From Streamlined Combinatorial Search to Efficient Constructive Procedures

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Computer Science

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Motivation

Background:

**Significant progress** in the area of **search, constraint satisfaction, and automated reasoning**.

These approaches have been evaluated on problems such as:

- \(N\)-Queens
- Round-Robin Tournament
- Orthogonal Latin squares
Motivation

Background:

**Significant progress** in the area of search, constraint satisfaction, and automated reasoning.

These approaches have been evaluated on problems such as:

- **N-Queens**
- **Round-Robin Tournament**
- **Orthogonal Latin squares**

Yet, 1) **constructions** have been found **by hand** for these problems, and

2) these techniques do not provide real **mathematical insights** on the structure of the problem and how to devise general construction rules.

**Goal:** Design a framework to discover **efficient constructive procedures**.
Example Domain: Design of Scientific Experiments

In the context of sustainability, assume we have 4 fertilizers, and we want to minimize their impact on the ‘dead-zones’ in the Gulf of Mexico.
Example Domain: Design of Scientific Experiments

In the context of **sustainability**, assume we have **4 fertilizers**, and we want to minimize their impact on the ‘dead-zones’ in the Gulf of Mexico.

In a more **general** context, assume we have **4 agronomic treatments** for growing beans and we want to **assess their effectiveness**.
Example Domain: Design of Scientific Experiments

In the context of **sustainability**, assume we have **4 fertilizers**, and we want to minimize their **impact on the ‘dead-zones’** in the Gulf of Mexico.

In a more **general** context, assume we have **4 agronomic treatments** for growing beans and we want to **assess their effectiveness**.

1) **We need to distribute** the treatments **evenly** over the test plots. 
Example Domain: 
Design of Scientific Experiments

In the context of sustainability, assume we have 4 fertilizers, and we want to minimize their impact on the ‘dead-zones’ in the Gulf of Mexico.

![Fertilizer, Nitrogen Based Fertilizers, Dead zones]

In a more general context, assume we have 4 agronomic treatments for growing beans and we want to assess their effectiveness.

1) We need to distribute the treatments evenly over the test plots.

2) We need to eliminate the correlation bias as much as possible, and factor in the geometry of the fields.
Example Domain:
The *Spatially-balanced Latin square (SBLS)* problem

**Problem Definition:**

An *SBLS* of order $n$ is an $n \times n$ square grid in which:

- Each symbol appears exactly once in each row and column (*Latin square* structure).

![SBLS of order 6](image)
Example Domain: The *Spatially-balanced Latin square (SBLS)* problem

**Problem Definition:**

An *SBLS* of order *n* is an *n x n* square grid in which:

- Each symbol appears exactly once in each row and column (*Latin square structure*).
- The average distance (column-wise) of a pair of symbols is the same for any pair (*Balanced structure*).

**Row Distance for pairs of colors**

<table>
<thead>
<tr>
<th>SBLS of order 6</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</table>

Total Row Distance (pair): 14  14  14  14  14
Average Row Distance (pair): 2.33  2.33  2.33  2.33  2.33
Example Domain: The *Spatially-balanced Latin square (SBLS)* problem

A computationally challenging combinatorial design problem:

<table>
<thead>
<tr>
<th>Approach</th>
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<th>Time (s)</th>
<th>Reference</th>
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<tbody>
<tr>
<td>Constraint Programming (CP)</td>
<td>9</td>
<td>241</td>
<td>[Gomes and Sellmann, CP’04]</td>
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<tr>
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<td>4.5</td>
<td>[Neveu et al., CP’04]</td>
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<tr>
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<td>5,434</td>
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<tr>
<td>Composition-based Streamlined CP</td>
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<td>1.2M</td>
<td>[Smith et al., IJCAI’05]</td>
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*The largest SBLS ever found (35x35)*
Example Domain: The *Spatially-balanced Latin square (SBLS)* problem

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The largest SBLS ever found (35x35)

**Conjecture** [Gomes, Sellmann et al., CPAIOR’04]

There exist *arbitrary large SBLSs*, and an *effective* way of *constructing* them.
Example Domain:
The *Spatially-balanced Latin square (SBLS)* problem

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**Conjecture** [Gomes, Sellmann et al., CPAIOR’04]

There exist *arbitrary large SBLSs*, and an *effective* way of *constructing* them.

