On the Efficiency of Markets with Two-sided Proportional Allocation Mechanisms

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Abstract. We analyze the performance of resource allocation mechanisms for markets in which there is competition amongst both consumers and suppliers (namely, two-sided markets). Specifically, we examine a natural generalization of both Kelly's proportional allocation mechanism for demand-competitive markets [9] and Johari and Tsitsiklis' proportional allocation mechanism for supply-competitive markets [7].

We first consider the case of a market for one divisible resource. Assuming that marginal costs are convex, we derive a tight bound on the price of anarchy of about 0.5887. This worst case bound is achieved when the demand-side of the market is highly competitive and the supply-side consists of a duopoly. As more firms enter the market, the price of anarchy improves to 0.64. In contrast, on the demand side, the price of anarchy improves when the number of consumers decreases, reaching a maximum of 0.7321 in a monopsony setting. When the marginal cost functions are concave, the above bound smoothly degrades to zero as the marginal costs tend to constants. For monomial cost functions of the form $C(x) = cx^{1+\frac{1}{d}}$, we show that the price of anarchy is $\Omega(\frac{1}{d^2})$.

We complement these guarantees by identifying a large class of two-sided single-parameter market-clearing mechanisms among which the proportional allocation mechanism uniquely achieves the optimal price of anarchy. We also prove that our worst case bounds extend to general multi-resource markets, and in particular to bandwidth markets over arbitrary networks.

1 Introduction

How to produce and allocate scarce resources is the most fundamental question in economics.[1] The standard tool for guiding production and allocation is a pricing mechanism. However, different mechanisms will have different performance attributes: no two mechanisms are equal. Of particular interest to computer scientists is the fact that there will typically be an inherent trade-off between the economic efficiency of a mechanism (measured in terms of social welfare) and its computational efficiency (both time and communication complexity). Socially optimal allocations can be achieved using pricing mechanisms based on classical VCG results, but implementing such mechanisms generally induces excessively high informational and computational costs [13]. In this paper, we study this tradeoff from the opposite viewpoint: we examine the level of social welfare that can be achieved by mechanisms performing *minimal* amounts of computation. In particular, we restrict our attention to so-called *scalar-parametrized* pricing mechanisms. Each participant submits only a single scalar bid that is used to set a unique market-clearing price for each good. Evidently, such mechanisms are computationally trivial to handle; more surprisingly, they can produce high welfare.

The chief practical motivation for considering scalar-parametrized mechanisms (both in our work and in the existing literature) is the problem of bandwidth sharing. Namely, how should we allocate capacity amongst users that want to transmit data over a network link? The use of market mechanisms for this task has been studied in Asynchronous Transfer Mode (ATM) networks [16] and the Internet [15]. The Internet is made up of smaller interconnected networks that buy capacities from each other,

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[1] In fact, economics is often defined as "the study of scarcity".

and the market mechanisms we consider are closely inspired by the structure of the Internet. Specifically, we are restricting our attention to mechanisms that are scalable to very large networks. This requirement for scalability forces us consider only simple mechanisms, such as those that set a unique market clearing price. The computational requirements of more complex systems, e.g. mechanisms that perform price discrimination, become impractical on large networks [1].

We remark that unique price mechanisms are also intuitively "fair", as every participant is treated equally. This fairness is appealing from a social and political perspective, and indeed these systems are used in many real-world settings, such as electricity markets [17].

1.1 Background and Previous Work

A basic method for resource allocation is the *proportional allocation mechanism* of Kelly [9]. In the context of networks, it operates as follows: each potential consumer submits a bid b_q; bandwidth is then allocated to the consumers in proportion to their bids. This simple idea has also been studied within economics by Shapley and Shubik [14] as a model for understanding pricing in market economies. In a groundbreaking result, Johari and Tsitsiklis [5] showed that the welfare loss incurred by this mechanism is at most 25% of optimal.

