(1 - rp(1-p)^{r-1}\right)}
\end{aligned} \tag{2.7}$$

Taking $p = \frac{1}{r}$,

$$x > \frac{-\log C}{\log\left(1 - \left(1 - \frac{1}{r}\right)^{r-1}\right)} = O\left(\log C\right) \tag{2.8}$$

So, there exists a proper bicolor covering of size $O(\log C)$.

The absolute value of the (negative) denominator in the above inequality for x shrinks from $\log_2 2 = 1$ for $r = 2$, and approaches $|\log_2\left(1 - \frac{1}{e}\right)|$ as r grows. □

2.2 Case with general proper bicoloring

Theorem 2.2. *The size of bicolor cover of an* r*-uniform hypergraph with* C *hyperedges is upper bounded by* $O(\log C)$.

Proof. We use similar notations as used in the previous section. Let $P(A_i^1)$ denote the probability that the ith hyperedge h_i is not properly bicolored by a random bicoloring.

A hyperedge h_i is not properly bicolored if it is monochromatic ,i.e., either all the vertices are colored white or all the vertices are colored black. Let the probability of a vertex being colored white is p and ,therefore, the probability of being colored black is $(1-p)$.

$$P\left(A_i^1\right) = p^r + (1-p)^r \tag{2.9}$$

Suppose, we repeat the bicoloring x times. The probability that none of the x strategies properly bicolor h_i is

$$P\left(A_i^x\right) = \left(p^r + (1-p)^r\right)^x \tag{2.10}$$

Let, b_i denote the indicator variable which equals 1 if hyperedge h_i is not satisfied (by any of the x strategies in the proposed solution) and 0, otherwise.

Let $B = \sum_{1}^{C} b_i$.

$B = 0$ if and only if the x randomly chosen strategies bicolor all the C hyperedges properly.

$$E\left(B\right) = E\left(\sum_{1}^{C} b_i\right) = \sum_{1}^{C} E\left(b_i\right) = C \times P\left(A_i^x\right) \tag{2.11}$$

Suppose, x is such that $E\left(B\right) < 1$. Since $E\left(B\right) < 1$, the integral random variable B should take the value 0 for some random choice of x strategies. So, an integral value of x satisfying the strict inequality is the sufficient number of strategies that together satisfy the hypergraph.

$$C \times P\left(A_i^x\right) < 1 \tag{2.12}$$

So, we have

$$C \times \left(p^r + (1-p)^r\right)^x < 1 \tag{2.13}$$

We now find x satisfying the above inequality as:

$$\begin{aligned} &\left(p^r + (1-p)^r\right)^x < \frac{1}{C} \\ &\Rightarrow x \log\left(p^r + (1-p)^r\right) < -\log C \\ &\Rightarrow x > \frac{-\log C}{\log\left(p^r + (1-p)^r\right)} \end{aligned} \tag{2.14}$$

If we consider $p = \frac{1}{2}$ such that a vertex is colored white or black with equal probability, then:

$$\begin{aligned} &x > \frac{-\log C}{\log\left(\frac{1}{2^r} + \frac{1}{2^r}\right)} \\ &\Rightarrow x > \frac{-\log C}{\log\left(\frac{1}{2^{r-1}}\right)} \\ &\Rightarrow x > \frac{\log C}{(r-1)\log 2} = O\left(\log C\right) \end{aligned} \tag{2.15}$$

From this, we can infer that there exists atleast one proper bicolor cover of the size given by the above bound which is $O(\log C)$, where C is the number of hyperedges in the hypergraph. □

2.3 Lower bound on the size of bicolor cover

In this section, we derive non-trivial and asymptotically increasing bounds on the size of the hypergraph bicolor cover for r-uniform complete hypergraphs, $r \geq 2$. The proof is essentially the same as provided in [2]. For each bicoloring that covers some of the hyperedges by properly bicoloring them, we define a partial function f from V to $\{w, b, -\}$. We say that strategy f properly colors the hyperedge h if $f(v) \in \{w, b\}$ for every vertex $v \in h$ and there exist $v_1, v_2 \in h$ such that $f(v_1) = w$ and $f(v_2) = b$. We also say that the bicoloring strategy f *satisfies* h, if the bicoloring f properly colors h. We define $f(v) =$'$-$' to indicate the bicoloring f is not defined for vertex v. This happens when v does not belong to any hyperedge properly colored by the bicoloring f.

Theorem 2.3. *The number of bicolorings required to cover a complete r-regular hypergraph K_n^r is lower bounded by $\left\lfloor \log_3 \left(\frac{n}{r-1}\right) \right\rfloor$.*

Proof. Let S be the set of bicolorings required to cover the hypergraph. Let $|S| = m$. Consider the m-tuples $[f_1(v_i), f_2(v_i), \cdots, f_m(v_i)]$ where each $f_j(v_i) \in \{b, w, \}$, $1 \leq j \leq m$, $1 \leq i \leq n$. Here, each partial function f_j is a bicoloring strategy in S. If $f_j(v_i) = -$, it implies that the j^{th} strategy is not defined for the v_i otherwise it implies the vertex v_i is colored white(w) or black(b) in the j^{th} strategy. We now generate n such m-tuples randomly and uniformly and assign them to the n vertices.