**Goal:** *Discover an efficient construction.*

*The largest SBLS ever found (35x35)*
Outline

• Motivation
• Example Domain
• Proposed Framework
  • Overview of Streamlined Search
  • Taking advantage of Human Insights
  • Formal Description and Overview
  • GUI for Human-guided Streamlined Search
• Application to the *Spatially-balanced Latin square* problem
• Application to the *Weak Schur Number* problem
• Conclusions and Future work
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Goal:
Exploit the structure of some solutions to dramatically boost the effectiveness of the propagation mechanisms.

Underlying Observation:
When one insists on maintaining the full solution set, there is a hard practical limit on the effectiveness of constraint propagation methods. Often, there is no compact representation for all the solutions.

Underlying Conjecture:
For many intricate combinatorial problems – if solutions exist – there will often be regular ones.
Proposed Framework: Overview of Streamlined Search

\( P_1 \) is substantially smaller than its complement \( P_2 \) and it will benefit from a stronger filtering thereafter.

Streamlined Search:

Strong branching mechanisms (by adding constraints based on structure properties) at high levels of the search tree.

[Gomes and Sellmann, CP’04]
Proposed Framework:
Taking advantage of Human Insights

Recognizing Patterns and Regularities:

Correcting Irregularities:

Generalizing / Formalizing Regularities:

[Cyclic Latin square of order 3]

[Cyclic Latin square of order 4]

[Source: Marijn J.H. Heule, 2009]
Proposed Framework: Formal Description and Overview

\[ \mathcal{O} \leftarrow \emptyset; \quad \text{\(\text{// Conjectured streamliners}\)} \]
\[ \Gamma \leftarrow \emptyset; \quad \text{\(\text{// Search streamliners}\)} \]
\[ \rho \leftarrow \rho_0; \quad \text{\(\text{// Search parameter}\)} \]
\[ S \leftarrow \emptyset; \quad \text{\(\text{// Solutions found}\)} \]
\[ \tau \leftarrow \text{false}; \quad \text{\(\text{// Timeout flag}\)} \]

repeat
  \[ \text{Solve}(P_\rho, \Gamma, t) \rightarrow (S', \tau); \quad \text{\(\text{// Search for new solutions}\)} \]
  if \( S' \cap S \neq \emptyset \) then
    \[ S \leftarrow S \cup S'; \quad \text{\(\text{// Case 1: successful search}\)} \]
    \[ \text{Analyze}(S) \rightarrow \mathcal{O}'; \quad \text{\(\text{// Conjecture new streamliners}\)} \]
    \[ \mathcal{O} \leftarrow \mathcal{O} \cup \mathcal{O}'; \]
    \[ \rho \leftarrow \rho + 1; \]
  else if \( \tau \text{ is true} \) then
    Select \( \Gamma' \subseteq \mathcal{O}'; \quad \text{\(\text{// Case 2: timed-out failed search}\)} \]
    \[ \Gamma \leftarrow \Gamma \cup \Gamma'; \quad \text{\(\text{// Strengthen streamliners}\)} \]
  else
    Select \( \Gamma' \subseteq \Gamma; \quad \text{\(\text{// Case 3: exhaustive failed search}\)} \]
    \[ \Gamma \leftarrow \Gamma \setminus \Gamma'; \quad \text{\(\text{// Weaken streamliners}\)} \]
    \[ \rho = \max\{\rho : S(\Gamma) \cap S(P_\rho) \neq \emptyset\} + 1; \]
    Select \( \Gamma'' \subseteq \Gamma'; \quad \text{\(\text{// Find next parameter of interest}\)} \]
    \[ \mathcal{O} \leftarrow \mathcal{O} \setminus \Gamma''; \quad \text{\(\text{// Drop unpromising streamliners}\)} \]
  until \( \mathcal{O} = \emptyset; \)

**Algorithm**: Discover-Construction procedure
for a given problem \( P \), with parameter set \( \rho \) and timeout \( t \).
Proposed Framework: Formal Description and Overview

1. Analyze smaller size solutions, and conjecture potential regularities in the solutions.

2. Validate through streamlining the observed regularities.

3. If the streamlined search does not give a larger size solution, the proposed regularity is quite likely accidental and one looks for a new pattern in the small scale solutions.

