From Streamlined Combinatorial Search to Efficient Constructive Procedures

Ronan Le Bras — Computer Science
Carla P. Gomes — Computer Science
Bart Selman — Computer Science

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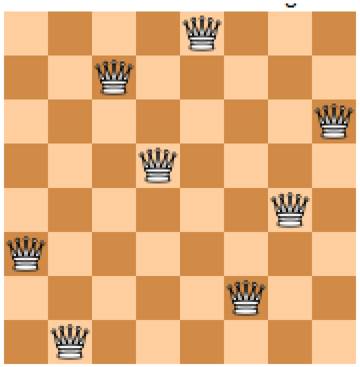
Motivation

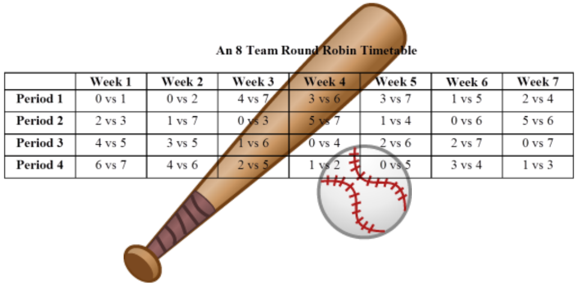
Background:

Significant progress in the area of **search**, **constraint satisfaction**, and **automated reasoning**.

These approaches have been evaluated on problems such as:



N-Queens



An 8 Team Round Robin Timetable

	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6	Week 7
Period 1	0 vs 1	0 vs 2	4 vs 7	3 vs 6	3 vs 7	1 vs 5	2 vs 4
Period 2	2 vs 3	1 vs 7	0 vs 3	5 vs 7	1 vs 4	0 vs 6	5 vs 6
Period 3	4 vs 5	3 vs 5	1 vs 6	0 vs 4	2 vs 6	2 vs 7	0 vs 7
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Round-Robin Tournament

Orthogonal Latin squares

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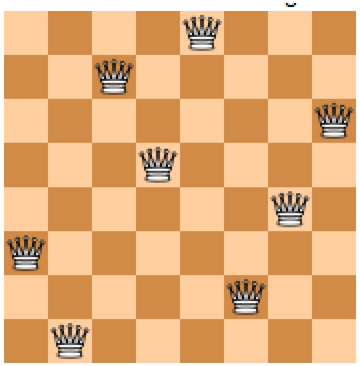

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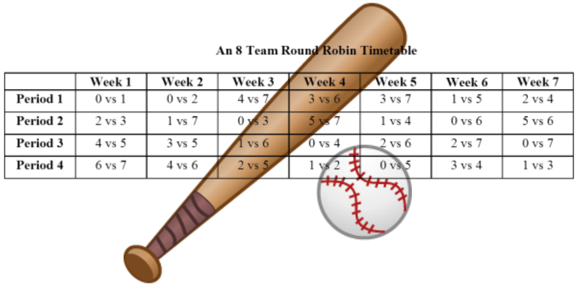

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Round-Robin Tournament

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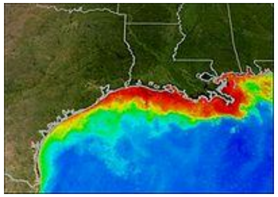
Yet, 1) **constructions** have been found **by hand** for these problems , and

2) these techniques do not provide real **mathematical insights** on the structure of the problem and how to devise general construction rules.

Goal: Design a framework to discover **efficient constructive procedures.**

Example Domain:
Design of Scientific Experiments

In the context of **sustainability**, assume we have **4 fertilizers**, and we want to minimize their **impact on the 'dead-zones'** in the Gulf of Mexico.



Nitrogen Based Fertilizers

Dead zones

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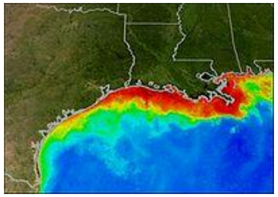


Nitrogen Based Fertilizers Dead zones

In a more **general** context, assume we have **4 agronomic treatments** for growing beans and we want to **assess their effectiveness.**

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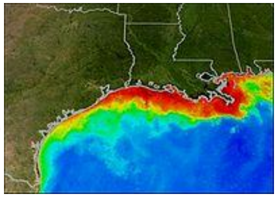

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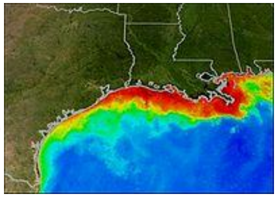


Nitrogen Based Fertilizers Dead zones

In a more **general** context, assume we have **4 agronomic treatments** for growing beans and we want to **assess their effectiveness.**

1) We need to **distribute** the treatments **evenly** over the test plots.

2) We need to **eliminate the correlation bias** as much as possible, and factor in the **geometry** of the fields

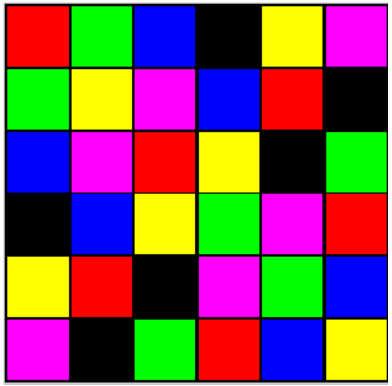
Example Domain:
The *Spatially-balanced Latin square (SBLS)* problem

Problem Definition:

An ***SBLS*** of order *n* is an *n* x *n* square grid in which:

- Each symbol appears exactly once in each row and column (*Latin square* structure).

SBLS of order 6



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- The average distance (column-wise) of a pair of symbols is the same for any pair *(Balanced structure)***.**

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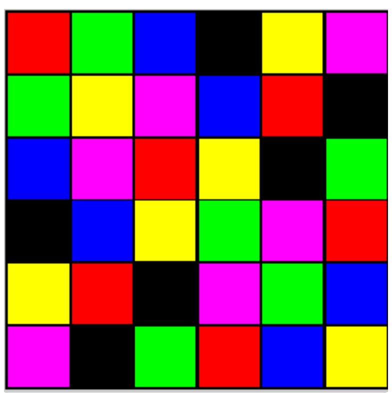

Row Distance for pairs of colors

	1	2	3	4	5
	4	1	1	3	2
	3	2	2	1	1
	2	4	5	3	1
	3	4	4	1	2
	1	1	1	2	3
Total Row Distance (pair)	14	14	14	14	14
Average Row Distance (pair)	2.33	2.33	2.33	2.33	2.33

Example Domain:
The *Spatially-balanced Latin square (SBLS)* problem

A computationally challenging combinatorial design problem:

Approach	Order	Time (s)	Reference
Constraint Programming (CP)	9	241	[Gomes and Sellmann, CP’04]
IDWalk *(metaheuristic)*	9	4.5	[Neveu et al., CP’04]
Self-symmetry-based Streamlined CP	14	5,434	[Gomes and Sellmann, CP’04]
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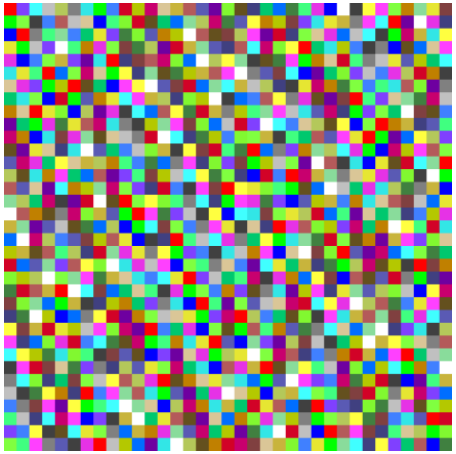
Streamlined Local Search	35	1.2M	[Smith et al., IJCAI’05]



The largest SBLS ever found (35x35)

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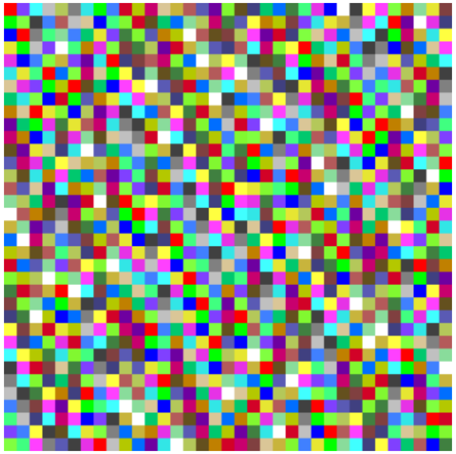

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Conjecture [Gomes, Sellmann et al., CPAIOR’04]

There exist **arbitrary large SBLSs**, and an **effective** way of **constructing** them.

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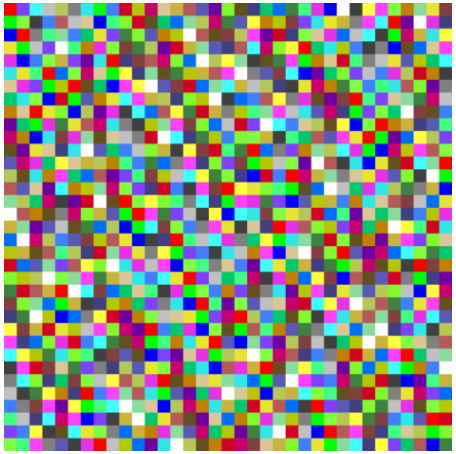

The largest SBLS ever found (35x35)

Conjecture [Gomes, Sellmann et al., CPAIOR'04]

There exist **arbitrary large SBLSs**, and an **effective** way of **constructing** them.