Let us assume that $m < \log_3 \left(\frac{n}{r-1}\right)$. We can write this as $(3^m \times (r-1)) < n$. Total number of such m-tuples possible is 3^m. Now, each of the n vertices is assigned one of the 3^m m-tuples. Therefore, there exists atleast one m-tuple that has been assigned to $\lceil \frac{n}{3^m} \rceil$ vertices. But,

$$\left\lceil \frac{n}{3^m} \right\rceil > r - 1 \tag{2.16}$$

So, the number of vertices that have been assigned the same color in all the m bicolorings is greater than $r-1$. Therefore, we conclude that there must be atleast one hyperedge h (set of r vertices), all of whose vertices are assigned the same color in all the m bicolorings. So, there exists atleast one hyperedge in the hypergraph which can not be properly bicolored using less than $\log_3 \left(\frac{n}{r-1}\right)$ bicolorings. Hence, for S to be a proper bicolor cover, $|S| > \log_3 \left(\frac{n}{r-1}\right)$. □

2.4 Tricolor cover of a hypergraph

Sometimes, it may not be possible to properly tricolor all the hyperedges of a hypergraph using only one tricoloring. But we can have a set of tricolorings such that each hyperedge is properly colored in at least one of the tricolorings. Such a set of tricolorings is called a *tricolor cover* of the hypergraph.

Theorem 2.4. *The number of tricolorings required to cover a r-uniform hypergraph is upper bounded by $O(\log C)$ where C is the number of hyperedges.*

Proof. We again use similar notations as used in the previous sections. $P(A_i^1)$ denotes the probability that the ith hyperedge h_i is not properly tricolored by a random tricoloring. A hyperedge h_i is not properly tricolored if it is does not have at least one vertex colored with each of the three colors.

$$P\left(A_i^1\right) = \frac{3 \times 2^r - 3}{3^r} = \frac{2^r - 1}{3^{r-1}} \tag{2.17}$$

On repeating the random tricoloring x times, the probability that the hyperedge h_i is not properly tricolored in any of the x tricolorings is given by :

$$P\left(A_i^x\right) = \left(\frac{2^r - 1}{3^{r-1}}\right)^x \tag{2.18}$$

Again, b_i denotes the indicator variable which equals 1 if hyperedge h_i is not satisfied (by any of the x strategies in the proposed solution) and 0, otherwise.
Let $B = \sum_1^C b_i$.
$B = 0$ if and only if the x randomly chosen strategies tricolor all the C hyperedges properly.

$$E\left(B\right) = E\left(\sum_1^C b_i\right) = \sum_1^C E\left(b_i\right) = C \times P\left(A_i^x\right) \tag{2.19}$$

Now, let x be such that $E\left(B\right) < 1$. Using similar arguments, since $E\left(B\right) < 1$, the integral random variable B should take the value 0 for some random choice of x strategies. So, an integral value of x satisfying the strict inequality is the sufficient number of strategies that together satisfy the hypergraph.

$$C \times P\left(A_i^x\right) < 1 \tag{2.20}$$

So, we have

$$C \times \left(\frac{2^r - 1}{3^{r-1}}\right)^x < 1 \tag{2.21}$$

We now find x satisfying the above inequality as:

$$\left(\frac{2^r - 1}{3^{r-1}}\right)^x < \frac{1}{C}$$

$$if \quad x \log\left(\frac{2^r - 1}{3^{r-1}}\right) < -\log C$$

$$
\begin{aligned}
if \quad x &> \frac{-\log C}{\log\left(2^r - 1\right) - \log\left(3^{r-1}\right)} \\
if \quad x &> \frac{\log C}{(r-1)\log 3 - \log\left(2^r - 1\right)} \\
if \quad x &> \frac{\log C}{(r-1)\log 3 - r\log 2} = O(\log C) \qquad (2.22)
\end{aligned}
$$

Therefore, using $O\left(\log C\right)$ tricolorings, we can cover all the hyperedges with proper tricoloring because the expected number of hyperedges not properly tricolored in any of the tricolorings is less than one, which essentially means zero because the number of hyperedges should be an integer. Hence, the size of the tricolor cover of a r-uniform hypergraph is upper bounded by $O\left(\log C\right)$. □

Chapter 3

Hypergraph c-coloring

A set system or hypergraph G(V, E) is a pair of two sets V and E. V is a set of n elements (vertices) and the set E containing m subsets $e \subseteq V$ of these elements, and $|e| \geq r$. Such subsets $e \in E$ are called hyperedges and such a set system $G(V, E)$ is called a hypergraph. We want to colour the vertices with some colors(say c colors) and wish to know whether a given hypergraph has a *proper c-coloring* (i.e. no hyperedge is colored using less than c colors).

3.1 Existence of proper bicoloring

Consider sparse hypergraphs such that $|E| < 2^{r-1}$, where $|e_i| \geq r$ for all $e_i \in E$. If we do a random bicoloring, then

the probability that a hyperedge is monochromatic $\leq 2 \times 2^{-r} = 2^{-(r-1)}$.