4. Otherwise, one proceeds by generating a number of new solutions that all contain the proposed structural regularity and are used to expand the solution set and to reveal new regularities.

Algorithm: Discover-Construction procedure for a given problem P, with parameter set ρ and timeout t.
Proposed Framework for Human-guided Streamlined Search
Overview on the *SBLS* problem

Start with first order of interest ($n=3$) and no streamliners ($\Gamma=\emptyset$)

- **Search Parameters**
  - $n=3$
  - $\Gamma=\emptyset$

- **Conjectured Streamliners**

- **Solution Set**
Solve for $n$ and $\Gamma$

Search Parameters

$n=3$

$\Gamma=\emptyset$

Conjectured Streamliners

Start with first order of interest ($n=3$) and no streamliners ($\Gamma=\emptyset$)

Solution Set

Proposed Framework for Human-guided Streamlined Search

Overview on the SBLS problem
Proposed Framework for Human-guided Streamlined Search Overview on the SBLS problem

Search Parameters

\[ n = 5 \]
\[ \Gamma = \emptyset \]

Conjectured Streamliners

\{Symmetric, Cyclic\}

Start with first order of interest (\( n=3 \)) and no streamliners (\( \Gamma = \emptyset \))

Solve for \( n \) and \( \Gamma \)

\{new solutions\}

Conjecture new streamliners

\{new streamliners, increase \( n \)\}

Solution Set
Proposed Framework for Human-guided Streamlined Search
Overview on the *SBLS* problem

**Search Parameters**

\[ n = 5 \]
\[ \Gamma = \{\text{Symmetric}\} \]

**Conjectured Streamliners**

\{Symmetric, Cyclic\}

**Solution Set**

\[
\begin{array}{ccc}
1 & 2 & 3 \\
2 & 1 & 3 \\
3 & 2 & 1 \\
\end{array}
\]

Start with first order of interest (\( n = 3 \)) and no streamliners (\( \Gamma = \{\}\)).

Solve for \( n \) and \( \Gamma \).

Conjecture new streamliners.

\{new solutions\}

Strengthen \( \Gamma \).

\{time-out\}

\{new streamliners, increase \( n \)\}
Proposed Framework for Human-guided Streamlined Search
Overview on the SBLS problem

Search Parameters

\[ n=6 \]
\[ \Gamma=\{\text{Symmetric}\} \]

Conjectured Streamliners

\{Symmetric, Cyclic, \textbf{Reduced}\}

\[ \text{Start with first order of interest (}n=3\text{) and no streamliners (}\Gamma=\{\}\text{)} \]

Solve for \(n\) and \(\Gamma\)

{new solutions}

\{time-out\}

Conjecture new streamliners

{new streamliners, increase \(n\)}

Strengthen \(\Gamma\)

Solution Set

\[ \begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{array} \]

\[ \begin{array}{ccc}
4 & 1 & 5 \\
5 & 3 & 1 \\
1 & 5 & 3 \\
\end{array} \]

\[ \begin{array}{ccc}
2 & 1 & 3 \\
3 & 4 & 2 \\
2 & 1 & 3 \\
\end{array} \]

\[ \begin{array}{ccc}
1 & 2 & 3 \\
4 & 1 & 5 \\
3 & 5 & 1 \\
\end{array} \]

\[ \begin{array}{ccc}
2 & 1 & 3 \\
2 & 5 & 4 \\
3 & 4 & 2 \\
\end{array} \]

\[ \begin{array}{ccc}
1 & 2 & 3 \\
1 & 3 & 2 \\
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\end{array} \]
Proposed Framework for Human-guided Streamlined Search Overview on the SBLS problem

Search Parameters

\( n=6 \)
\( \Gamma=\{\text{Symmetric, Cyclic}\} \)

Conjectured Streamliners

\{Symmetric, Cyclic, Reduced\}

Solve for \( n \) and \( \Gamma \)