Observe that Kelly's is a scalar-paramterized mechanism for a one-sided market: every participant is a consumer. Johari and Tsitsiklis [7] also examined one-sided markets with supply-side competition only. There, under a corresponding single-parameter mechanism, the welfare loss tends to zero as the level of competition increases. We remark that we cannot simply analyze supply-side competition by trying to model suppliers as demand-side consumers [3].

Of course, competition in markets typically occurs on *both* sides. Consequently, understanding the efficiency of *two-sided market*[2] mechanisms is an important problem. In this work, we analyze the price of anarchy in a mechanism for a two-sided market in which consumers and producers compete simultaneously to determine the production and allocation of goods. This mechanism was first proposed by Neumayer [10] and is the natural generalization of both the demand-side model of Kelly [9] and the supply-side model of Johari and Tsitsiklis [7].

In order to examine how the generalized proportional allocation mechanism performs in a two-sided market, it is important to note that there are three primary causes of welfare loss. First, the underlying allocation problem may be computationally hard. In other words, in some settings (such as combinatorial auctions, for example), it may be hard to compute the optimal allocation even when the players' utilities are known. Secondly, even if the allocation problem is computationally simple, the mechanism itself may still be insufficiently sophisticated to solve it. Thirdly, the mechanism may be susceptible to gaming; namely, the mechanism may incentivize selfish agents to behave in a manner that produces a poor overall outcome. As we will see in Section 4, the first two causes do not arise here: as long as the users do not behave strategically, the proportional allocation mechanism can quickly find optimal allocations in two-sided markets. Thus, we are concerned only with the third factor: how adversely is the proportional allocation mechanism affected by gaming agents? That is, the mechanism may be capable of producing an optimal solution, but how will the agents' selfish behaviour affect social welfare at the resultant equilibria?

In this paper we prove that the proportional allocation mechanism does perform well in two-sided markets. Specifically, under quite general assumptions, the mechanism admits a constant factor price of anarchy guarantee. Moreover, there exists a large family of mechanisms among which the proportional allocation mechanism uniquely achieves the best possible price of anarchy guarantee. We state our exact results in Section 3, after we have described the model and our assumptions.

[2] It should be noted that "two-sided market" often has a different meaning in the economics literature than the one we use here. There it refers to a specific class of markets where externalities occur between groups on the two sides of the market.

2 The Model

2.1 The two-sided proportional allocation mechanism

We now formally present the two-sided proportional allocation mechanism due to Neumayer [10]. There are Q consumers and R suppliers in the market. Each consumer q has a valuation function $V_q(d_q)$, where d_q is the amount of the resource allocated to consumer q, and each supplier r has a cost function $C_r(s_r)$, where s_r is the amount produced by supplier r. Consumers and suppliers respectively input bids b_q and b_r to the mechanism. Doing so, consumers are implicitly selecting b_q-parametrized demand functions of the form $D(b_q, p) = \frac{b_q}{p}$, and suppliers are selecting b_r-parametrized supply functions of the form $S(b_r, p) = 1 - \frac{b_r}{p}$. We can also interpret a high consumer bid as an indicator of high willingness to pay for the product, and a low supplier bid as an indicator of a high willingness to supply (alternatively, a high bid indicates a high cost supplier). The actual choice of constant used for the supply functions does not affect our results, and so we choose it to be 1.

Observe that the parametrized demand functions are identical to the ones in the demand-side mechanism of Kelly [9], and the supply functions are identical to the ones in the supply-side mechanism of Johari and Tsitsiklis [7]. The peculiar form of the supply functions comes from the interesting fact that for most scalar-parametrized mechanisms, in order to have a non-zero welfare ratio, the supply functions have to be bounded from above. In other words, suppliers' strategies must necessarily be constrained in order to obtain high welfare; see the full version of the paper for the precise statement of this fact. This rules out, for instance, Cournot-style mechanisms where suppliers directly submit the quantities they wish to produce.

More detailed justifications for this choice of model can be found in [10], as well as in [9] and [7]. Further justification for the mechanism will be provided by our results. Specifically, the proportional allocation mechanism generally produces high welfare allocations and, in addition, it is the optimal mechanism amongst a class of single-parameter mechanisms for two-sided markets.