Therefore, the probability that some hyperedge is monochromatic $\leq |E| \times 2^{-(r-1)} < 2^{r-1} \times 2^{-(r-1)} = 1$.

Hence, the probability that no hyperedge is monochromatic is non-zero for such a sparse graph. Therefore, there must be a proper bicoloring.

Further, we can also calculate the expected number of monochromatic hyperedges in the hypergraph. The probability of a particular hyperedge being monochromatic is $2^{-(r-1)}$.

Therefore, the expected number of monochromatic hyperedges $= \sum_{i=1}^{|E|} 2^{-(r-1)} < 2^{(r-1)} \times 2^{-(r-1)} = 1$. So, the expected number of monochromatic hyperedges is strictly less than 1. And therefore, there must be a proper bicoloring of the hypergraph.

3.2 Combinatorial discrepancy for bicoloring

In this section, we discuss the upper bound on the discrepancy for bicoloring[3]. For the hypergraph G(V, E), where $V = \{v_1, \cdots, v_n\}$ is the set of vertices and $E = \{e_1, \cdots, e_m\}$ is the set of hyperedges, we wish to color $v_i s$ using two colors, say red and blue, such that within each hyperedge e_i, no color outnumbers the other by too much. Formally, we can define *discrepancy* as

$$\chi(e_i) = \sum_{v_j \in e_i} \chi(v_j) \tag{3.1}$$

where $\chi(v_j) \in \{1, -1\}$ depending on the color of the vertex v_j. The *discrepancy* of the hypergraph under a given bicoloring is the maximum of $|\chi(e_i)|$ over all $e_i \in E$. When no particular bicoloring is specified, then the *discrepancy* of the hypergraph refers to the minimum discrepancy of the hypergraph over all possible bicolorings.

Upper bound on discrepancy

Lets consider e_i to be bad if $|\chi(e_i)| > \sqrt{2|e_i| \ln(2m)}$.

If $X = \sum_{i=1}^{n} x_i$ is the sum of n mutually independent random variables x_i uniformly distributed in $\{1, -1\}$, then, for any $\delta > 0$,

$$\text{Prob}\,[X \geq \delta] < e^{-\delta^2/2n} \tag{3.2}$$

Using the result of Eqn.3.2,

$$\text{Prob}\left[\chi(e_i) > \sqrt{2|e_i| \ln(2m)}\right] < e^{-2|e_i| \ln(2m)/(2|e_i|)} = 1/2m \tag{3.3}$$

Since, the random variable can assume two values, we take $2 \times 1/2m = 1/m$ as the limiting probability. Therefore, the probability that atleast one hyperedge is bad $< m \times 1/m = 1$. The probability that no hyperedge is bad is positive. So, the discrepancy of the hypergraph can not be more than $\sqrt{2n \ln(2m)}$.

Las Vegas algorithm for finding a bicoloring with bounded discrepancy

Again, if we consider e_i to be bad if $|\chi(e_i)| > \sqrt{3|e_i| \ln(2m)}$, then by the Chernoff's bound shown in eqn.3.2, probability that a particular e_i is bad $< m^{-3/2}$, and thus, the probability that atleast one e_i is bad $< 1/\sqrt{m}$. Therefore, a Las Vegas algorithm can be designed to find a bicoloring, within the above discrepancy, in $\frac{1}{1/\sqrt{m}} = \sqrt{m}$ steps.

If k independent rounds of random bicoloring are done, then the probability that all of

them have some bad hyperedge = $(1/\sqrt{m})^k = \frac{1}{m^{k/2}}$.
Therefore, probability of finding the desired discrepancy coloring in k trials $= 1 - \frac{1}{m^{k/2}}$.

3.3 Existence of proper tricoloring

A tricoloring is said to be *proper* if every hyperedge contains vertices colored with all the three colors.

Theorem 3.1. *For a hypergraph $G(V, E)$ with $|e_i| \geq r$ for all $e_i \in E$, a proper tricoloring exists if $|E| < \frac{3^{(r-1)}}{2^r}$*

Proof. Let us consider sparse hypergraphs such that $|E| < \delta$, where $|e_i| \geq r$ for all $e_i \in E$. If we do a random tricoloring, that is, color the vertices randomly with the three colors, then lets calculate the probability that a hyperedge is not trichromatic.
Let $M(l, k, c)$ denote the number of ways of coloring l vertices with exactly k colors out of c colors (i.e. each of the k colors is used atleast once).

$$M(l, 1, 3) = \binom{3}{1} \times 1^l = 3 \tag{3.4}$$

$M(l, 2, 3)$ will be number of ways of choosing 2 colors out of 3 colors times the number of ways of coloring the l vertices using both the colors atleast once (which is equal to the number of ways of coloring l vertices using 2 colors - number of such colorings in which only 1 color was used).

$$M(l, 2, 3) = \binom{3}{2} \times \left(2^l - 2\right) = 3 \times 2^l - 6 \tag{3.5}$$