Goal: Discover an **efficient construction.**

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Outline

- Motivation
- Example Domain
- Proposed Framework
  - Overview of Streamlined Search
  - Taking advantage of Human Insights
  - Formal Description and Overview
  - GUI for Human-guided Streamlined Search
- Application to the *Spatially-balanced Latin square* problem
- Application to the *Weak Schur Number* problem
- Conclusions and Future work

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Proposed Framework: Overview of Streamlined Search

Goal:

Exploit the **structure of some solutions** to dramatically **boost** the effectiveness of the **propagation mechanisms.**

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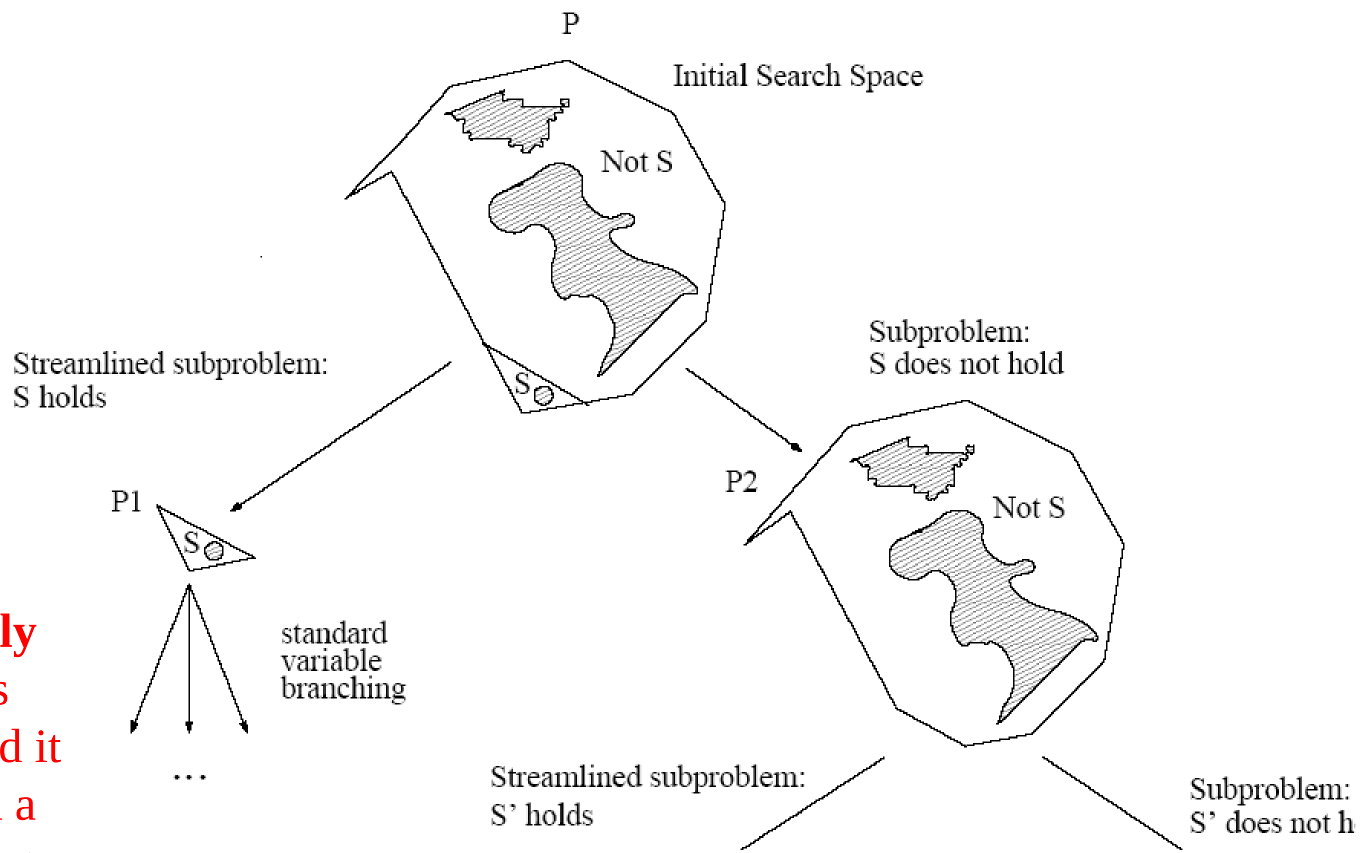
Underlying Observation:

When one insists on maintaining the **full solution set**, there is a **hard practical limit** on the effectiveness of **constraint propagation** methods. Often, there is **no compact representation** for all the solutions.

Underlying Conjecture:

For many intricate **combinatorial problems** – if **solutions exist** – there will often be **regular ones**.

Proposed Framework:
Overview of Streamlined Search



P_1 is **substantially smaller** than its complement P_2 and it will benefit from a **stronger filtering** thereafter**.**

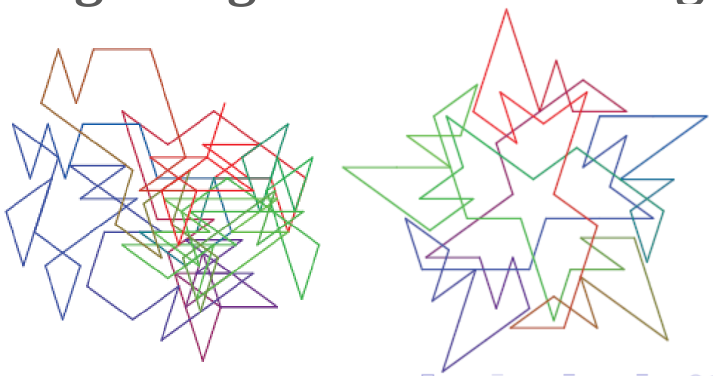
Streamlined Search:

Strong **branching mechanisms** (by adding constraints based on **structure properties**) at **high levels** of the search tree.

[Gomes and Sellmann, CP'04]

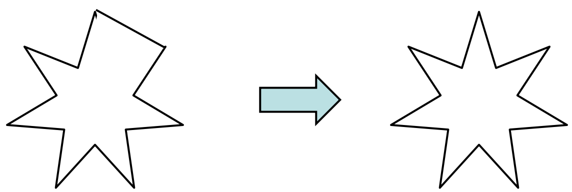
Proposed Framework:
Taking advantage of Human Insights

Recognizing Patterns and Regularities:



[Source: Marijn J.H. Heule, 2009]

Correcting Irregularities:



Generalizing / Formalizing Regularities:

1	2	3
3	1	2
2	3	1

Cyclic Latin square of order 3

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

Cyclic Latin square of order 4

Proposed Framework: Formal Description and Overview

$\mathcal{O} \leftarrow \emptyset$; // Conjectured streamliners
$\Gamma \leftarrow \emptyset$; // Search streamliners
$\rho \leftarrow \rho_0$; // Search parameter
$\mathcal{S} \leftarrow \emptyset$; // Solutions found
$\tau \leftarrow false$; // Timeout flag
repeat
  $Solve(P_\rho, \Gamma, t) \rightarrow (\mathcal{S}', \tau)$; // Search for new solutions
  if $\mathcal{S}' \cap \mathcal{S} \neq \emptyset$ **then**
    $\mathcal{S} \leftarrow \mathcal{S} \cup \mathcal{S}'$; // Case 1: successful search
    $Analyze(\mathcal{S}) \rightarrow \mathcal{O}'$; // Conjecture new streamliners
    $\mathcal{O} \leftarrow \mathcal{O} \cup \mathcal{O}'$;
    $\rho \leftarrow \rho + 1$;
  else if τ *is true* **then**
    Select $\Gamma' \subseteq \mathcal{O}$; // Case 2: timed-out failed search
    $\Gamma \leftarrow \Gamma \cup \Gamma'$; // Strengthen streamliners
  else
    Select $\Gamma' \subseteq \Gamma$; // Case 3: exhaustive failed search
    $\Gamma \leftarrow \Gamma \setminus \Gamma'$; // Weaken streamliners
    $\rho = max\{\rho : \mathcal{S}(\Gamma) \cap \mathcal{S}(P_\rho) \neq \emptyset\}+1$;
    Select $\Gamma'' \subseteq \Gamma'$; // Find next parameter of interest
    $\mathcal{O} \leftarrow \mathcal{O} \setminus \Gamma''$; // Drop unpromising streamliners
until $\mathcal{O} = \emptyset$;

Algorithm : `Discover-Construction` procedure for a given problem P, with parameter set ρ and timeout t.

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Algorithm : `Discover-Construction` procedure for a given problem P, with parameter set ρ and timeout t.

1. **Analyze** smaller size solutions, and **conjecture potential regularities** in the solutions.
2. Validate through **streamlining** the observed regularities.
3. If the streamlined search **does not give a larger size solution**, the proposed regularity is quite likely **accidental** and one looks for a new pattern in the small scale solutions.
4. Otherwise, one proceeds by generating a number of **new solutions** that all contain the proposed **structural regularity** and are used to expand the solution set and to **reveal new regularities**.

