On the Erdős Discrepancy Problem

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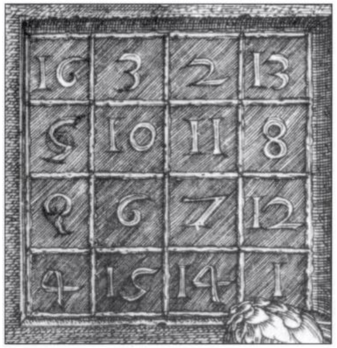
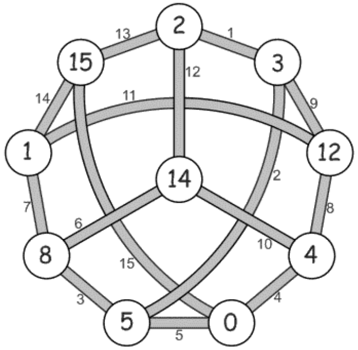
Motivation

There has been **significant progress** in the area of **search**, **constraint satisfaction**, and **automated reasoning**.

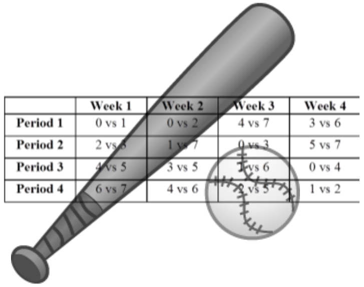
These approaches have been evaluated on problems such as:



Magic squares

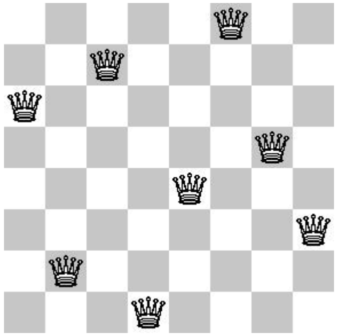


Graceful Graphs



	Week 1	Week 2	Week 3	Week 4
Period 1	0 vs 1	0 vs 2	4 vs 7	3 vs 6
Period 2	2 vs [illegible]	1 vs 7	0 vs 3	5 vs 7
Period 3	4 vs 5	3 vs 5	[illegible] vs 6	0 vs 4
Period 4	6 vs 7	4 vs 6	2 vs 5	1 vs 2

Round-Robin Tournament



N-Queens

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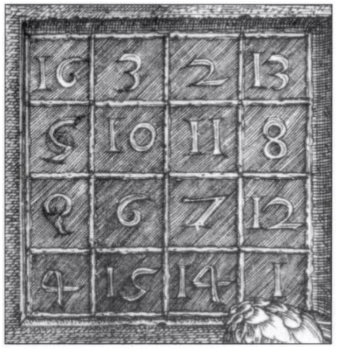

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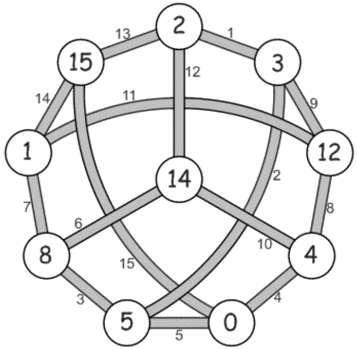

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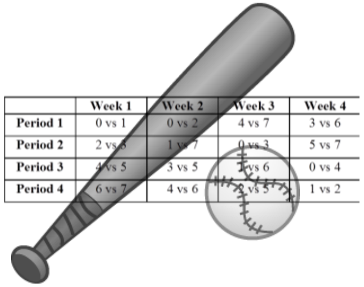

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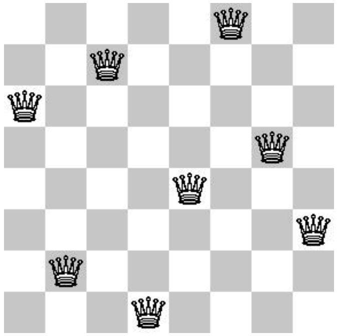

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Typically, these combinatorial objects are **highly regular**, and exhibit additional **hidden structure**, beyond the original structure of the problem.

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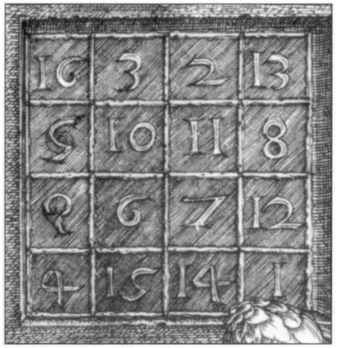

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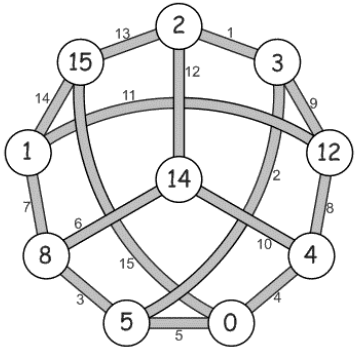