Therefore, the number of different tricolorings of a hyperedge (with l vertices) which are not proper $= M(l, 1, 3) + M(l, 2, 3)$. The total number of ways in which the hyperedge can be colored $= 3^l$. Let $\overline{P_3(e_i)}$ denote the probability that the hyperedge e_i is not trichromatic.

$$\begin{aligned}
\overline{P_3(e_i)} &= \frac{M(|e_i|, 1, 3) + M(|e_i|, 2, 3)}{3^{|e_i|}} \\
\Rightarrow \overline{P_3(e_i)} &= \frac{3 \times 2^{|e_i|} - 3}{3^{|e_i|}} \\
\Rightarrow \overline{P_3(e_i)} &< \frac{3 \times 2^{|e_i|}}{3^{|e_i|}} \\
\Rightarrow \overline{P_3(e_i)} &< \frac{3 \times 2^r}{3^r}
\end{aligned} \tag{3.6}$$

Therefore, the probability that some hyperedge is not trichromatic $< |E| \times \dfrac{3 \times 2^r}{3^r}$. Hence, for the probability that all hyperedges are trichromatic is non-zero for such a sparse graph, the probability that some hyperedge is not trichromatic should be strictly less than 1. This is true if :

$$\begin{aligned} |E| \quad & \times \frac{3 \times 2^r}{3^r} \leq 1 \\ \Rightarrow \quad |E| \quad & \leq \frac{3^{(r-1)}}{2^r} \end{aligned} \tag{3.7}$$

Since a random tricoloring in such a case yields a proper tricoloring with nonzero probability, there must be a proper tricoloring when $|e_i| \geq r$ for all $e_i \in E$ and $|E| < \frac{3^{(r-1)}}{2^r}$.

Again, we can also calculate the expected number of non-trichromatic hyperedges in the hypergraph to prove the existence of a proper tricoloring. The probability of a particular hyperedge being non-trichromatic is $\dfrac{2^r}{3^{(r-1)}}$. Therefore, the expected number of non-trichromatic hyperedges $< \sum_{i=1}^{|E|} \dfrac{2^r}{3^{(r-1)}} < \dfrac{3^{(r-1)}}{2^r} \times \dfrac{2^r}{3^{(r-1)}} = 1$. So, the expected number of non-trichromatic hyperedges is strictly less than 1. And therefore, there must be a proper tricoloring of the hypergraph. □

3.4 Existence of proper c-coloring

Let us now consider the case when we have to color using c colors, given that $|e_i| \geq r$. A *c-coloring* is said to be proper if in every hyperedge, there exist vertices colored with each of the c colors. $M(l, k, c)$ is the number of ways of coloring l vertices with exactly k colors out of c colors. $M(l, k, c)$ can be recursively defined as :

$$M(l,k,c) = \binom{c}{k} \times \left(k^l - \sum_{j=1}^{k-1} M(l,j,k) \right) \tag{3.8}$$

Let $\overline{P_c(e_i)}$ denote the probability that the hyperedge e_i is not properly *c-colored* in a random *c-coloring* where all the vertices are colored randomly using the c colors.
Therefore,

$$\begin{aligned} \overline{P_c(e_i)} \quad & = 1 - \frac{M(|e_i|, c, c)}{c^{|e_i|}} \\ \overline{P_c(e_i)} \quad & = \frac{c^{|e_i|} - M(|e_i|, c, c)}{c^{|e_i|}} \end{aligned}$$

$$\overline{P_c\left(e_i\right)} = \frac{\sum_{j=1}^{c-1} M\left(|e_i|, j, c\right)}{c^{|e_i|}} \tag{3.9}$$

The probability that some hyperedge is not properly *c-colored* in a random *c-coloring* becomes $|E| \times \overline{P_c\left(e_i\right)}$. Hence, to ensure that a proper *c-coloring* exists, this probability should be strictly less than 1.

$$\begin{aligned} |E| &\times \overline{P_c\left(e_i\right)} < 1 \\ \Rightarrow |E| &< \frac{1}{\overline{P_c\left(e_i\right)}} \end{aligned} \tag{3.10}$$

Let us use this relation to establish an upper bound on $|E|$ for the case when we are using 4 colors.

$$\begin{aligned} \Rightarrow \overline{P_4\left(e_i\right)} &= \frac{\sum_{j=1}^{3} M\left(|e_i|, j, 4\right)}{4^{|e_i|}} \\ \Rightarrow \overline{P_4\left(e_i\right)} &= \frac{M\left(|e_i|, 1, 4\right) + M\left(|e_i|, 2, 4\right) + M\left(|e_i|, 3, 4\right)}{4^{|e_i|}} \\ \Rightarrow \overline{P_4\left(e_i\right)} &= \frac{4 + 6 \times \left(2^{|e_i|} - 2\right) + 4 \times \left(3^{|e_i|} - 3 \times 2^{|e_i|} + 3\right)}{4^{|e_i|}} \\ \Rightarrow \overline{P_4\left(e_i\right)} &= \frac{4 \times 3^{|e_i|} - 6 \times 2^{|e_i|} + 4}{4^{|e_i|}} \\ \Rightarrow \overline{P_4\left(e_i\right)} &< \frac{4 \times 3^r - 6 \times 2^r + 4}{4^r}, \forall r \geq 3 \end{aligned} \tag{3.11}$$

$$|E| < \frac{1}{\overline{P_4\left(e_i\right)}}$$

$$if \quad |E| < \frac{4^{|e_i|}}{4 \times 3^{|e_i|} - 6 \times 2^{|e_i|} + 4} \tag{3.12}$$

$$if \quad |E| < \frac{4^r}{4 \times 3^r - 6 \times 2^r + 4}, \forall r \geq 3 \tag{3.13}$$