Start with first order of interest (\( n=3 \)) and no streamliners (\( \Gamma=\{\} \))

Conjecture new streamliners

\{new solutions\}

Strengthen \( \Gamma \)

\{time-out\}

\{new streamliners, increase \( n \)\}

Solution Set
Proposed Framework for Human-guided Streamlined Search

Overview on the SBLS problem

Search Parameters

\[ n = 3 \]
\[ \Gamma = \{ \text{Symmetric, Cyclic} \} \]

Conjectured Streamliners

\{ \text{Symmetric, Cyclic, Reduced} \}

Solution Set

Solve for \( n \) and \( \Gamma \)

Start with first order of interest (\( n=3 \)) and no streamliners (\( \Gamma = \{ \} \))

\{ new solutions \} \rightarrow \{ time-out \} \rightarrow \{ new streamliners, increase n \} \rightarrow \{ set next interesting n \}

\{ exhaustive failed search \} \rightarrow \{ weaken \( \Gamma \) \}

\{ strengthen \( \Gamma \) \}

Conjecture new streamliners
GUI for Human-guided Streamlined Search
GUI for Human-guided Streamlined Search

Conjecture new streamliners
GUI for Human-guided Streamlined Search

Select streamliners, parameters, and perform the search
GUI for Human-guided Streamlined Search

Select solutions
GUI for Human-guided Streamlined Search

Analyze solutions
Outline

• Motivation
• Example Domain
• Proposed Framework
  • Application to the *Spatially-balanced Latin square* problem
    • Successful Streamliners
    • Constructive Procedure 1
    • Constructive Procedure 2
  • Application to the *Weak Schur Number* problem
• Conclusions and Future work
**Successful Key Streamliners:**

{Diagonal symmetry, Reduced form, Assignments of columns 2 and \( n \), Multiples of \( i \) in row \( i \), Second sequence decreasing}

<table>
<thead>
<tr>
<th>Streamliners</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>9</th>
<th>11</th>
<th>14</th>
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</thead>
<tbody>
<tr>
<td>( \Gamma_1 = \emptyset )</td>
<td>5760</td>
<td>15878</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Gamma_2 = \Gamma_1 \cup { \text{Symmetric} } )</td>
<td>240</td>
<td>8447</td>
<td>714</td>
<td>43</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Gamma_3 = \Gamma_2 \cup { \text{Reduced} } )</td>
<td>2</td>
<td>14</td>
<td>14</td>
<td>51</td>
<td></td>
<td></td>
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<tr>
<td>( \Gamma_4 = \Gamma_3 \cup { \text{Columns 2 &amp; n} } )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
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<tr>
<td>( \Gamma_5 = \Gamma_4 \cup { \text{Multiples of } i } )</td>
<td>1</td>
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</tbody>
</table>

*Fig: Number of SBLSs generated in 60 seconds, by order and streamliners (Bold indicates exhaustive search).*
Application to the *SBLS* problem: Construction 1
### Application to the *SBLS* problem: Construction 1

<table>
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**Notes:**
- **a**
- **a+i**
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Application to the *SBLS* problem: Construction 1

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for row $i = 1, \ldots, N$ do

$k = 1$;  \hspace{1cm} // Sequence number

$j = 1$;  \hspace{1cm} // Column index

$a_{i,j} = i$;  \hspace{1cm} // First symbol of row $i$

while $j < N$ do

if $k$ is odd then

while $a_{i,j} + i \leq N$ and $j < N$ do

$a_{i,j+1} = a_{i,j} + i$;

$j = j + 1$;

else

while $a_{i,j} - i \geq 1$ and $j < N$ do

$a_{i,j+1} = a_{i,j} - i$;

$j = j + 1$;

if $j < N$ then

if $k$ is odd then

$a_{i,j+1} = 2N + 1 - i - a_{i,j}$;

else

$a_{i,j+1} = i - a_{i,j}$;

$k = k + 1$;

$j = j + 1$;

Algorithm: *SBLS*-sequence procedure for SBLS of order $N$, when $2N + 1$ is prime.
Application to the *SBLS* problem: Construction 1