Given the bids, the mechanism sets a price $p(\mathbf{b})$ that clears the market; i.e. that satisfies the supply equals demand equation: $\sum_{q=1}^{Q} \frac{b_q}{p} = \sum_{r=1}^{R}(1 - \frac{b_r}{p})$. The price therefore gets set to $p(\mathbf{b}) = \frac{\sum_q b_q + \sum_r b_r}{R}$. Consumer q then receives d_q units of the resource, and pays pd_q, while supplier r produces s_r units and receives a payment of ps_r. In the game induced by this mechanism, the payoff (or utility) to consumer q placing a bid b_q is defined to be

$$\Pi_q(b_q) = \begin{cases} V_q\left(\dfrac{b_q}{\sum_{q\in Q} b_q + \sum_r b_{r\in R}} R\right) - b_q & \text{if } b_q > 0 \\ V_q(0) & \text{if } b_q = 0 \end{cases}$$

and the payoff to supplier r placing a bid b_r is defined as

$$\Pi_r(b_r) = \begin{cases} \dfrac{\sum_{q\in Q} b_q + \sum_{r\in R} b_r}{R} - b_r - C_r\left(1 - \dfrac{b_r}{\sum_{q\in Q} b_q + \sum_{r\in R} b_r} R\right) & \text{if } b_r > 0 \\ \dfrac{\sum_{q\neq r} b_q + \sum_{r\in R} b_r}{R} - C_r(1) & \text{if } b_r = 0 \end{cases}$$

2.2 The Welfare Ratio

Given a vector of bids $\mathbf{b}$, the *social welfare* at the resulting mechanism allocation is defined to be

$$\mathcal{W}(\mathbf{b}) = \sum_{q=1}^{Q} V_q(d_q(\mathbf{b})) - \sum_{r=1}^{R} C_r(s_r(\mathbf{b}))$$

If the agents do not strategically anticipate the effects of their actions on the price, that is if they act as "price-takers", we show in Section 4 that the mechanism maximizes social welfare. However, since the price is a function of their bid, each agent is a "price-maker". If agents attempt to exploit this market power, then a welfare loss may occur at a Nash equilibrium. Consequently we are interested in maximizing (over all equilibria) the *welfare ratio*, more commonly known as the *price of anarchy*, $\frac{\mathcal{W}^{\mathrm{NE}}}{\mathcal{W}^{\mathrm{OPT}}}$. Equivalently, we wish to minimize the *welfare loss*, $1 - \frac{\mathcal{W}^{\mathrm{NE}}}{\mathcal{W}^{\mathrm{OPT}}}$.

2.3 Assumptions

We make the following assumption on the valuation and cost functions.

Assumption 1 *For each consumer q, the valuation function $V_q(d_q) : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing and concave. For each supplier r, the cost function $C_r(s_r) : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing and convex.*

Assumption 1 corresponds to decreasing marginal valuations and increasing marginal costs. The assumption is standard in the literature. It certainly may not hold in every market[3], but without it there will be a natural incentive for the number of agents to decline on both sides of the market. In this paper, we will also assume that our functions are differentiable over their entire domain; this property is assumed primarily for clarity and is not essential.

Assumption 1, however, is not sufficient to ensure a large welfare ratio. In fact, the welfare ratio depends upon the curvature of the *marginal cost functions*. Specifically, if the marginal cost functions are convex, then we show in Section 4 that the welfare ratio is at least 0.58. Concave marginal cost functions also exhibit constant welfare ratios, provided the corresponding total cost function is sufficiently non-linear. However, in the limit as the total cost functions become linear, the welfare ratio degrades to zero (see Section 5 for more details).

Our main result thus concerns convex marginal cost functions. Formally, for most of the paper, we assume that

Assumption 2 *For each supplier r, the marginal cost function $C_r'(s_r)$ is convex. Furthermore, we assume that $C_r(0) = C_r'(0) = 0$.*

Convex marginal cost functions are extremely common in both the theoretical and the practical literature on industrial theory [18], so this assumption is not particularly restrictive. In Assumption 2 we also set $C_r'(0) = 0$, but as we show in the full version of the paper, constant welfare ratios still arise whenever $C_r'(0)$ is bounded below one (it cannot be higher than one or the firm is uncompetitive).