3.5 Bounded discrepancy tricoloring

We use another definition of discrepancy to calculate the discrepancy in case of tricoloring. Let ϵ be the upper bound on the discrepancy of an edge of the tricoloring we want so as to ensure that a tricoloring with discrepancy $\chi \leq \epsilon$ exists. Therefore, $P\left[\chi(e_j) > \epsilon\right]$ should be less than some p that ensures that the probability that there is atleast one bad edge (edge with discrepancy greater than ϵ) is strictly less than 1, and thus the probability that there is no bad edge is greater than zero, thereby ensuring that there exists atleast one tricoloring with discrepancy less than the bound ϵ.

For the hypergraph G(V, E), where $V = \{v_1, \cdots, v_n\}$ is the set of vertices and $E = \{e_1, \cdots, e_m\}$ is the set of hyperedges, we wish to color v_i's using three colors, say C_1, C_2 and C_3, such that within each hyperedge e_i, no color outnumbers the other by too much. Let $\chi_{v_j} \in \{1, \omega, \omega^2\}$ depending on the color of the vertex v_j, where 1, ω and ω^2 are cube roots of unity. Say, $\chi_{v_j} = 1$ if the vertex v_j is colored with C_1, ω, if it is colored with C_2 and ω^2, if it is colored with C_3. The *discrepancy* in this case can be defined as

$$\chi(e_i) = \max\left(|X_{i,1}|, |X_{i,\omega}|, |X_{i,\omega^2}|\right) \tag{3.14}$$

$$X_i = \sum_{v_j \in e_i} \chi_{v_j} \tag{3.15}$$

where, $X_{i,1}$, $X_{i,\omega}$ and X_{i,ω^2} are the projections of the vector representing X_i on the vectors representing 1, ω and ω^2, respectively, in the complex plane. The *discrepancy* of the hypergraph under a given tricoloring is the maximum of $|\chi(e_i)|$ over all $e_i \in E$. When no particular tricoloring is specified, then the *discrepancy* of the hypergraph refers to the minimum discrepancy of the hypergraph over all possible tricolorings.

Take for example a hyperedge e_i with 9 vertices which are to be colored with 3 colors (say R, G & B). Suppose 3 vertices are colored with R, another 3 with G and the remaining 3 with blue. In this case, $X_i = 3 + 3\omega + 3\omega^2 = 0$. So, $X_{i,1}$, $X_{i,\omega}$ and X_{i,ω^2} are all 0 and thus, the discrepancy $\chi(e_i) = 0$, as expected. Now, suppose the color distribution is skewed so that there are 7 R, 1 G and 1 B vertices and R, G and B correspond to 1, ω and ω^2, respectively. Therefore, $X_i = 7 + \omega + \omega^2 = 6$ and thus $X_{i,1} = 6$, $X_{i,\omega} = -3$ and $X_{i,\omega^2} = -3$. Hence, $\chi(e_i) = 6$ in this case. The value of $\chi(e_i) \in [0, |e_i|]$.

Theorem 3.2. *The discrepancy in tricoloring of a hypergraph cannot be more than* $\sqrt{\left(\frac{3}{2} n \log(6m)\right)}$.

Proof. Let us first consider an edge e_i and take the projection of X_i on the x-axis.
Using Markov's inequality,

$$\text{Prob}\left[X_{i,1} \geq \delta\right] = \text{Prob}\left[e^{\lambda X_{i,1}} \geq e^{\lambda\delta}\right] \leq e^{-\lambda\delta}\mathbf{E}\left[e^{\lambda X_{i,1}}\right] \tag{3.16}$$

$$\mathbf{E}\left[e^{\lambda X_{i,1}}\right] = \mathbf{E}\left[e^{\lambda \sum_{v_j \in e_i} Re(\chi(v_j))}\right] \tag{3.17}$$

Each of $\chi(v_j)$ is an independent random variable. Therefore,

$$\begin{aligned}
\mathbf{E}\left[e^{\lambda X_{i,1}}\right] &= \mathbf{E}\left[\prod_{v_j \in e_i} e^{\lambda Re(\chi(v_j))}\right] \\
&= \prod_{v_j \in e_i} \mathbf{E}\left[e^{\lambda Re(\chi(v_j))}\right] \\
&= \left(\mathbf{E}\left[e^{\lambda Re(\chi(v_j))}\right]\right)^{|e_i|} \\
&= \left[\frac{1}{3}\left(e^{\lambda} + e^{\frac{-\lambda}{2}} + e^{\frac{-\lambda}{2}}\right)\right]^{|e_i|} \\
&= \left[\frac{1}{3}\left(\sum_{i=0}^{\infty}\left(\frac{\lambda^i}{i!} + 2 \times \frac{(-\lambda/2)^i}{i!}\right)\right)\right]^{|e_i|} \qquad (3.18)
\end{aligned}$$