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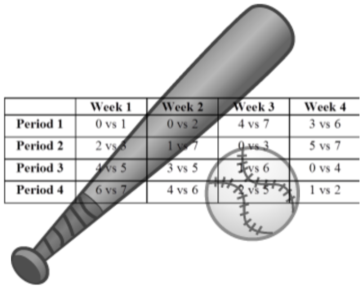

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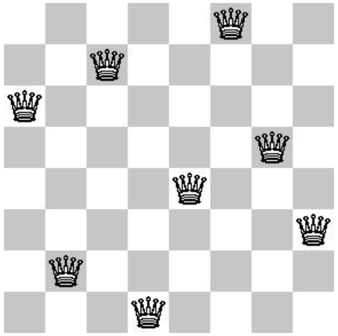

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Typically, these combinatorial objects are **highly regular**, and exhibit additional **hidden structure**, beyond the original structure of the problem.

How to **uncover** and **exploit** these regularities in the solutions?

Outline

- Motivation
- Framework
- Problem Definition
- Streamliners
- Recursive Construction Rule
- Conclusions and Future Directions

Framework: User-guided Streamlined Search

- Goal -

Exploit the **structure of some solutions** to dramatically **boost** the effectiveness of the **propagation mechanisms.**

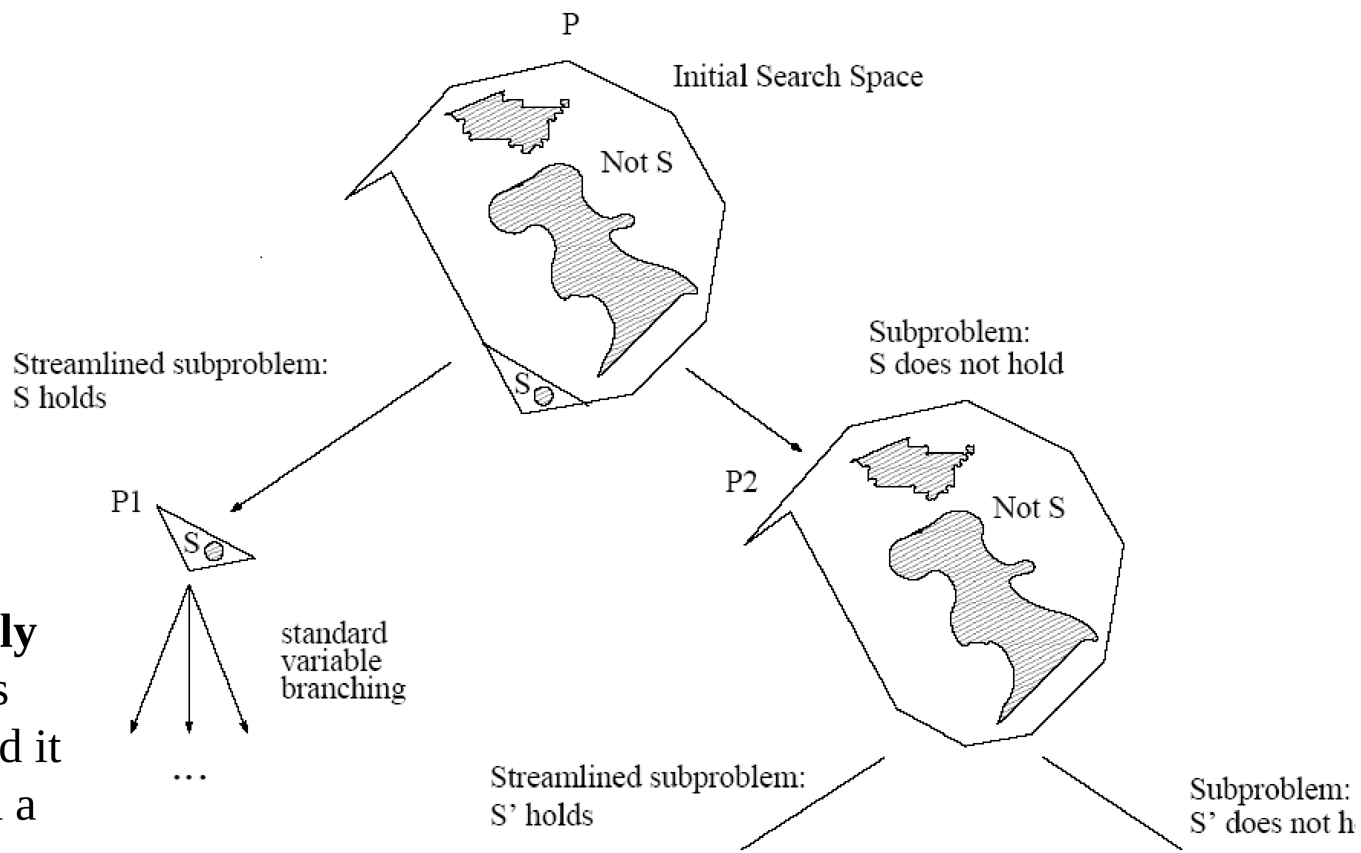
- Underlying Observation -

When one insists on maintaining the **full solution set**, there is a **hard practical limit** on the effectiveness of **constraint propagation** methods. Often, there is **no compact representation** for all the solutions.