Proposed Framework for Human-guided Streamlined Search
Overview on the *SBLS* problem

Search Parameters

$n=3$

$\Gamma=\{\}$

Conjectured Streamliners

Solution Set

Start with first order of interest (n=3) and no streamliners ($\Gamma=\{\}$)

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Solve for n and Γ

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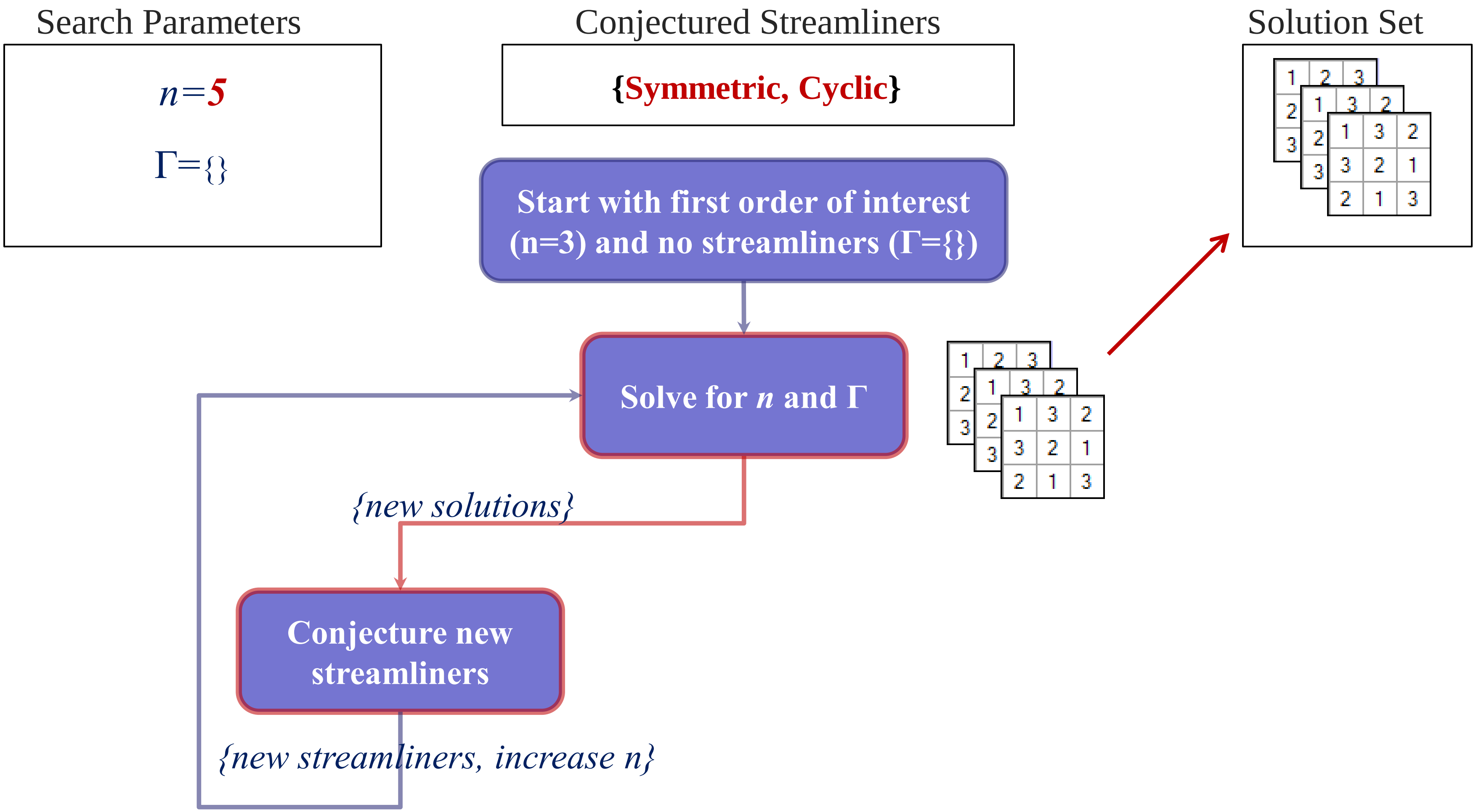


Proposed Framework for Human-guided Streamlined Search
Overview on the *SBLS* problem

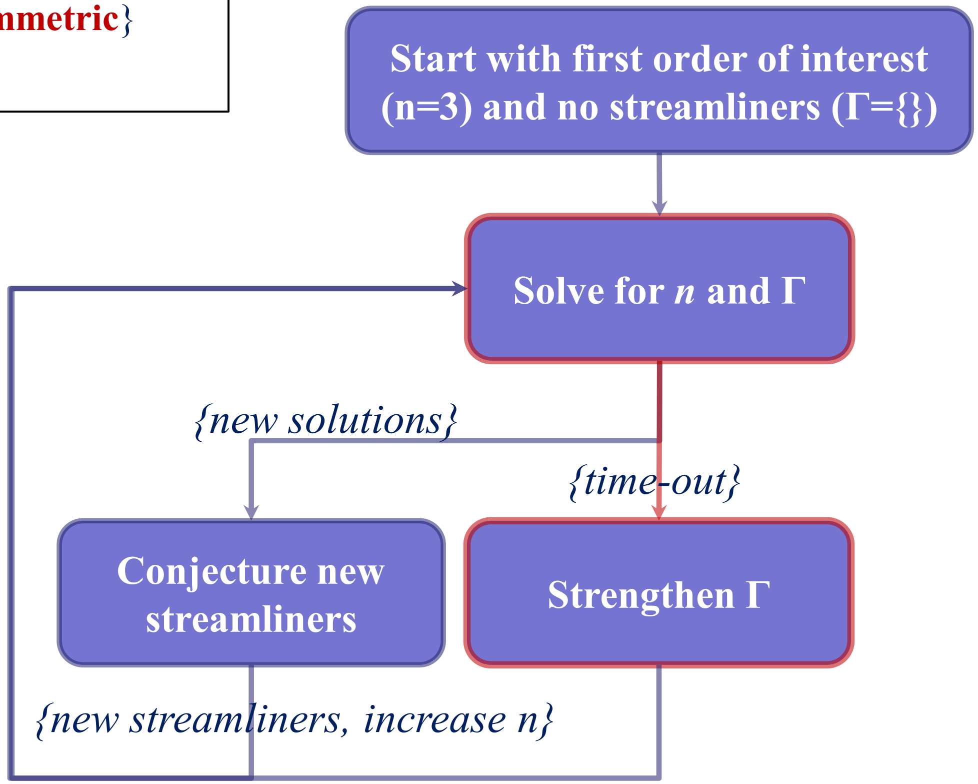
Search Parameters

$n=5$

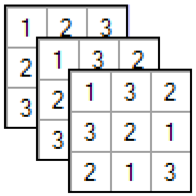
$\Gamma=${**Symmetric**}

Conjectured Streamliners

{Symmetric, Cyclic}



Solution Set



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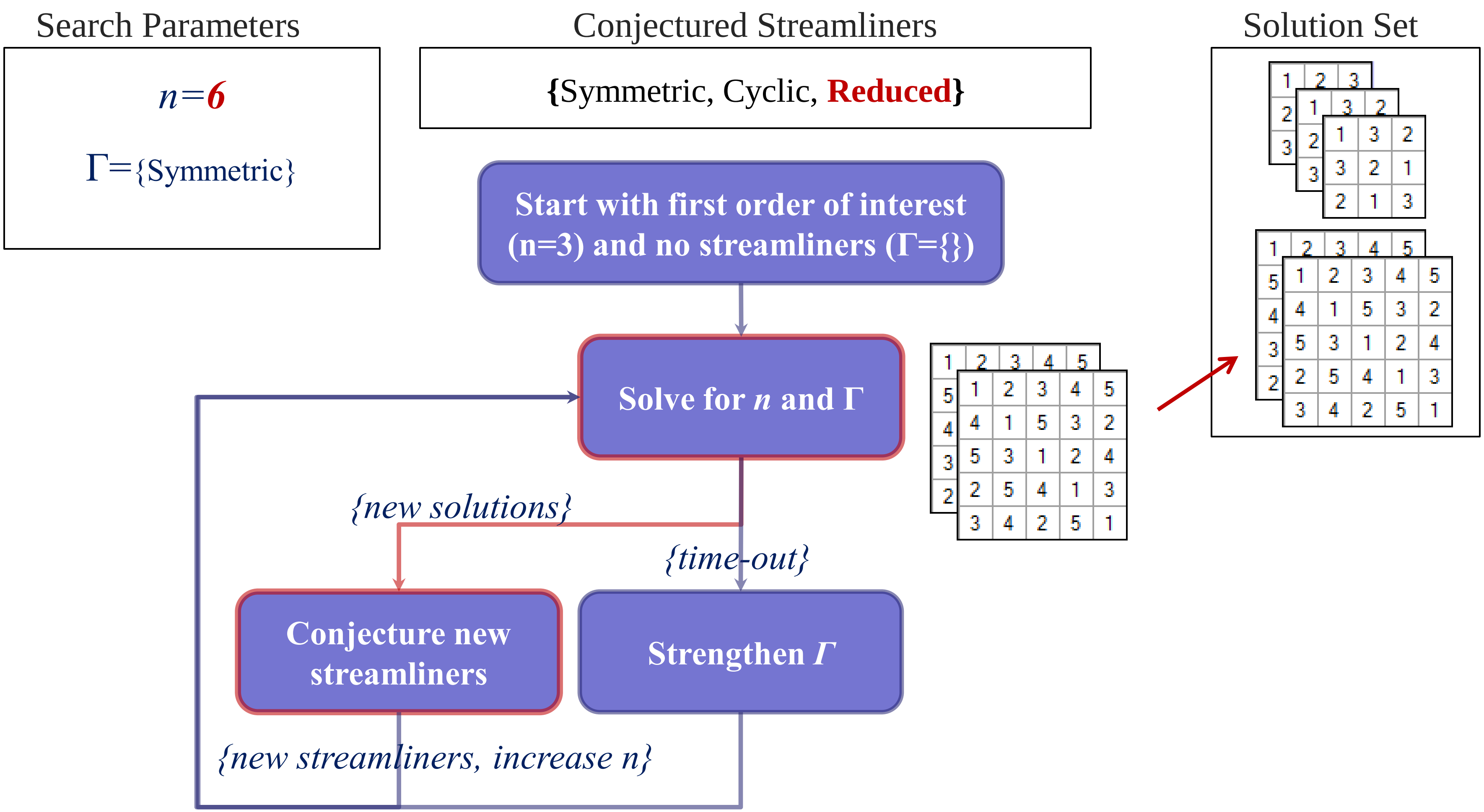


Proposed Framework for Human-guided Streamlined Search
Overview on the *SBLS* problem