Taking the first two terms out of the summation and then combining the consecutive even and odd terms,

$$\begin{aligned}
\mathbf{E}\left[e^{\lambda X_{i,1}}\right] &= \left[\frac{1}{3}\left(3 + \sum_{i=2}^{\infty}\left(\frac{\lambda^i}{i!} + 2 \times \frac{(-\lambda/2)^i}{i!}\right)\right)\right]^{|e_i|} \\
&= \left[\frac{1}{3}\left(3 + \sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{(2i)!} + \frac{\lambda^{2i+1}}{(2i+1)!} + \frac{2(-\lambda/2)^{2i}}{(2i)!} + \frac{2(-\lambda/2)^{2i+1}}{(2i+1)!}\right)\right)\right]^{|e_i|} \\
&= \left[\frac{1}{3}\left(3 + \sum_{i=1}^{\infty}\left(\frac{1}{(2i)!}\left(\lambda^{2i} + \frac{\lambda^{2i+1}}{2i+1} + \frac{\lambda^{2i}}{2^{2i-1}} - \frac{\lambda^{2i+1}}{2^{2i}(2i+1)}\right)\right)\right)\right]^{|e_i|} \\
&= \left[\frac{1}{3}\left(3 + \sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{(2i)!}\left(1 + \frac{\lambda}{2i+1} + \frac{1}{2^{2i-1}} - \frac{\lambda}{2^{2i}(2i+1)}\right)\right)\right)\right]^{|e_i|} \\
&< \left[\frac{1}{3}\left(3 + \sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{(2i)!}\left(1 + \frac{\lambda}{2i+1} + \frac{1}{2^{2i-1}}\right)\right)\right)\right]^{|e_i|} \\
&< \left[\frac{1}{3}\left(3 + \sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{(2i)!}\left(1 + \frac{1}{2i+1} + \frac{1}{2^{2i-1}}\right)\right)\right)\right]^{|e_i|} \qquad (\text{as } \lambda \leq 1) \\
&= \left[\frac{1}{3}\left(3 + \sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{(2i)!}\left(\frac{2^{2i}i + 2^{2i-1} + 2^{2i-1} + 2i + 1}{(2i+1)2^{2i-1}}\right)\right)\right)\right]^{|e_i|} \\
&= \left[\frac{1}{3}\left(3 + \sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{(2i)!}\left(\frac{2^{2i}i + 2^{2i} + 2i + 2 - 1}{(2i+1)2^{2i-1}}\right)\right)\right)\right]^{|e_i|} \\
&< \left[\frac{1}{3}\left(3 + \sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{(2i)!}\left(\frac{(2^{2i}+2)(i+1)}{(2i+1)2^{2i-1}}\right)\right)\right)\right]^{|e_i|} \\
&= \left[\frac{1}{3}\left(3 + \sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{(2i)!}\left(\left(2 + \frac{1}{2^{2i-2}}\right)\frac{i+1}{2i+1}\right)\right)\right)\right]^{|e_i|}
\end{aligned}$$

$$
\begin{aligned}
&< \left[\frac{1}{3}\left(3+\sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{(2i)!}\left(\frac{3\,(i+1)}{2i+1}\right)\right)\right)\right]^{|e_i|} \\
&= \left[1+\sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{(2i)!}\frac{(i+1)}{(2i+1)}\right)\right]^{|e_i|} \\
&= \left[1+\sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{i!}\frac{(i+1)}{\prod_{j=1}^{i}(i+j)\times(2i+1)}\right)\right]^{|e_i|} \\
&= \left[1+\sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{i!}\frac{1}{\prod_{j=0}^{i-1}(i+2+j)}\right)\right]^{|e_i|} \\
&< \left[1+\sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{i!}\frac{1}{(i+2)^{i}}\right)\right]^{|e_i|} \\
&< \left[1+\sum_{i=1}^{\infty}\left(\frac{\lambda^{2i}}{i!}\frac{1}{3^{i}}\right)\right]^{|e_i|} \\
&= \left[\sum_{i=0}^{\infty}\frac{\lambda^{2i}}{3^{i}i!}\right]^{|e_i|} \\
&= \left[e^{\frac{\lambda^2}{3}}\right]^{|e_i|} \\
&= e^{\frac{|e_i|\lambda^2}{3}}
\end{aligned}
$$

Substituting $\lambda = \left(\frac{\delta}{|e_i|}\right)$, which is less than 1 as used in the above proof because the discrepancy cannot be more than the total number of vertices in the hyperedge, in the above equation, we get :

$$\text{Prob}\left[X_{i,1} \geq \delta\right] < e^{-\frac{\delta^2}{|e_i|}+\frac{\delta^2}{3|e_i|}} = e^{-\frac{2\delta^2}{3|e_i|}} \tag{3.19}$$