- Underlying Conjecture -

For many intricate **combinatorial problems** – if **solutions exist** – there will often be **regular ones**.

Framework: User-guided Streamlined Search



P_1 is **substantially smaller** than its complement P_2 and it will benefit from a **stronger filtering** thereafter**.**

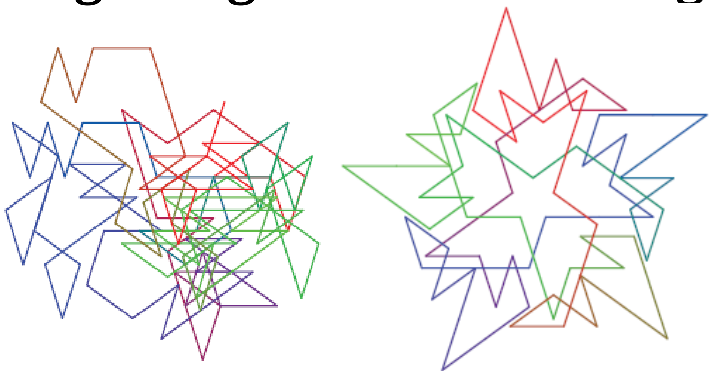
Streamlined Search:

Strong **branching mechanisms** (by adding constraints based on **structure properties**) at **high levels** of the search tree.

[Gomes and Sellmann, CP'04]

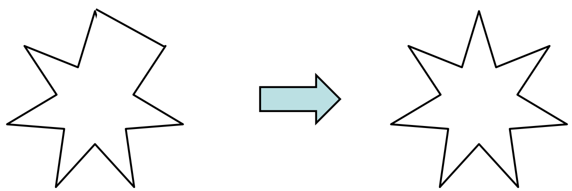
Framework: User-guided Streamlined Search

Recognizing Patterns and Regularities:



[Source: Marijn J.H. Heule, 2009]

Correcting Irregularities:



Generalizing / Formalizing Regularities:

1	2	3
3	1	2
2	3	1

Cyclic Latin square of order 3

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

Cyclic Latin square of order 4

Framework: User-guided Streamlined Search

$\mathcal{O} \leftarrow \emptyset$; // Conjectured streamliners
$\Gamma \leftarrow \emptyset$; // Search streamliners
$\rho \leftarrow \rho_0$; // Search parameter
$\mathcal{S} \leftarrow \emptyset$; // Solutions found
$\tau \leftarrow false$; // Timeout flag
repeat
  $Solve(P_\rho, \Gamma, t) \rightarrow (\mathcal{S}', \tau)$; // Search for new solutions
  if $\mathcal{S}' \cap \mathcal{S} \neq \emptyset$ **then**
    $\mathcal{S} \leftarrow \mathcal{S} \cup \mathcal{S}'$; // Case 1: successful search
    $Analyze(\mathcal{S}) \rightarrow \mathcal{O}'$; // Conjecture new streamliners
    $\mathcal{O} \leftarrow \mathcal{O} \cup \mathcal{O}'$;
    $\rho \leftarrow \rho + 1$;
  else if τ *is true* **then**
    Select $\Gamma' \subseteq \mathcal{O}$; // Case 2: timed-out failed search
    $\Gamma \leftarrow \Gamma \cup \Gamma'$; // Strengthen streamliners
  else
    Select $\Gamma' \subseteq \Gamma$; // Case 3: exhaustive failed search
    $\Gamma \leftarrow \Gamma \setminus \Gamma'$; // Weaken streamliners
    $\rho = max\{\rho : \mathcal{S}(\Gamma) \cap \mathcal{S}(P_\rho) \neq \emptyset\}+1$;
    Select $\Gamma'' \subseteq \Gamma'$; // Find next parameter of interest
    $\mathcal{O} \leftarrow \mathcal{O} \setminus \Gamma''$; // Drop unpromising streamliners
until $\mathcal{O} = \emptyset$;

Algorithm : `Discover-Construction` procedure for a given problem P, with parameter set ρ and timeout t.

Outline:

1) **Analyze** smaller size solutions, and **conjecture potential regularities** in the solutions.

2) Validate through **streamlining** the observed regularities.

3) If the streamlined search **does not give a larger size solution**, the proposed regularity is quite likely **accidental** and one looks for a new pattern in the small scale solutions.

4) Otherwise, one proceeds by generating a number of **new solutions** that all contain the proposed **structural regularity** and are used to expand the solution set and to **reveal new regularities**.