We also remark that Assumption 2 was used in Johari, Mannor and Tsitsiklis [4] in their analysis of the demand-side proportional allocation mechanism with elastic supply. Most of the results of Johari and Tsitsiklis [6] and Tobias and Harks [2] on demand-side Cournot competition with elastic supply also hold under the assumption of convex marginal costs.

3 Our Results

Our first results are concerned with the performance of the mechanism when the users act as price-takers. Under Assumption 1, we prove that:

Theorem 1. *A unique competitive equilibrium exists for the two-sided proportional allocation mechanism. The social welfare attained at the competitive equilibrium is optimal.*

This property was exhibited by Kelly's original proportional allocation mechanism, and has been a feature of all subsequent generalizations by Johari and Tsitsiklis. It is very appealing from a practical point of view, as in actual networks, users are likely to have little information about each other, making it difficult to manipulate the system.

In many other settings however, users will be incentivized to act strategically. In that case, we need to use the stronger solution concept of a Nash equilibrium to analyze the resulting game. Our second result establishes the existence and uniqueness of such equilibria under Assumption 1.

[3] For example, in markets exhibiting economies of scale.

Theorem 2. *The two-sided proportional allocation mechanism has a unique Nash equilibrium for $R \geq 2$.*

Our main result measures the loss of welfare at that unique Nash equilibrium under Assumption 2.

Theorem 3. *The worst case welfare ratio for the mechanism involving $R \geq 2$ suppliers equals*

$$\frac{s^2((R-1)^2 + 4(R-1)s + 2s^2)}{(R-1)(R-1+2s)}$$

where s is the unique positive root of the quartic polynomial $\gamma(s) = 16s^4 + (R-1)s^2(49s-24) + 10(R-1)^2 s(3s-2) + (R-1)^3(5s-4)$. Furthermore, this bound is tight.

It follows that the mechanism admits a constant bound on the price of anarchy. Moreover, Theorem 3 allows us to measure the effects of market competition on social welfare. The following two corollaries are concerned with that relationship.

Corollary 1. *The worst possible price of anarchy is achieved when the supply side is a duopoly ($R = 2$). It evaluates numerically to about 0.588727.*

Corollary 2. *When the supply side is fully competitive ($R \to \infty$) the price of anarchy equals precisely 0.64.*

Consequently, as supply-side competition increases, the welfare ratio improves. In contrast, the welfare ratio *decreases* as demand-side competition increases. Although this fact may seem surprising at first, it turns out to have a simple intuitive explanation. The optimal demand-side allocation consists in giving the entire production to the user which derives from it the highest utility. When more consumers are present in the market, they selfishly request more of the resource for themselves, leaving less for the most needy user and reducing the overall social welfare.

The best welfare ratios thus arise when there is only one consumer ($Q = 1$), that is, in the case of a *monopsony*. In the two-sided proportional allocation mechanism, the best possible price of anarchy over all possible values of Q and R is given by the next corollary.

Corollary 3. *In a market in which a monopsonist faces a fully competitive supply side, the price of anarchy equals $\sqrt{3} - 1$, which is about 0.7321.*

Recall that in the one-sided proportional allocation mechanism for suppliers facing a fixed demand, the welfare loss tends to zero when the supply side is fully competitive [7]. In contrast, Corollary 3 implies that in two-sided markets, that result no longer holds and that full efficiency cannot be achieved.

So far, our results assumed the convexity of marginal costs. Dropping that assumption, we find that the welfare ratio equals zero when the providers' total cost functions are linear. However, the price of anarchy remains bounded for a class of concave marginal cost functions, and degrades smoothy to zero as the total costs become linear.