Using similar argument, the same bounds exist for $X_{i,\omega}$ and X_{i,ω^2}. Now each of the $X_{i,1}$, $X_{i,\omega}$ and X_{i,ω^2} can either be positive or negative with maximum absolute value. Therefore,

$$\text{Prob}\left[\chi\left(e_i\right) \geq \delta\right] < 6e^{-\frac{2\delta^2}{3|e_i|}} \tag{3.20}$$

If we consider a hyperedge e_i to be bad if $\chi(e_i) > \left(\frac{3}{2}|e_i|\log(6m)\right)^{1/2}$, then :

$$\text{Prob}\left[\chi(e_i) > \left(\frac{3}{2}|e_i|\log(6m)\right)^{1/2}\right] < \frac{1}{m} \tag{3.21}$$

Hence, the probability that a hyperedge is bad is strictly less than $\frac{1}{m}$. Therefore, the probability that atleast one hyperedge is bad $< m \times 1/m = 1$. The probability that no hyperedge is bad is non-zero. So, the discrepancy of the hypergraph can not be more than $\sqrt{\left(\frac{3}{2}n\log(6m)\right)}$. □

3.6 Lower bound on discrepancy for tricoloring

Consider a m-uniform hypergraph $G(V, E)$ with $2m$ vertices where m is even. Let the hyperedges e_1 and e_2 do not have any common vertex. We construct the other edges in such a way that each edge contains $\frac{m}{2}$ common vertices with both e_1 and e_2. We include all such possible hyperedges. We now show that the discrepancy for tricoloring of such a hypergraph is always greater than $\frac{m}{4}$.

Theorem 3.3. *The discrepancy for tricoloring of such a hypergraph is always greater than* $\frac{m}{4}$.

Proof. Let the three colors be represented by R, G and B. Consider any tricoloring of the hypergraph.
If either of e_1 or e_2 has $x > \frac{m}{2}$ vertices of the same color, then its discrepancy will be $x - \frac{m-x}{2} = \frac{3x}{2} - \frac{m}{2} > \frac{m}{4}$.
Otherwise, each color has less than or equal to $\frac{m}{2}$ vertices in e_1 and e_2, each. Let the number of vertices colored with R in e_1 and e_2 be r_1 and r_2, respectively. Both r_1 and r_2 are less than $\frac{m}{2}$. Without loss of generality, lets assume that maximum number of vertices are colored with R. Therefore, $r_1 + r_2 \geq \frac{2m}{3}$. Now, consider a hyperedge e_j which contains all the r_1 vertices of e_1 that are colored with R and all the r_2 vertices of e_2 that are colored with R. The number of vertices colored with R in e_j is therefore $r_1 + r_2 \geq \frac{2m}{3}$ which is greater than the number of vertices colored with G or B in e_j.
Hence, the discrepancy of $e_j = (r_1 + r_2) - \frac{m-(r_1+r_2)}{2} = \frac{3(r_1+r_2)}{2} - \frac{m}{2} \geq \frac{m}{2} > \frac{m}{4}$.
Hence, the discrepancy of such a hypergraph is always greater than $\frac{m}{4}$. □

3.7 Combinatorial discrepancy for c-coloring

Let us try to define discrepancy for c-coloring by extending the definition that we have used for tricoloring. In the discrepancy upper bound for tricoloring, the vectors representing the cube roots of unity can be seen as the vectors from the center to the vertices of a

2-simplex. So, if we consider *c-coloring* of hypergraphs, we can use an regular $(c-1)$-simplex and use the vectors from its center to the vertices to denote each color.

For the hypergraph G(V, E), where $V = \{v_1, \cdots, v_n\}$ is the set of vertices and $E = \{e_1, \cdots, e_m\}$ is the set of hyperedges, we wish to color the vertices using c colors now. Let C_i denotes the i^{th} color. We want to color in such a way that within each hyperedge e_i, no color outnumbers the other by too much. Let $\chi(v_j) \in \{\omega_0, \omega_1, \cdots, \omega_{c-1}\}$ depending on the color of the vertex v_j, where ω_k denotes a vector from the centre to a vertex of a $(c-1)$-simplex. Say, $\chi(v_j) = \omega_k$ if the vertex v_j is colored with color C_k. The *discrepancy* can now be defined as

$$\chi(e_i) = \max\left(\left|X_{i,\omega_0}\right|, \left|X_{i,\omega_1}\right|, \cdots, \left|X_{i,\omega_{c-1}}\right|\right) \quad (3.22)$$

$$X_i = \sum_{v_j \in e_i} \chi_{v_j} \quad (3.23)$$

where, X_{i,ω_k} denotes the projection of the vector X_i on the vector ω_k. The *discrepancy* of the hypergraph under a given tricoloring is the maximum of $|\chi(e_i)|$ over all $e_i \in E$. When no particular c-coloring is specified, then the *discrepancy* of the hypergraph refers to the minimum discrepancy of the hypergraph over all possible c-colorings.

We can utilise the following two properties of a regular n-dimensional simplex :
1. For a regular simplex, the distances of its vertices to its center are equal.
2. The angle subtended by any two vertices of an n-dimensional simplex through its center is $\arccos\left(\frac{-1}{n}\right)$.