Overview of the results

A simple yet effective approach…

1	2	3	4	5	6	7	8	9	10	11
2	4	6	8	10	11	9	7	5	3	1
3	6	9	11	8	5	2	1	4	7	10
4	8	11	7	3	1	5	9	10	6	2
5	10	8	3	2	7	11	6	1	4	9
6	11	5	1	7	10	4	2	8	9	3
7	9	2	5	11	4	3	10	6	1	8
8	7	1	9	6	2	10	5	3	11	4
9	5	4	10	1	8	6	3	11	2	7
10	3	7	6	4	9	1	11	2	8	5
11	1	10	2	9	3	8	4	7	5	6

1 2 4 8 11 22
3 5-7 19 21 23
9 10 12-18 20

Spatially Balanced Latin Squares	**Weak Schur Numbers**
[Smith et al., IJCAI'05] $n \leq 35$	[Eliahou et al., Computers & Math Applications'12] $WS(6) \geq 575$
[L. et al, AAAI'12] Any n s.t. 2n+1 is prime	[L. et al, AAAI'12] $WS(6) \geq 581$

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6	11	5	1	7	10	4	2	8	9	3
7	9	2	5	11	4	3	10	6	1	8
8	7	1	9	6	2	10	5	3	11	4
9	5	4	10	1	8	6	3	11	2	7
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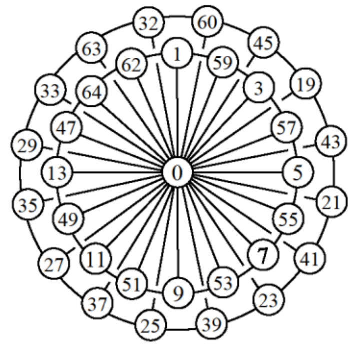

1	20	71	40	51	41	70	30	90	91
92	12	29	62	49	59	32	79	82	9
8	83	23	38	53	48	68	73	13	98
94	17	74	34	47	57	64	24	87	7
6	85	26	65	45	55	36	76	15	96
5	86	75	66	46	56	35	25	16	95
97	84	77	37	54	44	67	27	14	4
93	88	28	63	58	43	33	78	18	3
99	19	22	69	42	52	39	72	89	2
10	11	80	31	60	50	61	21	81	100

Spatially Balanced Latin Squares	**Weak Schur Numbers**	**Graceful Double-Wheel Graphs**	**Diagonally ordered Magic Squares**
[Smith et al., IJCAI'05] $n \leq 35$	[Eliahou et al., Computers & Math Applications'12] $WS(6) \geq 575$	[Heule & Walsh, AAAI'10] $n \leq 24$	[Gomes & Sellmann, CP'04] $n \leq 19$
[L. et al, AAAI'12] Any n s.t. 2n+1 is prime	[L. et al, AAAI'12] $WS(6) \geq 581$	[L. et al, IJCAI'13] Any $n > 3$	[L. et al, IJCAI'13] Any n s.t. n is doubly even

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1	2	3	4	5	6	7	8	9	10	11
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8	7	1	9	6	2	10	5	3	11	4
9	5	4	10	1	8	6	3	11	2	7
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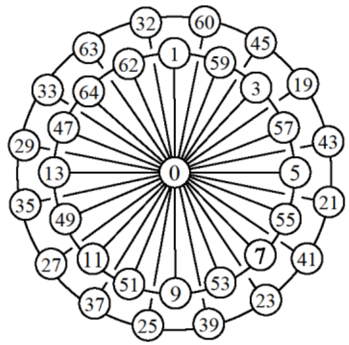

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Spatially Balanced Latin Squares	Weak Schur Numbers	Graceful Double-Wheel Graphs	Diagonally ordered Magic Squares	Erdos Discrepancy Sequences
[Smith et al., IJCAI'05] $n \leq 35$	[Eliahou et al., Computers & Math Applications'12] $WS(6) \geq 575$	[Heule & Walsh, AAAI'10] $n \leq 24$	[Gomes & Sellmann, CP'04] $n \leq 19$	[Konev & Lisitsa, SAT'14] $n \leq 13{,}900$
[L. et al, AAAI'12] Any n s.t. 2n+1 is prime	[L. et al, AAAI'12] $WS(6) \geq 581$	[L. et al, IJCAI'13] Any $n > 3$	[L. et al, IJCAI'13] Any n s.t. n is doubly even	[L. et al, CP'14] $n \leq 127{,}645$

Erdős Discrepancy Problem

Discrepancy theory: distributing points uniformly over some geometric object, and studying how irregularities inevitably occur in these distributions.

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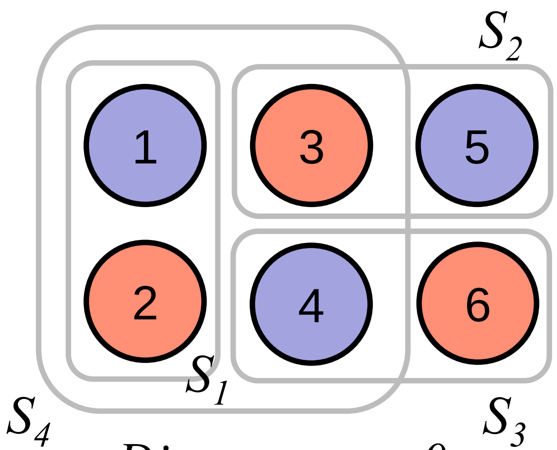
Discrepancy: Given *m* subsets $S_1, S_2, \ldots, S_m$ of $\{1, \ldots, n\}$, the discrepancy of a two-coloring of the elements is the maximum difference between the number of elements of one color and the number of elements of the other color.