Corollary 4. *The welfare ratio for cost functions $C_r(s_r) = c_r s_r^{1+\frac{1}{d}}$ where $c_r > 0$ and $d \geq 1$ is $\Omega(\frac{1}{d^2})$.*

Like its one-sided versions, the two-sided mechanism can be generalized to multi-resource markets. An important multi-resource setting is that of bandwidth shared on a network of links. The same guarantees as in the single-resource setting hold for the network version of our market, as well as for more general multi-resource markets (see the full version of the paper for more details).

Theorem 4. *The welfare ratio in networks equals that of the single-resource model.*

Theorem 5. *The welfare guarantees hold for more general multi-resource markets.*

Finally, we show that the proportional allocation mechanism is optimal in the following way:

Theorem 6. *In two-sided markets, the proportional allocation mechanism provides the best welfare ratio amongst a class of single-parameter market-clearing mechanisms.*

Our proof techniques are inspired by the approaches and techniques developed to analyze single-sided markets by Johari [3], Johari and Tsitsiklis ([5], [8] and [7]), Johari, Mannor and Tsitsiklis [4], Tobias and Harks [2], and Roughgarden [12]. Due to space limitations, most of our results will be deferred to the full version of the paper. Here, we will focus upon the proof of Theorem 3.

4 Optimization in Eight Steps

The proof of the main result, Theorem 3, is presented below in eight steps. We formulate the efficiency loss problem as an optimization program in Step III. To be able to formulate this we first need to understand the structure of optimal solutions and of equilibria under this mechanism. This we do in Steps I and II, where we give necessary and sufficient conditions for optimal solutions and for equilibrium. This leads us to an optimization problem that initially appears slightly formidable, so we then attempt to simplify it. In Steps IV and V, we show how to simplify the demand constraints in the program, and in Steps VI and VII, we simplify the supply constraints. This produces an optimization program in a form more amenable to quantitive analysis; we perform this analysis in Step VIII.

Step I: Optimality Conditions. The best possible allocation is the solution to the system:

$$\begin{aligned}
\text{(OPT)} \qquad \max \ & \sum_{q=1}^{Q} V_q(d_q^{\text{OPT}}) - \sum_{r=1}^{R} C_r(s_r^{\text{OPT}}) \\
\text{s.t} \ & \sum_{q=1}^{Q} d_q^{\text{OPT}} = \sum_{r=1}^{R} s_r^{\text{OPT}} \\
& 0 \le s_r^{\text{OPT}} \le 1 \\
& d_q^{\text{OPT}} \ge 0
\end{aligned}$$

Since the constraints are linear, there exists an optimal solution at which the Karush-Kuhn-Tucker (KKT) conditions hold. As the objective function is concave, the following first order conditions are both necessary and sufficient:

$$\begin{aligned}
C_r'\left(s_r^{\text{OPT}}\right) &\le \lambda && \text{if } 0 < s_r^{\text{OPT}} \le 1 \\
C_r'\left(s_r^{\text{OPT}}\right) &\ge \lambda && \text{if } 0 \le s_r^{\text{OPT}} < 1 \\
V_q'\left(d_q^{\text{OPT}}\right) &\le \lambda && \text{if } d_q^{\text{OPT}} = 0 \\
V_q'\left(d_q^{\text{OPT}}\right) &= \lambda && \text{if } d_q^{\text{OPT}} > 0
\end{aligned}$$

We have used λ to denote the dual variable corresponding to the equality constraint.

Step II: Equilibria Conditions. Here we describe necessary and sufficient conditions for a set of bids $\mathbf{b}$ to form Nash equilibrium.

First, observe that there must be at least two suppliers, that is $R \ge 2$. If not, then we have a monopolist k whose payoff is is strictly increasing in b_k. Specifically,

$$\Pi_k(b_k, b_{-k}) = \sum_q b_q - C_k\left(1 - \frac{b_k}{b_k + \sum_q b_q}\right) = \sum_q b_q - C_k\left(\frac{\sum_q b_q}{b_k + \sum_q b_q}\right)$$