Let χ_{v_j,ω_k} denote the projection of χ_{v_j} on ω_k. Again, using the Markov's inequality,

$$\text{Prob}\left[X_{i,\omega_0} \geq \delta\right] = \text{Prob}\left[e^{\lambda X_{i,\omega_0}} \geq e^{\lambda\delta}\right] \leq e^{-\lambda\delta}\mathbf{E}\left[e^{\lambda X_{i,\omega_0}}\right] \quad (3.24)$$

$$\mathbf{E}\left[e^{\lambda X_{i,\omega_0}}\right] = \mathbf{E}\left[e^{\lambda\sum_{j=1}^{n}\chi_{v_j,\omega_0}}\right] \quad (3.25)$$

Since χ_{v_j,ω_0} are independent random variables,

$$\begin{aligned}\mathbf{E}\left[e^{\lambda X_{i,\omega_0}}\right] &= \mathbf{E}\left[\prod_{j=1}^{n} e^{\lambda\chi_{v_j,\omega_0}}\right] \\ &= \prod_{j=1}^{n}\mathbf{E}\left[e^{\lambda\chi_{v_j,\omega_0}}\right]\end{aligned}$$

$$\begin{aligned} &= \left(\mathbf{E}\left[e^{\lambda\chi_{v_j,\omega_0}}\right]\right)^{|e_i|} \\ &= \left[\frac{1}{c}\left(e^{\lambda} + e^{\frac{-\lambda}{c-1}} + \cdots + e^{\frac{-\lambda}{c-1}}\right)\right]^{|e_i|} \\ &= \left[\frac{1}{c}\left(e^{\lambda} + (c-1)\,e^{\frac{-\lambda}{c-1}}\right)\right]^{|e_i|} \end{aligned} \tag{3.26}$$

If c is known, then we can proceed to find the upper bound on the discrepancy. Lets consider the case with $c = 4$.

$$\begin{aligned} E\left[e^{\lambda X_{i,\omega_0}}\right] &= \left[\frac{1}{4}\left(e^{\lambda} + 3e^{\frac{-\lambda}{3}}\right)\right]^{|e_i|} \\ &< \left[e^{\frac{\lambda^2}{4}}\right]^{|e_i|}, \forall |e_i| > 0 \\ &= e^{\frac{|e_i|\lambda^2}{4}} \end{aligned} \tag{3.27}$$

Again, substituting $\lambda = \left(\frac{\delta}{|e_i|}\right)$, we get

$$\text{Prob}\left[X_{i,\omega_0} \geq \delta\right] < e^{-\frac{\delta^2}{|e_i|} + \frac{\delta^2}{4|e_i|}} = e^{-\frac{3\delta^2}{4|e_i|}} \tag{3.28}$$

The same bound holds for ω_1, ω_2 and ω_3. And again, any of $\omega_0, \omega_1, \omega_2$ and ω_3 can be either positive or negative with maximum absolute value. Hence,

$$\text{Prob}\left[\chi\left(e_i\right) \geq \delta\right] < 8e^{-\frac{3\delta^2}{4|e_i|}} \tag{3.29}$$

If we consider a hyperedge e_i to be bad if $\chi(e_i) > \left(\frac{4}{3}|e_i| \log\left(8m\right)\right)^{\frac{1}{2}}$, then

$$\text{Prob}\left[\chi(e_i) > \left(\frac{4}{3}|e_i| \log\left(8m\right)\right)^{\frac{1}{2}}\right] < \frac{1}{m} \tag{3.30}$$

So, the probability that a hyperedge is bad is strictly less than $\frac{1}{m}$ and thus the probability of atleast one hyperedge being bad is strictly less than one. Therefore the discrepancy can not be more than $\sqrt{\frac{4}{3}n \log\left(8m\right)}$.

Thus, using the above arguments, the discrepancy for *c-coloring* can be upper bounded.

Chapter 4

Conclusion and Future Work

This thesis contains work on mainly three problems on hypergraphs. The first one is the size of the set of colorings required to cover a given hypergraph. The second problem relates to providing some conditions that will ensure the existence of a proper c-coloring. The third problem is establishing bounds on the discrepancy for $c-$coloring of hypergraphs. In this work, we have established an upper bound on the size of the general bicolor cover of hypergraphs. We have then extended the work for tricoloring of hypergraphs. Then, we proved an upper bound on the number of hyperedges of a hypergraph (with $|e_i| > r$) that ensures presence of a proper tricoloring and extended the result for c-coloring where $c > 3$. Next, we have defined discrepancy for tricoloring and c-coloring where $c > 3$ and established an upper bound on the discrepancy for tricoloring. For a special class of hypergraphs, we have established a lower bound on the discrepancy for tricoloring. Regarding the discrepancy for c-coloring of hypergraphs, we have given a scheme to upper bound the discrepancy for c-coloring where $c > 3$.

The future plan is to devise a Las Vegas algorithm to find a tricoloring under bounded discrepancy. Regarding the upper bound on the number of hyperedges that ensure the existence of proper c-coloring, we can try to prove a closed-form expression in c as the upper bound for any value of c. Similar generalization can be done for the upper bound on the discrepancy of c-coloring.

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