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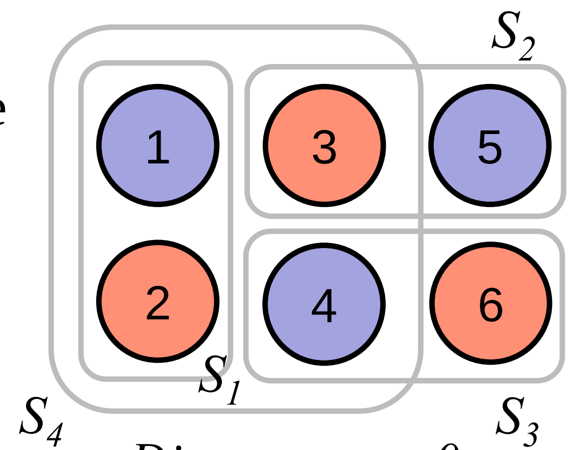

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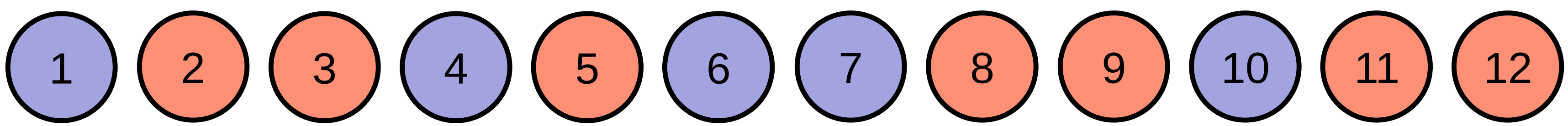
Erdős Discrepancy Problem: Given the subsets $S_{d,k} = \{d, 2d, \ldots, kd\}$, $1 \leq d \leq n$ and $1 \leq k \leq n/d$, does there exist a two-coloring of the elements $\{1, \ldots, n\}$ of discrepancy C?



$S_{3,4} = \{3, 6, 9, 12\}$ has discrepancy 2, so this coloring is of discrepancy at least 2.

Erdős Discrepancy Problem

Constructing a two-coloring of *{1,..,12}* of discrepancy at most *1*:

1 2 3 4 5 6 7 8 9 10 11 12

1) Start with arbitrary color for *1* (say blue)

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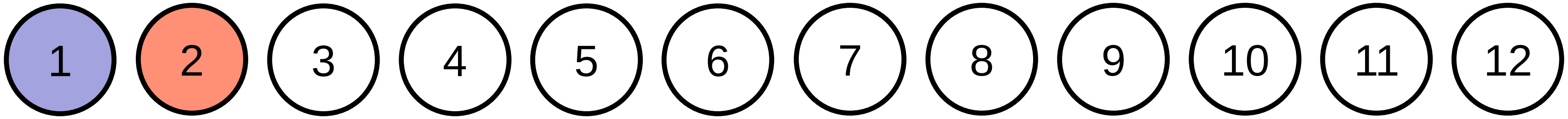

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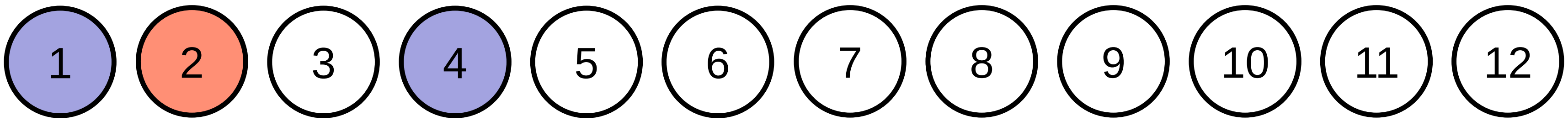

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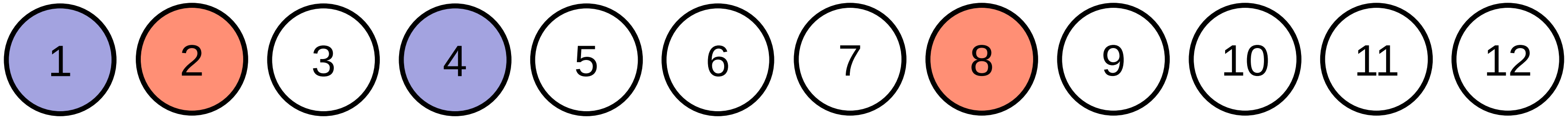

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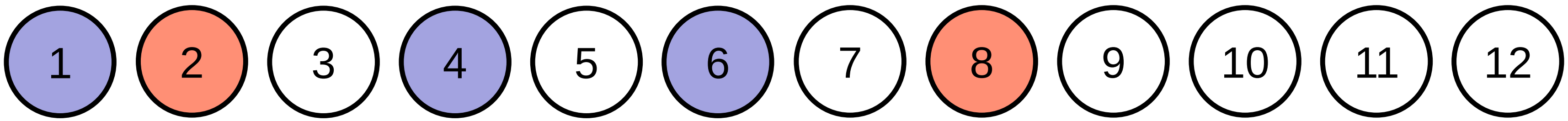