Next, we show that if $\mathbf{b}$ is a Nash equilibrium, then at least two bids must be positive. Suppose for a contradiction that we have a supplier k and $\sum_{r \neq k} b_r = \sum_q b_q = 0$. Then $\Pi_k(0) = -C_k(1)$, and $\Pi_k(b_k) = -\frac{R-1}{R} b_k$ when $b_k > 0$. For the second expression, we used the fact that $C_k(x) = 0$ for any $x \leq 0$. Observe that if $b_k = 0$ then the firm can profitably deviate by increasing b_k infinitesimally; on the other hand, if $b_k > 0$ then the firm should infinitesimally decrease b_k. Thus, there is no equilibrium in which either all bids are zero, or a single supplier is the only agent to make a positive bid. Thus there must be at least two positive bids at equilibrium.

Since at least two bids are positive, the payoffs Π_k are differentiable and concave, and the following conditions are necessary and sufficient for the existence of a Nash equilibrium. For the suppliers,

$$C_r'(s_r)\left(1+\frac{s_r^{\mathrm{NE}}}{R-1}\right) \geq p \quad \text{if } 0 < b_r \leq p \qquad\qquad C_r'(s_r)\left(1+\frac{s_r^{\mathrm{NE}}}{R-1}\right) \leq p \quad \text{if } 0 \leq b_r < p$$

For the consumers, $V_q'(0) \leq p$ and $V_q'(d_q^{\mathrm{NE}})\left(1-\frac{d_q}{R}\right) = p$ if $d_q^{\mathrm{NE}} > 0$.

Step III: An optimization problem. We can now formulate the welfare ratio as an optimization problem.

$$\min \quad \frac{\sum_{q=1}^{Q} V_q(d_q^{\mathrm{NE}}) - \sum_{r=1}^{R} C_r(s_r^{\mathrm{NE}})}{\sum_{q=1}^{Q} V_q(d_q^{\mathrm{OPT}}) - \sum_{r=1}^{R} C_r(s_r^{\mathrm{OPT}})} \tag{1}$$

$$\text{s.t.} \quad V_q'(d_q^{\mathrm{NE}})\left(1-\frac{d_q^{\mathrm{NE}}}{R}\right) \geq p \quad \forall q \text{ s.t. } d_q^{\mathrm{NE}} > 0 \tag{2}$$

$$V_q'(d_q^{\mathrm{NE}})\left(1-\frac{d_q^{\mathrm{NE}}}{R}\right) \leq p \quad \forall q \tag{3}$$

$$C_r'(s_r^{\mathrm{NE}})\left(1+\frac{s_r^{\mathrm{NE}}}{R-1}\right) \leq p \quad \forall r \text{ s.t. } 0 < s_r^{\mathrm{NE}} \leq 1 \tag{4}$$

$$C_r'(s_r^{\mathrm{NE}})\left(1+\frac{s_r^{\mathrm{NE}}}{R-1}\right) \geq p \quad \forall r \text{ s.t. } 0 \leq s_r^{\mathrm{NE}} < 1 \tag{5}$$

$$\sum_{q=1}^{Q} d_q^{\mathrm{NE}} = \sum_{r=1}^{R} s_r^{\mathrm{NE}} \tag{6}$$

$$C_r'(s_r^{\mathrm{OPT}}) \leq \lambda \quad \forall r \text{ s.t. } 0 < s_r^{\mathrm{OPT}} \leq 1 \tag{7}$$

$$C_r'(s_r^{\mathrm{OPT}}) \geq \lambda \quad \forall r \text{ s.t. } 0 \leq s_r^{\mathrm{OPT}} < 1 \tag{8}$$

$$V_q'(d_q^{\mathrm{OPT}}) \leq \lambda \quad \forall q \text{ s.t. } d_q^{\mathrm{OPT}} = 0 \tag{9}$$

$$V_q'(d_q^{\mathrm{OPT}}) = \lambda \quad \forall q \text{ s.t. } d_q^{\mathrm{OPT}} > 0 \tag{10}$$

$$\sum_{q=1}^{Q} d_q^{\mathrm{OPT}} = \sum_{r=1}^{R} s_r^{\mathrm{OPT}} \tag{11}$$