Application to the *SBLS* problem: Construction 2

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\[
c_{1,1} = 1; \quad \text{\textit{Generate 1st row of the conjugate}} \\
\text{\textbf{for column} } j = 2, \ldots, N \text{ do} \quad \text{\textit{Observed pattern 1, 2, 4, ...}} \\
\quad \text{if } 2c_{1,j-1} \leq N \text{ then} \\
\quad \quad c_{1,j} = 2c_{1,j-1}; \\
\quad \text{else} \\
\quad \quad c_{1,j} = 2N + 1 - 2c_{1,j-1}; \\
\text{\textbf{for row} } i = 2, \ldots, N \text{ do} \quad \text{\textit{Subsequent rows}} \\
\quad c_{i,1} = c_{i-1,N}; \quad \text{\textit{Shifted version of previous column}} \\
\text{\textbf{for column} } j = 2, \ldots, N \text{ do} \\
\quad c_{i,j} = c_{i-1,j-1}; \\
\text{\textbf{for row} } i = 1, \ldots, N \text{ do} \quad \text{\textit{Generate SBLS from conjugate}} \\
\quad \text{\textbf{for column} } j = 1, \ldots, N \text{ do} \\
\quad a_{i,c_{i,j}} = j; \]

\textbf{Algorithm: SBLS-Cyclic procedure.}
Application to the Weak Schur problem

Problem Definition:

■ A set is \textit{(weakly) sum free} if for any two (distinct) elements of this set, their sum does not belong to the set.

■ The \textbf{Weak Schur Number} of order $k$, $WS(k)$, is the largest integer $n$ for which there exists a partition of $[1,n]$ into $k$ weakly sum-free sets.

\begin{center}
\begin{tabular}{|c|c|}
\hline
$k=3$ sets & \begin{tabular}{|c|c|c|c|c|}
1 & 2 & 4 & 8 & 11 \ 22 \\
3 & 5-7 & 19 & 21 & 23 \\
9 & 10 & 12-18 & 20 \\
\hline
\end{tabular} \\
\hline
\end{tabular}
\end{center}

Each of the 3 sets is such that, for any 2 elements of the set, their sum does not belong to the same set.

\textit{Fig:} Partition of $[1,23]$ into 3 weakly sum-free sets, proving $WS(3) \geq 23$
Application to the *Weak Schur* problem

Best known lower bounds:

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<th>WS(6)</th>
<th>Reference</th>
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<td>-</td>
<td>[G.W. Walker, AMM’50]</td>
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<td>[J.H. Braun, AMM’50]</td>
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<td>[Fonlupt et al., EA’11]</td>
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<td>[Eliahou et al., Computers &amp; Math Applications’12] <em>(revised)</em></td>
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</table>
Application to the *Weak Schur* problem

**Successful Key Streamliners:**

{Ordered sets, constrained minimum of each set, partial assignments, sequences of consecutive integers, sequence interleaving}
Application to the *Weak Schur* problem

**Successful Key Streamliners:**

{Ordered sets, constrained minimum of each set, partial assignments, sequences of consecutive integers, sequence interleaving}

| 1 2 4 8 11 22 25 50 63 68 139 149 154 177 182 192 198 393 398 408 413 436 450 455 521 526 540 563 568 578 |
| 24 26-49 153 155-176 178 412 414-435 437 539 541-562 564 |
| 67 69-135 454 456-520 522 |
| 196 197 199-392 394 |

*Fig:* Partition of [1,581] into 6 weakly sum-free sets, proving \( WS(6) \geq 581. \)

Although **not** an example of a **fully constructive procedure** yet, any progress on Schur numbers is quite **significant** given their long history.
Conclusion and Future work

- General framework that integrates specialized search techniques (so-called *streamlining*) with human insight in an iterative approach to discover efficient constructive procedures.

- Provides the first constructive procedures for the *Spatially-Balanced Latin Square* problem.

- Improves the best known lower bound for the *Weak Schur Number* problem.

- One exciting extension would be to crowd-source the search for regularities in the solution set.
Extra slides
### SBLS of order 50 – Construction 1

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |

The table above represents the SBLS of order 50, constructed using Construction 1.
Results of \textit{SBLS-sequence} (U&B)

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