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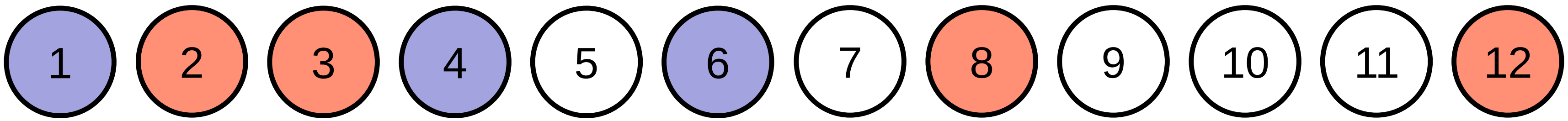

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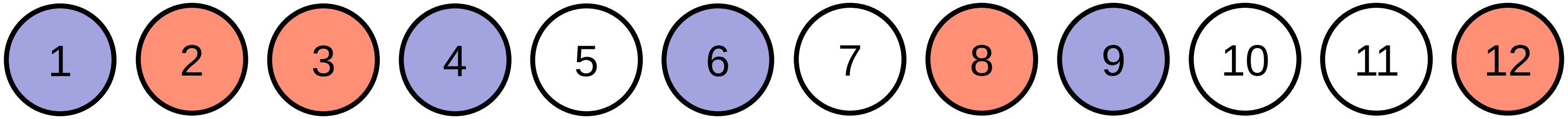

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7) *9* should be blue given the set *{3,6,9,12}*

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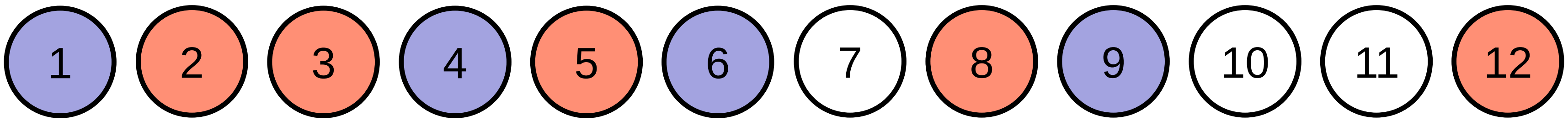

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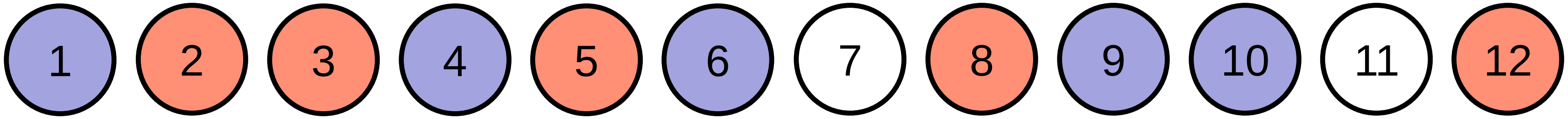

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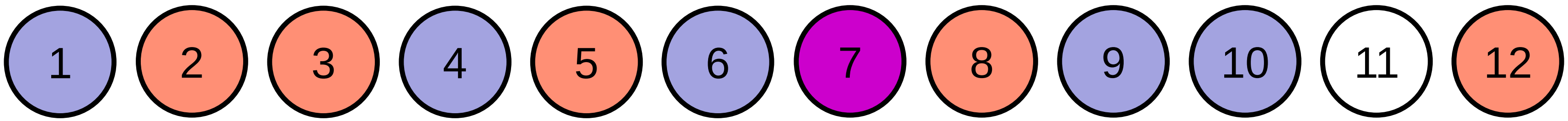
Erdős Discrepancy Problem

Constructing a two-coloring of *{1,..,12}* of discrepancy at most *1*:



1) Start with arbitrary color for *1* (say blue)
2) *2* should be red given the set *{1,2}*
3) *4* should be blue given the set *{2,4}*
4) *8* should be red given the set *{4,8}*
5) *6* should be blue given the set *{2,4,6,8}*
6) *3* should be red given the set *{3,6}*, and *12* should be red given *{6,12}*
7) *9* should be blue given the set *{3,6,9,12}*
8) *5* should be red given the set *{1,2,3,4,5,6}*
9) *10* should be blue given the set *{5,10}*
10) *7* should be blue given the set *{1,2,3,4,5,6,7,8}* and red given the set *{1,2,3,4,5,6,7,8,9,10}* - *IMPOSSIBLE*

Erdős Discrepancy Problem

Theorem: For any two-coloring of $\{1,\ldots,n\}$ where $n>11$, there is at least one subset $S_{d,k}=\{d,2d,\ldots,kd\}$ of discrepancy strictly greater than 1.

In the following, we replace the two colors with the labels $+1$ and -1.