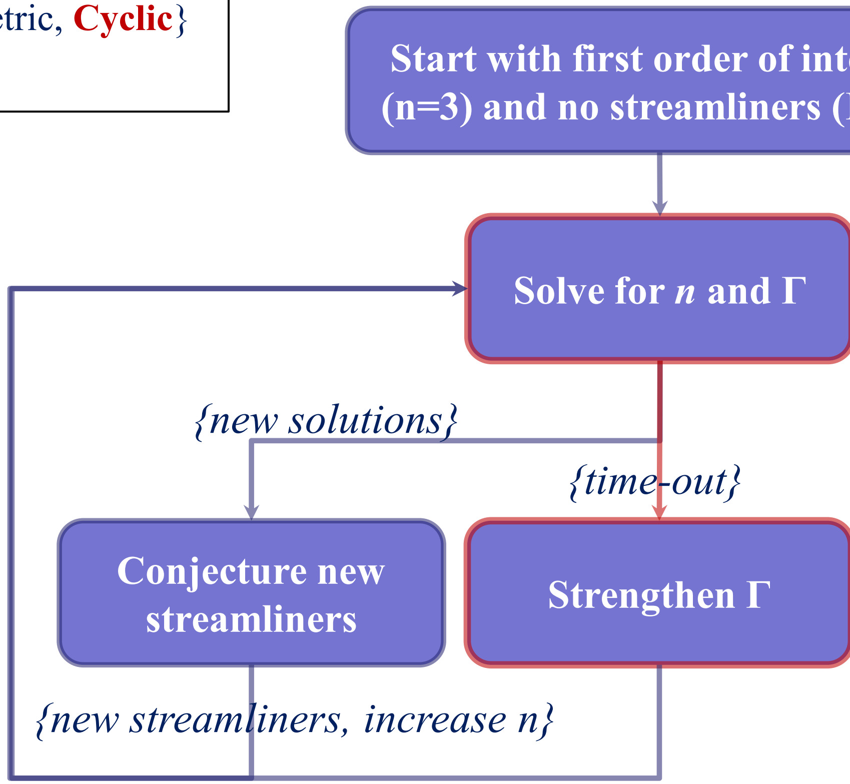
Search Parameters

n=6

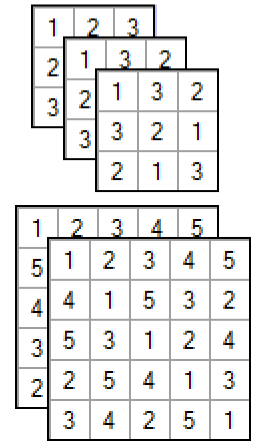
Γ={Symmetric, **Cyclic**}

Conjectured Streamliners

{Symmetric, Cyclic, Reduced}



Solution Set



Proposed Framework for Human-guided Streamlined Search
Overview on the *SBLS* problem