$$d_q^{\mathrm{OPT}}, d_q^{\mathrm{NE}} \geq 0 \quad \forall q \tag{12}$$

$$0 \leq s_r^{\mathrm{NE}}, s_r^{\mathrm{OPT}} \leq 1 \quad \forall q, r \tag{13}$$

$$p, \lambda \geq 0 \tag{14}$$

Given the cost and valuation functions, the constraints (2)-(6) are necessary and sufficient conditions for a Nash equilibrium by Step II, and constraints (7)-(11) are the optimality conditions from Step I. We now want to find the worst-case cost and valuation functions for the mechanism.

Step IV: Linear Valuation Functions. To evaluate this intimidating looking program we attempt to simplify it. First, efficiency loss is worst when each consumer has a linear valuation function. This is simple to show using a standard trick (see, for example, [5]). Thus, we restrict ourselves to linear functions of the form $V_q(d_q) = \alpha_q d_q$. Without loss of generality, we may assume that $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_Q$ and that $\max_q \alpha_q = 1$ after we normalize the functions by $1/\max_q \alpha_q$. Observe that this implies that $d_1^{\text{OPT}} = \sum_r s_r^{\text{OPT}}$ and $d_q^{\text{OPT}} = 0$ for $q > 1$. As a result the objective function becomes $\left(d_1^{\text{NE}} + \sum_{q=2}^{Q} \alpha_q d_q^{\text{NE}} - \sum_{r=1}^{R} C_r(s_r^{\text{NE}})\right) / \left(\sum_{r=1}^{R} s_r^{\text{OPT}} - \sum_{r=1}^{R} C_r(s_r^{\text{OPT}})\right)$, and the optimality constraints become $C_r'(s_r^{\text{OPT}}) \leq 1, \forall r$ s.t. $0 < s_r^{\text{OPT}} \leq 1$ and $C_r'(s_r^{\text{OPT}}) \geq 1, \forall r$ s.t. $0 \leq s_r^{\text{OPT}} < 1$. With linear valuations, the new optimality constraints ensure s_r^{OPT} is optimal by setting the marginal cost of each supplier to the marginal valuation, $\alpha_1 = 1$, of the first consumer.

Step V: Eliminating the Demand Constraints. In this step, we describe how to eliminate the demand constraints from the program. First we show that we can transform constraint (14) into $0 \leq p < 1$. Since $\alpha_q \leq 1, \forall q$, we see that constraint (2) implies that $p \leq 1$. Furthermore, if $p = 1$, then (2) can never be satisfied, and so we must have $d_q^{\text{NE}} = 0, \forall q$. The supply equals demand constraint (6) then gives $s_r^{\text{NE}} = 0, \forall r$. This gives a contradiction as the resulting allocation is not a Nash equilibrium: any supplier can increase its profits by providing a bid slightly smaller than p (remember that $C_r'(0) = 0$ by Assumption 2). Thus $p < 1$. This, in turn, implies that $d_1^{\text{NE}} > 0$. To see this, note that if $d_1^{\text{NE}} = 0$ then (3) cannot be satisfied for $q = 1$. Consequently, constraints (2) and (3) must hold with equality for $q = 1$. In fact, without loss of generality, constraints (2) and (3) hold with equality for $q > 1$. If constraint (2) does not hold with equality, we can reduce α_q, and this does not increase the value of the objective function. If $d_q^{\text{NE}} = 0$ and constraint (3) does not hold with equality, we can set $\alpha_q = p$ and the objective function will be unaffected. So, $\alpha_q = \frac{p}{1-d_q^{\text{NE}}/R}$ for all q. Substituting into the objective function:

$$\min \quad \frac{d_1^{\text{NE}} + p\sum_{q=2}^{Q} \frac{d_q^{\text{NE}}}{1-d_q^{\text{NE}}/R} - \sum_{r=1}^{R} C_r(s_r^{\text{NE}})}{\sum_{r=1}^{R} s_r^{\text{OPT}} - \sum_{r=1}^{R} C_r(s_r^{\text{OPT}})} \tag{15}$$