Erdos Conjecture 1 (1930s): For any C, in any infinite ± 1-sequence (x_n), there is at least one subset $S_{d,k}=\{d,2d,\ldots,kd\}$ of discrepancy strictly greater than C.

Erdos Conjecture 2 (1930s): For any C, in any infinite ± 1-sequence (x_n) *s.t.* $x_{pq}=x_p*x_q$ (*completely multiplicative*), there is at least one subset $S_{d,k}=\{d,2d,\ldots,kd\}$ of discrepancy strictly greater than C.

The previous theorem proves that both conjectures are true for $C=1$.

Erdős Discrepancy Problem

[B. Konev and A. Lisitsa, A SAT Attack on the Erdos Discrepancy Conjecture, SAT'14]

Recent development proved the first conjecture for *C=2*, and provided a ±1-sequence of length ***1,160*** as the longest sequence of discrepancy ***2***.

They were also able to generate a ±1-sequence of discrepancy *3* of length ***13,900*** in about **214 hours** of CPU time, as the longest known sequence of discrepancy ***3***.

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In this work, using **streamlined search** to **boost a SAT solver**, we generate a sequence of discrepancy ***3*** of length ***127,645*** in about **1.5 hours** and claim it is tight bound for the completely multiplicative case.

	Size	Time
[Konev and Lisitsa]	13,900	214 hours
This work	127,645*	1.5 hours

Longest sequence of discrepancy 3. (*:tight bound for completely multiplicative sequence)

SAT Encoding

Given C, *a* ± 1 *sequence* $(x_1, \ldots, x_n)$,

Let p_i be the proposition corresponding to $x_i = +1$.

A proposition that tracks the state of the automaton for $(x_d, x_{2d}, \ldots, x_{\lfloor n/d \rfloor d})$:

$$\phi(n, C, d) = s_0^{(1,d)} \bigwedge_{m=1}^{n/d} \Bigg(\bigwedge_{-C \leq j < C} \left(s_j^{(m,d)} \wedge p_{id} \rightarrow s_{j+1}^{(m+1,d)}\right) \wedge$$

$$\bigwedge_{-C < j \leq C} \left(s_j^{(m,d)} \wedge \overline{p_{id}} \rightarrow s_{j+1}^{(m+1,d)}\right) \wedge$$

$$\left(s_C^{(m,d)} \wedge p_{id} \rightarrow s_B\right) \wedge$$

$$\left(s_{-C}^{(m,d)} \wedge \overline{p_{id}} \rightarrow s_B\right) \Bigg)$$

$s_j^{(m,d)}$ is true if the automaton is in state $\sum_{i=1}^{m-1} x_{i \cdot d}$ after $(x_d, \ldots, x_{(m-1)d})$.

s_B captures whether the sequence has exceeded the discrepancy C.

SAT Encoding

In addition, the automaton is in exactly one state:

$$\chi(n,C) = \bigwedge_{1\le d\le n/C, 1\le m\le n/d} \left(\bigvee_{-C\le j\le C} s_j^{(i,d)} \wedge \bigwedge_{-C\le j_1,j_2\le C} \left(\overline{s}_{j_1}^{(i,d)} \vee \overline{s}_{j_2}^{(i,d)}\right)\right)$$

$$\boxed{\mathbf{EDP}_1(n,C) : \overline{s}_B \wedge \chi(n,C) \wedge \bigwedge_{d=1}^{n} \phi(n,C,d)}$$

For the completely multiplicative case $(x_{id} = x_i x_d)$,

$$\mathcal{M}(i,d) = (p_i \vee p_d \vee p_{id}) \wedge (\overline{p_i} \vee \overline{p_d} \vee p_{id}) \wedge (p_i \vee \overline{p_d} \vee \overline{p_{id}}) \wedge (\overline{p_i} \vee p_d \vee \overline{p_{id}})$$

$$\boxed{\mathbf{EDP}_2(n,C) : \overline{s}_B \wedge \chi(n,C) \wedge \phi(n,C,1) \bigwedge_{1\le d\le n, 1\le i\le n/d} \mathcal{M}(i,d)}$$

Streamliners

+	−	−	+	−	+	+	−	−	+	−	−	+
−	+	+	−	+	+	−	−	+	−	+	+	−
+	+	−	−	+	−	+	+	−	−	+	−	−
+	−	+	+	−	+	+	−	−	+	−	+	+
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−	+	+	+	−	+	−	−	+	−	+	+	−
−	+	−	−	+	+	+	−	−	+	+	−	−

...