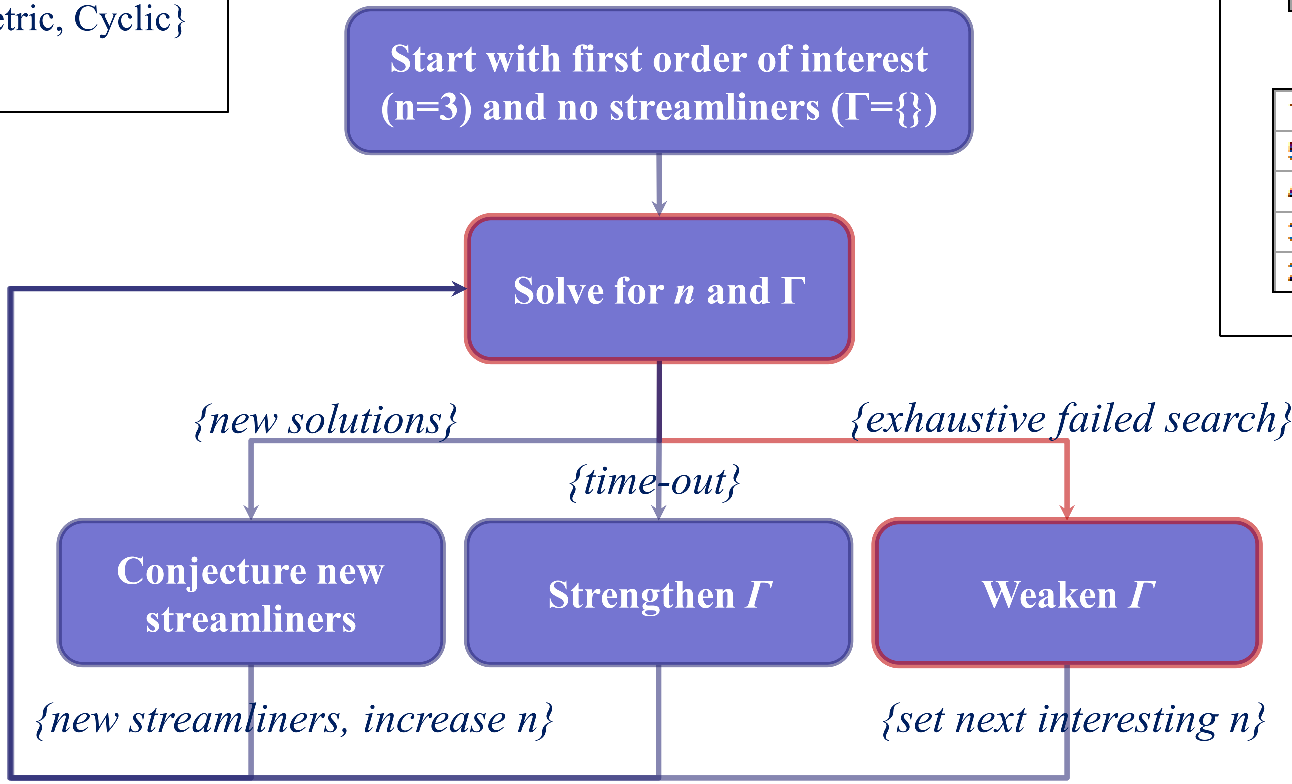
Search Parameters

$n=\cancel{6}3$

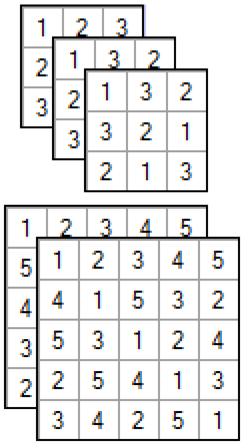
Γ={~~Symmetric~~, Cyclic}

Conjectured Streamliners

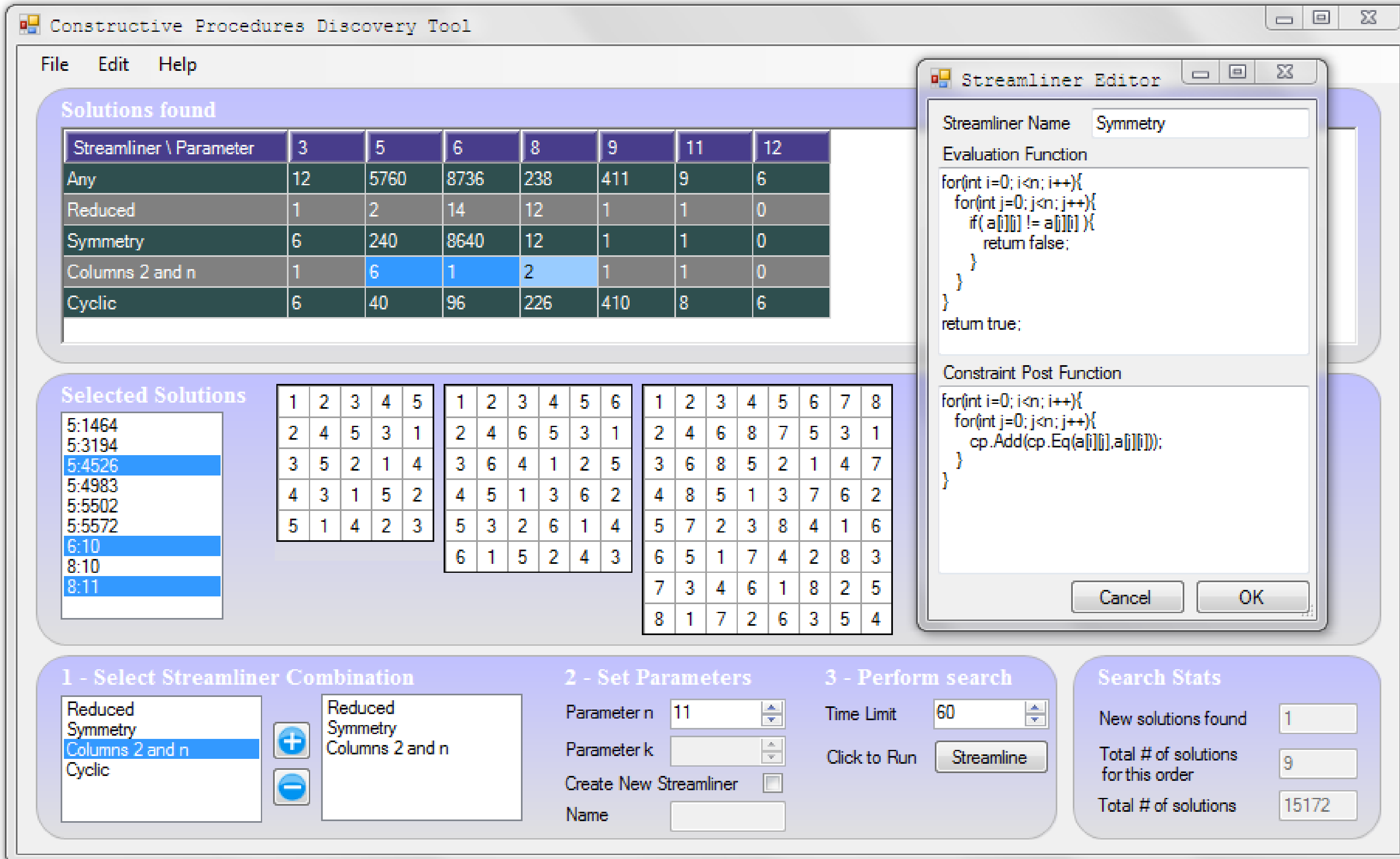
{Symmetric, Cyclic, Reduced}



Solution Set



GUI for Human-guided Streamlined Search



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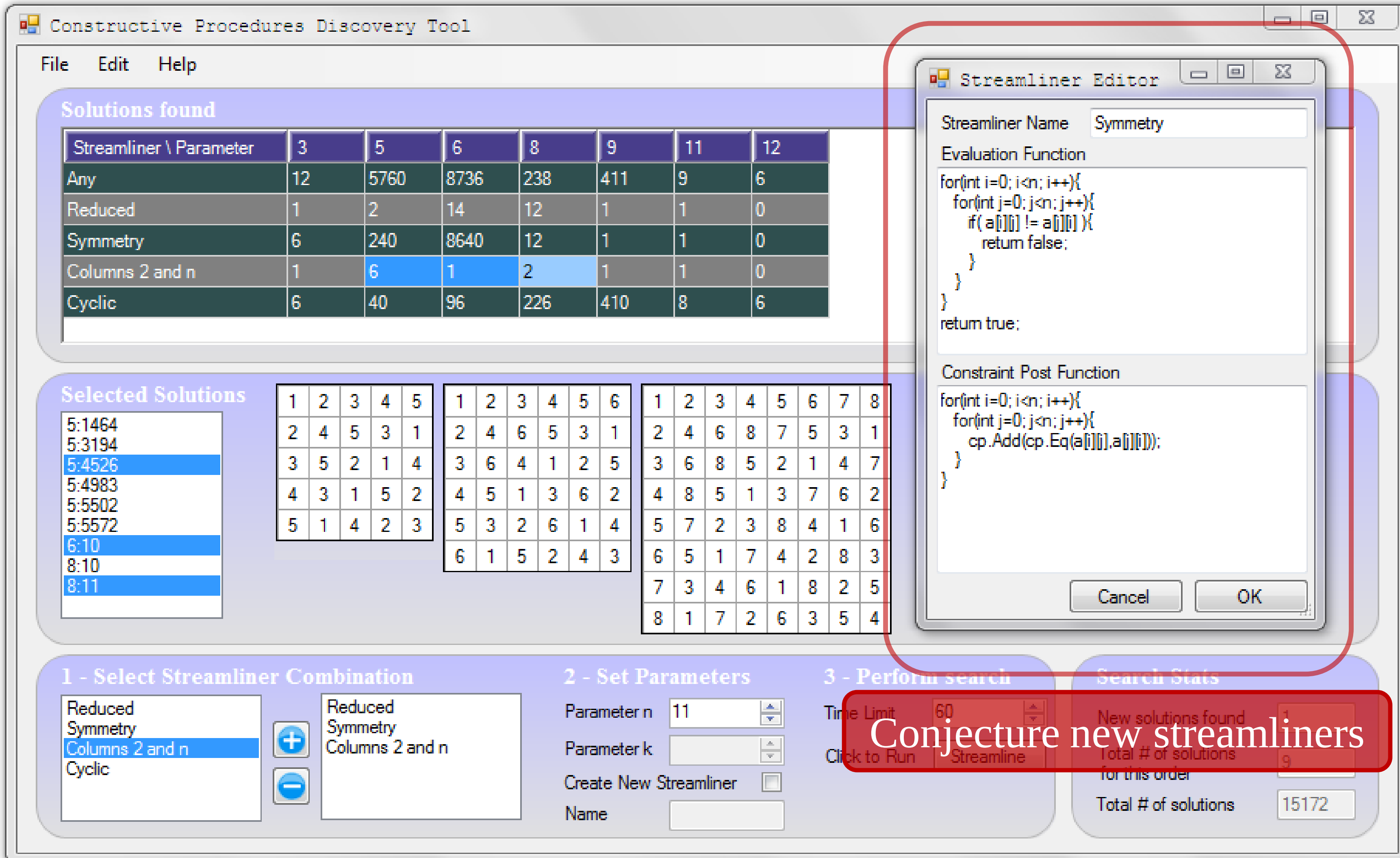

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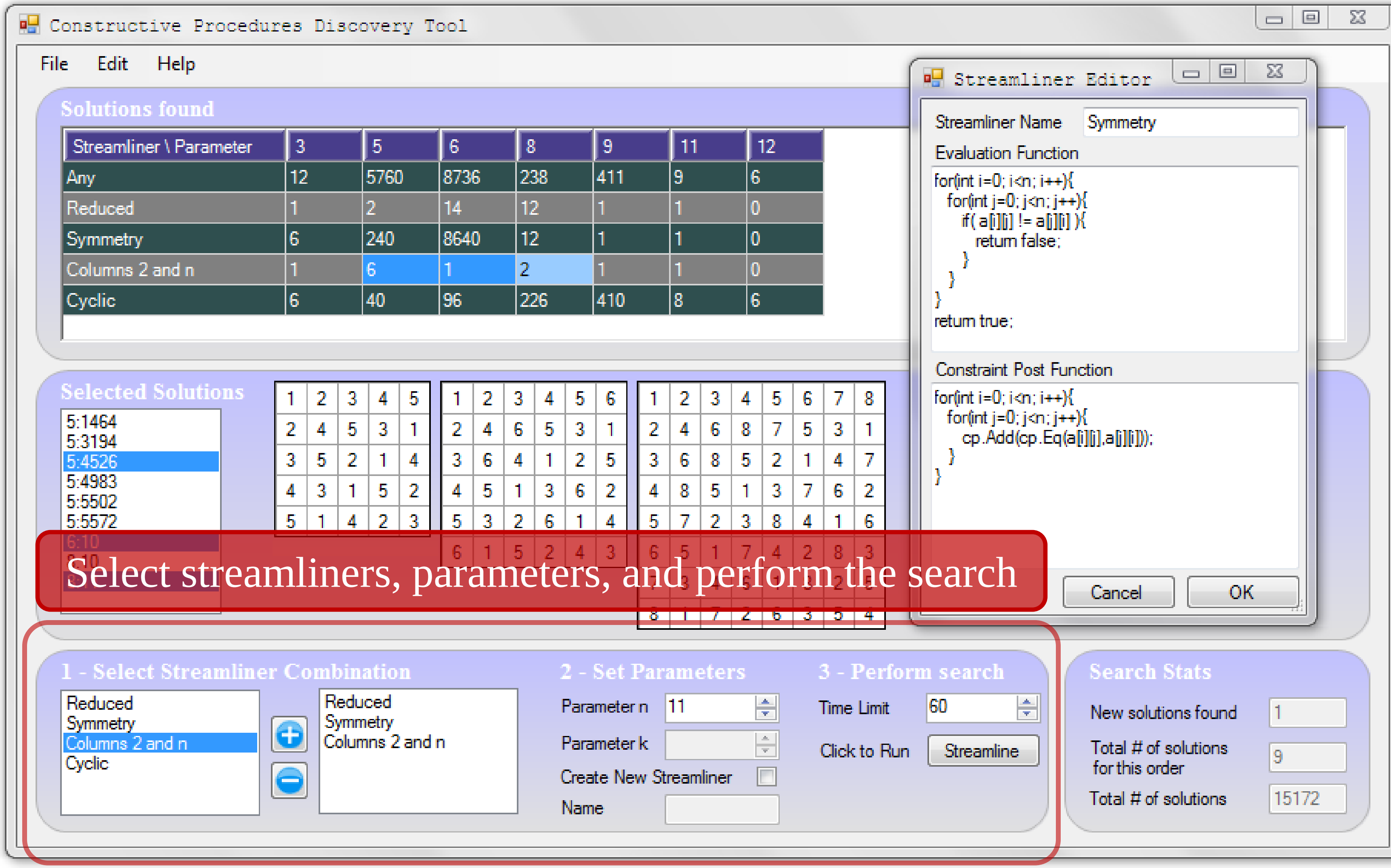
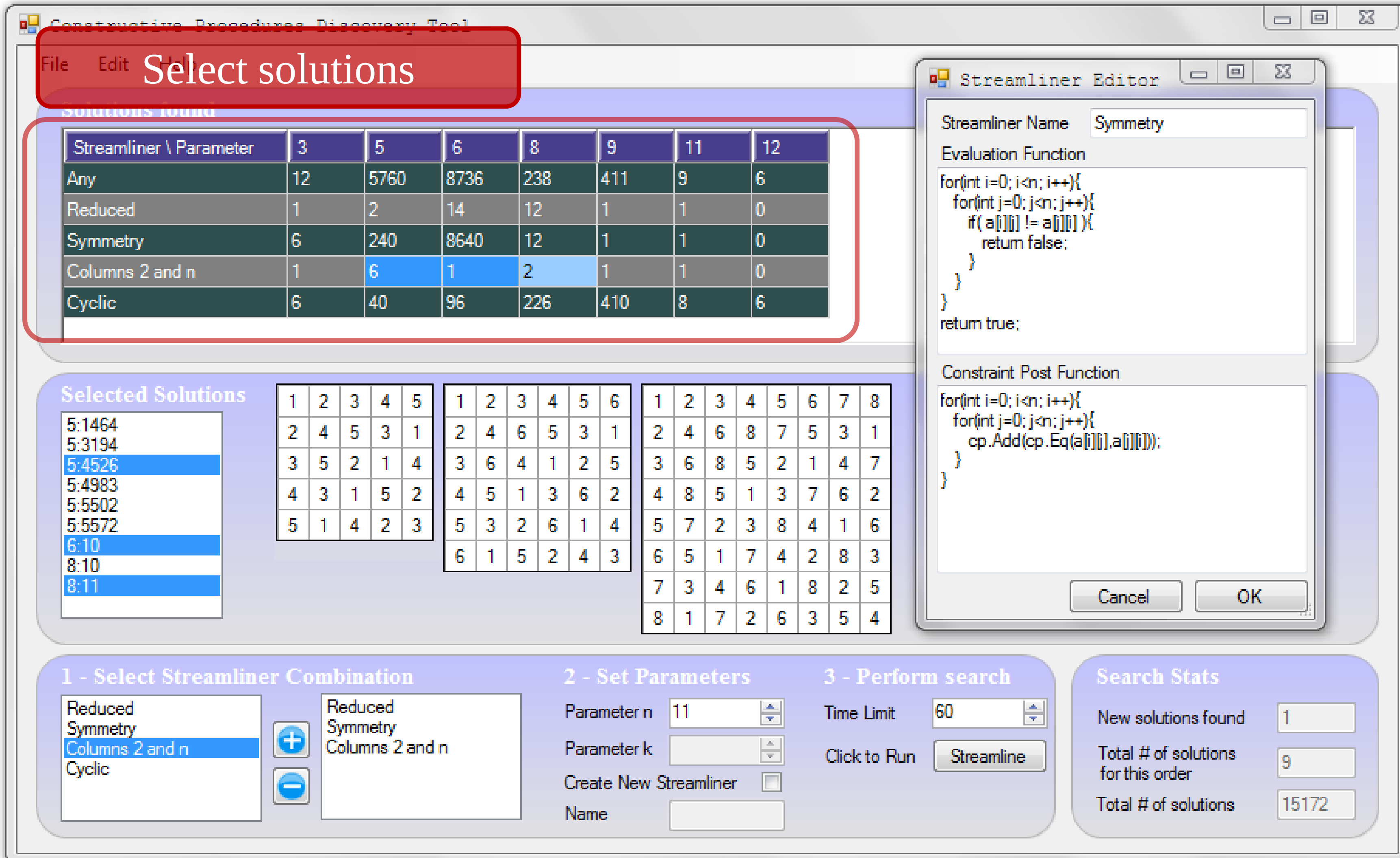


Select streamliners, parameters, and perform the search

GUI for Human-guided Streamlined Search



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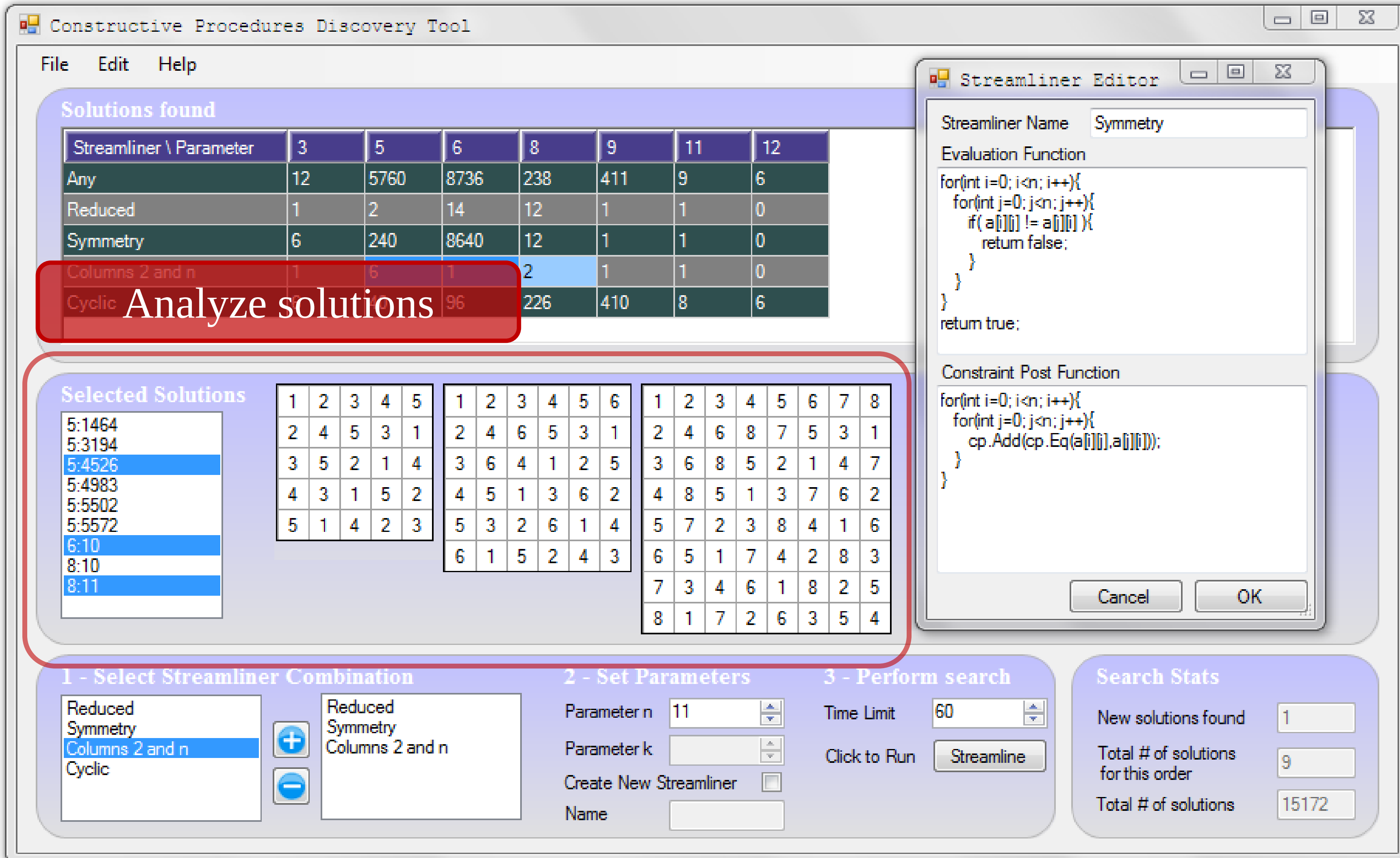

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  - Successful Streamliners
  - Constructive Procedure 1
  - Constructive Procedure 2
- Application to the *Weak Schur Number* problem
- Conclusions and Future work

Application to the *SBLS* problem: Construction 1

Successful Key Streamliners:

{Diagonal symmetry, Reduced form, Assignments of columns 2 and *n*, Multiples of *i* in row *i*, Second sequence decreasing}

Streamliners	5	6	8	9	11	14
$\Gamma_1 = \emptyset$	**5760**	15878	-	-	-	-
$\Gamma_2 = \Gamma_1 \cup \{Symmetric\}$	**240**	8447	714	43	-	-
$\Gamma_3 = \Gamma_2 \cup \{Reduced\}$	**2**	**14**	**14**	51	-	-
$\Gamma_4 = \Gamma_3 \cup \{Columns\ 2\ \&\ \mathrm{n}\}$	**1**	**1**	**2**	**1**	**1**	-
$\Gamma_5 = \Gamma_4 \cup \{Multiples\ of\ i\}$	**1**	**1**	**2**	**1**	**1**	1

Fig: Number of SBLSs generated in 60 seconds, by order and streamliners (Bold indicates exhaustive search).

Application to the *SBLS* problem: Construction 1

1	2	3	4	5	6	7	8
2	4	6	8	7	5	3	1
3	6	8	5	2	1	4	7
4	8	5	1	3	7	6	2
5	7	2	3	8	4	1	6
6	5	1	7	4	2	8	3
7	3	4	6	1	8	2	5
8	1	7	2	6	3	5	4

1	2	3	4	5	6	7	8	9
2	4	6	8	9	7	5	3	1
3	6	9	7	4	1	2	5	8
4	8	7	3	1	5	9	6	2
5	9	4	1	6	8	3	2	7
6	7	1	5	8	2	4	9	3
7	5	2	9	3	4	8	1	6
8	3	5	6	2	9	1	7	4
9	1	8	2	7	3	6	4	5

1	2	3	4	5	6	7	8	9	10	11
2	4	6	8	10	11	9	7	5	3	1
3	6	9	11	8	5	2	1	4	7	10
4	8	11	7	3	1	5	9	10	6	2
5	10	8	3	2	7	11	6	1	4	9
6	11	5	1	7	10	4	2	8	9	3
7	9	2	5	11	4	3	10	6	1	8
8	7	1	9	6	2	10	5	3	11	4
9	5	4	10	1	8	6	3	11	2	7
10	3	7	6	4	9	1	11	2	8	5
11	1	10	2	9	3	8	4	7	5	6

1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	4	6	8	10	12	14	13	11	9	7	5	3	1
3	6	9	12	14	11	8	5	2	1	4	7	10	13
4	8	12	13	9	5	1	3	7	11	14	10	6	2
5	10	14	9	4	1	6	11	13	8	3	2	7	12
6	12	11	5	1	7	13	10	4	2	8	14	9	3
7	14	8	1	6	13	9	2	5	12	10	3	4	11
8	13	5	3	11	10	2	6	14	7	1	9	12	4
9	11	2	7	13	4	5	14	6	3	12	8	1	10
10	9	1	11	8	2	12	7	3	13	6	4	14	5
11	7	4	14	3	8	10	1	12	6	5	13	2	9
12	5	7	10	2	14	3	9	8	4	13	1	11	6
13	3	10	6	7	9	4	12	1	14	2	11	5	8
14	1	13	2	12	3	11	4	10	5	9	6	8	7