Fig. : First elements of a sequence of length 1160 and of discrepancy 2.

$$\boxed{mult(x, m, l) : x_{i \cdot d} = x_i x_d \; \forall 2 \leq d \leq m, 1 \leq i \leq n/d, i \leq l}$$

Streamliners

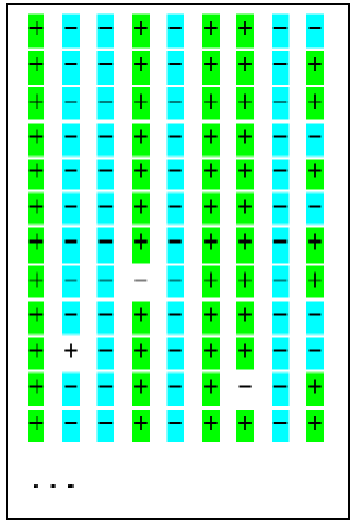
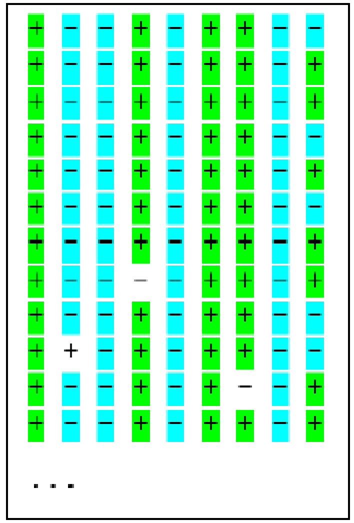


Fig. : First elements of a sequence of length 1160 and of discrepancy 2.

$$\boxed{period(x,p,t) : x_i = x_{i \bmod p}\ \forall 1 \le i \le t, i \not\equiv 0 \bmod p}$$

Streamliners



$$\mu_3(i) = \begin{cases} +1, & \text{if } i \text{ is 1 mod 3} \\ -1, & \text{if } i \text{ is 2 mod 3} \\ -\mu_3(i/3), & \text{otherwise.} \end{cases}$$

Fig. : First elements of a sequence of length 1160 and of discrepancy 2.

$$walters(x, w) : x_i = \mu_3(i)\ \forall 1 \leq i \leq w$$

Results

Encoding	*Streamliners*	*Size of sequence*	*Runtime (in sec)*
EDP_1	-	13,000	286,247
	-	13,500	560,663
	-	13,900	770,122
	mult(120,2000)	15,600	4,535
	mult(150,2000)	18,800	8,744
	mult(200,1000)	23,900	12,608
	mult(700,10000)	27,000	45,773
	mult(700,20000)	31,500	51,144
EDP_2	walters(800)	81,000	1,364
	walters(800)	108,000	4,333
	walters(700)	112,000	5,459
	walters(730)	127,645	4,501

Table : Solution runtimes of searches with and without streamliners.

Results

For **EDP$_2$**(n=127646,C=3):

- The SAT solver Lingeling proves **UNSAT** after about **60 hours** of computation.
- It generates a DRUP proof of approximately **29GB.**
- DRAT-trim, an independent satisfiability proof checker verifies the **88 million lemmas** of the proof in about **45 hours**.

> **Theorem:** Any **completely multiplicative** sequence of **discrepancy 3** is **finite.** (Erdos Conjecture is true for C=3).

Recursive Construction Rule

Let C be the discrepancy of $z = (z_1, z_2, ..., z_k)$ and C' the discrepancy of $(y_1, ..., y_{p-1})$.

$$\begin{aligned} x = (&y_1, y_2, \ldots, y_{p-2}, y_{p-1}, z_1 \\ &y_1, y_2, \ldots, y_{p-2}, y_{p-1}, z_2 \\ &\ldots \\ &y_1, y_2, \ldots, y_{p-2}, y_{p-1}, z_k) \end{aligned}$$

Definition (Discrepancy $\bmod p$). *Given two integers p and C', does there exist a ± 1 sequence $(y_1, \ldots, y_{p-1})$ such that:*

$$|\sum_{i=1}^{m} y_{i \cdot d \bmod p}| \leq C', \ \forall 1 \leq d \leq n, m < \frac{p}{gcd(d,p)}$$

$$\sum_{i=1}^{\frac{p}{gcd(d,p)}-1} y_{i \cdot d \bmod p} = 0, \ \forall 1 \leq d \leq n$$

Recursive Construction Rule - Results

There exist sequences whose discrepancy mod p is 1 for:

p = 1, 3, 5, 7, 9

There exist sequences whose discrepancy mod p is 2 for:

p = 11, 13, 15, 17, 25, 27, 45, 81

Therefore, $\mathbf{EDP_1}$(n=9*127645,C=4) is satisfiable.

Lemma: The longest sequence of **discrepancy 4** is of size **at least** 1,148,805.

Conclusions and Future Direction

Contributions

1) We improve the **lower bound** of the Erdos Discrepancy Problem for **discrepancy 3** from 13,901 to **127,646**.

2) We prove that this **bound is tight** for the **completely multiplicative** case.

3) We provide a **recursive construction** to inductively generate longer sequences of limited discrepancy.

Future Research Directions

1) Can we automate the **discovery of streamliners** and how to **combine them**?
2) Can we define **human intelligence tasks** for anyone to provide valuable **insights** about the problem **structure**?
3) How **small** can the **proof** of the conjecture be? What is the size of the smallest strong **backdoor** set?
4) Fundamental research question: Can any **combinatorial problem** whose sole **input** is its parameter **size** be solved in polynomial time?

Thank you.

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