Application to the *SBLS* problem: Construction 1

1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	4	6	8	10	12	14	13	11	9	7	5	3	1
3	6	9	12	14	11	8	5	2	1	4	7	10	13
4	8	12	13	9	5	1	3	7	11	14	10	6	2
5	10	14	9	4	1	6	11	13	8	3	2	7	12
6	12	11	5	1	7	13	10	4	2	8	14	9	3
7	14	8	1	6	13	9	2	5	12	10	3	4	11
8	13	5	3	11	10	2	6	14	7	1	9	12	4
9	11	2	7	13	4	5	14	6	3	12	8	1	10
10	9	1	11	8	2	12	7	3	13	6	4	14	5
11	7	4	14	3	8	10	1	12	6	5	13	2	9
12	5	7	10	2	14	3	9	8	4	13	1	11	6
13	3	10	6	7	9	4	12	1	14	2	11	5	8
14	1	13	2	12	3	11	4	10	5	9	6	8	7

a	a+i

a	a-i

Application to the *SBLS* problem: Construction 1

1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	4	6	8	10	12	14	13	11	9	7	5	3	1
3	6	9	12	14	11	8	5	2	1	4	7	10	13
4	8	12	13	9	5	1	3	7	11	14	10	6	2
5	10	14	9	4	1	6	11	13	8	3	2	7	12
6	12	11	5	1	7	13	10	4	2	8	14	9	3
7	14	8	1	6	13	9	2	5	12	10	3	4	11
8	13	5	3	11	10	2	6	14	7	1	9	12	4
9	11	2	7	13	4	5	14	6	3	12	8	1	10
10	9	1	11	8	2	12	7	3	13	6	4	14	5
11	7	4	14	3	8	10	1	12	6	5	13	2	9
12	5	7	10	2	14	3	9	8	4	13	1	11	6
13	3	10	6	7	9	4	12	1	14	2	11	5	8
14	1	13	2	12	3	11	4	10	5	9	6	8	7

a	a+i

a	a-i

for *row* $i = 1, \ldots, N$ **do**
  $k = 1;$ // Sequence number
  $j = 1;$ // Column index
  $a_{i,j} = i;$ // First symbol of row i
  while $j < N$ **do**
    if k *is odd* **then** // Odd sequence
      while $a_{i,j} + i \leq N$ *and* $j < N$ **do**
        $a_{i,j+1} = a_{i,j} + i;$
        $j = j + 1;$
    else // Even sequence
      while $a_{i,j} - i \geq 1$ *and* $j < N$ **do**
        $a_{i,j+1} = a_{i,j} - i;$
        $j = j + 1;$
    if $j < N$ **then** // Switch sequence
      if k *is odd* **then**
        $a_{i,j+1} = 2N + 1 - i - a_{i,j};$
      else
        $a_{i,j+1} = i - a_{i,j};$
      $k = k + 1;$
      $j = j + 1;$

Algorithm : `SBLS-sequence` procedure for SBLS of order N, when $2N + 1$ is prime.

Application to the *SBLS* problem: Construction 1

1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	4	6	8	10	12	14	13	11	9	7	5	3	1
3	6	9	12	14	11	8	5	2	1	4	7	10	13
4	8	12	13	9	5	1	3	7	11	14	10	6	2
5	10	14	9	4	1	6	11	13	8	3	2	7	12
6	12	11	5	1	7	13	10	4	2	8	14	9	3
7	14	8	1	6	13	9	2	5	12	10	3	4	11
8	13	5	3	11	10	2	6	14	7	1	9	12	4
9	11	2	7	13	4	5	14	6	3	12	8	1	10
10	9	1	11	8	2	12	7	3	13	6	4	14	5
11	7	4	14	3	8	10	1	12	6	5	13	2	9
12	5	7	10	2	14	3	9	8	4	13	1	11	6
13	3	10	6	7	9	4	12	1	14	2	11	5	8
14	1	13	2	12	3	11	4	10	5	9	6	8	7

a	a+i	a	a-i

for *row* $i = 1, \dots, N$ **do**
  $k = 1;$ // Sequence number
  $j = 1;$ // Column index
  $a_{i,j} = i;$ // First symbol of row i
  while $j < N$ **do**
    if k *is odd* **then** // Odd sequence
      while $a_{i,j} + i \leq N$ *and* $j < N$ **do**
        $a_{i,j+1} = a_{i,j} + i;$
        $j = j + 1;$
    else // Even sequence
      while $a_{i,j} - i \geq 1$ *and* $j < N$ **do**
        $a_{i,j+1} = a_{i,j} - i;$
        $j = j + 1;$
    if $j < N$ **then** // Switch sequence
      if k *is odd* **then**
        $a_{i,j+1} = 2N + 1 - i - a_{i,j};$
      else
        $a_{i,j+1} = i - a_{i,j};$
      $k = k + 1;$
      $j = j + 1;$

Algorithm : `SBLS-sequence` procedure for SBLS of order N, when $2N + 1$ is prime.

Proof of Correctness in [R. Le Bras, A. Perrault, and C. Gomes, *Polynomial Time Construction for Spatially Balanced Latin Squares*, 2012]

Application to the *SBLS* problem: Construction 2

1	2	6	3	9	7	13	4	11	10	12	8	5	14
2	3	7	4	10	8	14	5	12	11	13	9	6	1
3	4	8	5	11	9	1	6	13	12	14	10	7	2
4	5	9	6	12	10	2	7	14	13	1	11	8	3
5	6	10	7	13	11	3	8	1	14	2	12	9	4
6	7	11	8	14	12	4	9	2	1	3	13	10	5
7	8	12	9	1	13	5	10	3	2	4	14	11	6
8	9	13	10	2	14	6	11	4	3	5	1	12	7
9	10	14	11	3	1	7	12	5	4	6	2	13	8
10	11	1	12	4	2	8	13	6	5	7	3	14	9
11	12	2	13	5	3	9	14	7	6	8	4	1	10
12	13	3	14	6	4	10	1	8	7	9	5	2	11
13	14	4	1	7	5	11	2	9	8	10	6	3	12
14	1	5	2	8	6	12	3	10	9	11	7	4	13

$c_{1,1} = 1;$ // Generate 1st row of the conjugate
for *column* $j = 2, \ldots, N$ **do** // Observed pattern $1, 2, 4, \ldots$
  if $2c_{1,j-1} \leq N$ **then**
    $c_{1,j} = 2c_{1,j-1};$
  else
    $c_{1,j} = 2N + 1 - 2c_{1,j-1};$
for *row* $i = 2, \ldots, N$ **do** // Subsequent rows
  $c_{i,1} = c_{i-1,N};$ // Shifted version of previous
  for *column* $j = 2, \ldots, N$ **do**
    $c_{i,j} = c_{i-1,j-1};$
for *row* $i = 1, \ldots, N$ **do** // Generate SBLS from conjugate
  for *column* $j = 1, \ldots, N$ **do**
    $a_{i,c_{i,j}} = j;$

Algorithm: `SBLS-Cyclic` procedure.

Application to the *Weak Schur* problem

Problem Definition:

- A set is ***(weakly)* sum free** if for any two *(distinct)* elements of this set, their sum does not belong to the set.

- The ***Weak Schür Number*** of order *k*, *WS(k)*, is the largest integer *n* for which there exists a partition of [1,*n*] into *k* weakly sum-free sets.

k=3 sets

1 2 4 8 11 22
3 5-7 19 21 23
9 10 12-18 20

Each of the 3 sets is such that, for any 2 elements of the set, their sum does not belong to the sdame set.

Fig: Partition of [1,23] into 3 weakly sum-free sets, proving $WS(3) \geq 23$

Application to the *Weak Schur* problem

Best known lower bounds:

Approach	***WS*(5)**	***WS*(6)**	**Reference**
(not disclosed)	196	-	[G.W. Walker, AMM'50]
Theoretical bound *(not proved)*	188	554	[J.H. Braun, AMM'50]
SAT	196	572	[Eliahou et al., Computers & Math Applications'12]
Multi-level Tabu-Search	196	574	[Fonlupt et al., EA'11]
SAT *(no certificate)*	196	575	[Eliahou et al., Computers & Math Applications'12] *(revised)*

Application to the *Weak Schur* problem

Successful Key Streamliners:

{Ordered sets, constrained minimum of each set, partial assignments, sequences of consecutive integers, sequence interleaving}

Application to the *Weak Schur* problem

Successful Key Streamliners:

{Ordered sets, constrained minimum of each set, partial assignments, sequences of consecutive integers, sequence interleaving}

1 2 4 8 11 22 25 50 63 68 139 149 154 177 182 192 198 393 398 408 413 436 450 455 521 526 540 563 568 578
3 5-7 19 21 23 51-53 64-66 136-138 150-152 179-181 193-195 395-397 409-411 438-440 451-453 523-525 536-538 565-567 579-581
9 10 12-18 20 54-62 140-148 183-191 399-407 441-449 527-535 569-577
24 26-49 153 155-176 178 412 414-435 437 539 541-562 564
67 69-135 454 456-520 522
196 197 199-392 394

Fig: Partition of [1,581] into 6 weakly sum-free sets, proving $WS(6) \geq 581$.

- Although **not** an example of a **fully constructive procedure** yet, any progress on Schur numbers is quite **significant** given their long history.

Conclusion and Future work

- General **framework** that integrates specialized search techniques (so-called **streamlining**) with **human insight** in an **iterative** approach to discover **efficient constructive procedures**.

- Provides the **first constructive procedures** for the ***Spatially-Balanced Latin Square*** problem.

- Improves the best known **lower bound** for the ***Weak Schur Number*** problem.

- One exciting extension would be to **crowd-source** the search for regularities in the solution set.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	49	47	45	43	41	39	37	35	33	31	29	27	25	23	21	19	17	15	13	11	9	7	5	3	1
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	50	47	44	41	38	35	32	29	26	23	20	17	14	11	8	5	2	1	4	7	10	13	16	19	22	25	28	31	34	37	40	43	46	49
4	8	12	16	20	24	28	32	36	40	44	48	49	45	41	37	33	29	25	21	17	13	9	5	1	3	7	11	15	19	23	27	31	35	39	43	47	50	46	42	38	34	30	26	22	18	14	10	6	2
5	10	15	20	25	30	35	40	45	50	46	41	36	31	26	21	16	11	6	1	4	9	14	19	24	29	34	39	44	49	47	42	37	32	27	22	17	12	7	2	3	8	13	18	23	28	33	38	43	48
6	12	18	24	30	36	42	48	47	41	35	29	23	17	11	5	1	7	13	19	25	31	37	43	49	46	40	34	28	22	16	10	4	2	8	14	20	26	32	38	44	50	45	39	33	27	21	15	9	3
7	14	21	28	35	42	49	45	38	31	24	17	10	3	4	11	18	25	32	39	46	48	41	34	27	20	13	6	1	8	15	22	29	36	43	50	44	37	30	23	16	9	2	5	12	19	26	33	40	47
8	16	24	32	40	48	45	37	29	21	13	5	3	11	19	27	35	43	50	42	34	26	18	10	2	6	14	22	30	38	46	47	39	31	23	15	7	1	9	17	25	33	41	49	44	36	28	20	12	4
9	18	27	36	45	47	38	29	20	11	2	7	16	25	34	43	49	40	31	22	13	4	5	14	23	32	41	50	42	33	24	15	6	3	12	21	30	39	48	44	35	26	17	8	1	10	19	28	37	46
10	20	30	40	50	41	31	21	11	1	9	19	29	39	49	42	32	22	12	2	8	18	28	38	48	43	33	23	13	3	7	17	27	37	47	44	34	24	14	4	6	16	26	36	46	45	35	25	15	5
11	22	33	44	46	35	24	13	2	9	20	31	42	48	37	26	15	4	7	18	29	40	50	39	28	17	6	5	16	27	38	49	41	30	19	8	3	14	25	36	47	43	32	21	10	1	12	23	34	45
12	24	36	48	41	29	17	5	7	19	31	43	46	34	22	10	2	14	26	38	50	39	27	15	3	9	21	33	45	44	32	20	8	4	16	28	40	49	37	25	13	1	11	23	35	47	42	30	18	6
13	26	39	49	36	23	10	3	16	29	42	46	33	20	7	6	19	32	45	43	30	17	4	9	22	35	48	40	27	14	1	12	25	38	50	37	24	11	2	15	28	41	47	34	21	8	5	18	31	44
14	28	42	45	31	17	3	11	25	39	48	34	20	6	8	22	36	50	37	23	9	5	19	33	47	40	26	12	2	16	30	44	43	29	15	1	13	27	41	46	32	18	4	10	24	38	49	35	21	7
15	30	45	41	26	11	4	19	34	49	37	22	7	8	23	38	48	33	18	3	12	27	42	44	29	14	1	16	31	46	40	25	10	5	20	35	50	36	21	6	9	24	39	47	32	17	2	13	28	43
16	32	48	37	21	5	11	27	43	42	26	10	6	22	38	47	31	15	1	17	33	49	36	20	4	12	28	44	41	25	9	7	23	39	46	30	14	2	18	34	50	35	19	3	13	29	45	40	24	8
17	34	50	33	16	1	18	35	49	32	15	2	19	36	48	31	14	3	20	37	47	30	13	4	21	38	46	29	12	5	22	39	45	28	11	6	23	40	44	27	10	7	24	41	43	26	9	8	25	42
18	36	47	29	11	7	25	43	40	22	4	14	32	50	33	15	3	21	39	44	26	8	10	28	46	37	19	1	17	35	48	30	12	6	24	42	41	23	5	13	31	49	34	16	2	20	38	45	27	9
19	38	44	25	6	13	32	50	31	12	7	26	45	37	18	1	20	39	43	24	5	14	33	49	30	11	8	27	46	36	17	2	21	40	42	23	4	15	34	48	29	10	9	28	47	35	16	3	22	41
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21	42	38	17	4	25	46	34	13	8	29	50	30	9	12	33	47	26	5	16	37	43	22	1	20	41	39	18	3	24	45	35	14	7	28	49	31	10	11	32	48	27	6	15	36	44	23	2	19	40
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23	46	32	9	14	37	41	18	5	28	50	27	4	19	42	36	13	10	33	45	22	1	24	47	31	8	15	38	40	17	6	29	49	26	3	20	43	35	12	11	34	44	21	2	25	48	30	7	16	39
24	48	29	5	19	43	34	10	14	38	39	15	9	33	44	20	4	28	49	25	1	23	47	30	6	18	42	35	11	13	37	40	16	8	32	45	21	3	27	50	26	2	22	46	31	7	17	41	36	12
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26	49	23	3	29	46	20	6	32	43	17	9	35	40	14	12	38	37	11	15	41	34	8	18	44	31	5	21	47	28	2	24	50	25	1	27	48	22	4	30	45	19	7	33	42	16	10	36	39	13
27	47	20	7	34	40	13	14	41	33	6	21	48	26	1	28	46	19	8	35	39	12	15	42	32	5	22	49	25	2	29	45	18	9	36	38	11	16	43	31	4	23	50	24	3	30	44	17	10	37
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38	25	13	50	12	26	37	1	39	24	14	49	11	27	36	2	40	23	15	48	10	28	35	3	41	22	16	47	9	29	34	4	42	21	17	46	8	30	33	5	43	20	18	45	7	31	32	6	44	19
39	23	16	46	7	32	30	9	48	14	25	37	2	41	21	18	44	5	34	28	11	50	12	27	35	4	43	19	20	42	3	36	26	13	49	10	29	33	6	45	17	22	40	1	38	24	15	47	8	31
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41	19	22	38	3	44	16	25	35	6	47	13	28	32	9	50	10	31	29	12	48	7	34	26	15	45	4	37	23	18	42	1	40	20	21	39	2	43	17	24	36	5	46	14	27	33	8	49	11	30
42	17	25	34	8	50	9	33	26	16	43	1	41	18	24	35	7	49	10	32	27	15	44	2	40	19	23	36	6	48	11	31	28	14	45	3	39	20	22	37	5	47	12	30	29	13	46	4	38	21
43	15	28	30	13	45	2	41	17	26	32	11	47	4	39	19	24	34	9	49	6	37	21	22	36	7	50	8	35	23	20	38	5	48	10	33	25	18	40	3	46	12	31	27	16	42	1	44	14	29
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45	11	34	22	23	33	12	44	1	46	10	35	21	24	32	13	43	2	47	9	36	20	25	31	14	42	3	48	8	37	19	26	30	15	41	4	49	7	38	18	27	29	16	40	5	50	6	39	17	28
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Extra slides

SBLS of order 50 – Construction 1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
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4	8	12	16	20	24	28	32	36	40	44	48	49	45	41	37	33	29	25	21	17	13	9	5	1	3	7	11	15	19	23	27	31	35	39	43	47	50	46	42	38	34	30	26	22	18	14	10	6	2
5	10	15	20	25	30	35	40	45	50	46	41	36	31	26	21	16	11	6	1	4	9	14	19	24	29	34	39	44	49	47	42	37	32	27	22	17	12	7	2	3	8	13	18	23	28	33	38	43	48
6	12	18	24	30	36	42	48	47	41	35	29	23	17	11	5	1	7	13	19	25	31	37	43	49	46	40	34	28	22	16	10	4	2	8	14	20	26	32	38	44	50	45	39	33	27	21	15	9	3
7	14	21	28	35	42	49	45	38	31	24	17	10	3	4	11	18	25	32	39	46	48	41	34	27	20	13	6	1	8	15	22	29	36	43	50	44	37	30	23	16	9	2	5	12	19	26	33	40	47
8	16	24	32	40	48	45	37	29	21	13	5	3	11	19	27	35	43	50	42	34	26	18	10	2	6	14	22	30	38	46	47	39	31	23	15	7	1	9	17	25	33	41	49	44	36	28	20	12	4
9	18	27	36	45	47	38	29	20	11	2	7	16	25	34	43	49	40	31	22	13	4	5	14	23	32	41	50	42	33	24	15	6	3	12	21	30	39	48	44	35	26	17	8	1	10	19	28	37	46
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18	36	47	29	11	7	25	43	40	22	4	14	32	50	33	15	3	21	39	44	26	8	10	28	46	37	19	1	17	35	48	30	12	6	24	42	41	23	5	13	31	49	34	16	2	20	38	45	27	9
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29	43	14	15	44	28	1	30	42	13	16	45	27	2	31	41	12	17	46	26	3	32	40	11	18	47	25	4	33	39	10	19	48	24	5	34	38	9	20	49	23	6	35	37	8	21	50	22	7	36
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31	39	8	23	47	16	15	46	24	7	38	32	1	30	40	9	22	48	17	14	45	25	6	37	33	2	29	41	10	21	49	18	13	44	26	5	36	34	3	28	42	11	20	50	19	12	43	27	4	35
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Results of *SBLS-sequence* (U&B)

Order	CP	CPSS	CPCS	LSS	U&B
6	0.06	0.05	0.02	0.00	0.00
8	16.00	0.88		0.00	0.00
9	241.00	0.91		0.00	0.00
11		9.84		0.00	0.00
12		531.00	14.40	0.00	
14		5,434.00		0.02	0.00
15				0.01	0.00
17				0.25	
18			107,000.00	2.30	0.00
20				16.00	0.00
21				16.00	0.00
23				104.00	0.00
24				281.00	
26				609.00	0.00
27				4,000.00	
29				23,000.00	0.00
30				160,000.00	0.00
32				1,200,000.00	
33				1,200,000.00	0.00
35				1,200,000.00	0.00
36					0.00
39					0.00
41					0.00
44					0.00
48					0.00
50					0.00
51					0.00
53					0.00
54					0.00
56					0.00
...					...
999					0.02