$$\text{s.t.} \quad \left(1 - \frac{d_1^{\text{NE}}}{R}\right) = p \tag{16}$$

$$C_r'(s_r^{\text{NE}})\left(1 + \frac{s_r^{\text{NE}}}{R-1}\right) \leq p \qquad \forall r \text{ s.t. } 0 < s_r^{\text{NE}} \leq 1 \tag{17}$$

$$C_r'(s_r^{\text{NE}})\left(1 + \frac{s_r^{\text{NE}}}{R-1}\right) \geq p \qquad \forall r \text{ s.t. } 0 \leq s_r^{\text{NE}} < 1 \tag{18}$$

$$\sum_{q=1}^{Q} d_q^{\text{NE}} = \sum_{r=1}^{R} s_r^{\text{NE}} \tag{19}$$

$$C_r'(s_r^{\text{OPT}}) \leq 1 \qquad \forall r \text{ s.t. } 0 < s_r^{\text{OPT}} \leq 1 \tag{20}$$

$$C_r'(s_r^{\text{OPT}}) \geq 1 \qquad \forall r \text{ s.t. } 0 \leq s_r^{\text{OPT}} < 1 \tag{21}$$

$$d_q^{\text{NE}} \geq 0 \qquad \forall q \geq 2 \tag{22}$$

$$d_1^{\text{NE}} > 0 \tag{23}$$

$$0 \leq s_r^{\text{NE}}, s_r^{\text{OPT}} \leq 1 \qquad \forall r \tag{24}$$

$$0 \leq p < 1 \tag{25}$$

Now, observe that the objective function is convex and symmetric in the variables $d_2, ..., d_Q$, when all the other variables are held fixed. Convexity holds because our function is a sum of functions $\frac{d_q^{\text{NE}}}{1-d_q^{\text{NE}}/R}$, $q = 2, ..., Q$, that are convex on the range $[0, R]$; note that $d_q^{\text{NE}} \leq R$ by (6), (12) and (13).

Therefore, for any given fixed assignment to the other variables, we must have $d_2 = ... = d_Q := x$. Otherwise, we could reshuffle the variable labels and obtain a second minimum, which is impossible by the convexity of the objective function. So, after replacing every d_q by x, constraint (19) becomes $x = \left(\sum_{r=1}^{R} s_r^{\mathrm{NE}} - d_1^{\mathrm{NE}}\right) / (Q-1)$. After inserting constraint (16) and the new constraint (19), the numerator of the objective function (15) becomes

$$
\begin{aligned}
&(1-p)R + p(Q-1)\frac{x}{1-x/R} - \sum_{r=1}^{R} C_r(s_r^{\mathrm{NE}}) \\
&= (1-p)R + p(Q-1)\frac{\left(\sum_{r=1}^{R} s_r^{\mathrm{NE}} - d_1^{\mathrm{NE}}\right) / (Q-1)}{1 - \frac{1}{R}\left(\sum_{r=1}^{R} s_r^{\mathrm{NE}} - d_1^{\mathrm{NE}}\right) / (Q-1)} - \sum_{r=1}^{R} C_r(s_r^{\mathrm{NE}}) \\
&= (1-p)R + p\frac{\sum_{r=1}^{R} s_r^{\mathrm{NE}} - (1-p)R}{1 - \left(\sum_{r=1}^{R} s_r^{\mathrm{NE}} - d_1^{\mathrm{NE}}\right) / R\,(Q-1)} - \sum_{r=1}^{R} C_r(s_r^{\mathrm{NE}})
\end{aligned}
$$

Finally, observe that if we increase Q by one, the objective function (1) cannot increase, since we can set $d_{Q+1} = 0$ and at least keep the same objective function value as before. Therefore, without loss of generality, we can take the limit as $Q \to \infty$. Note that this only changes the objective function, as all the constraints that contained Q have been inserted into the function and can be eliminated. After these changes, the optimization problem becomes

$$
\min \quad \frac{(1-p)^2 R + p\sum_{r=1}^{R} s_r^{\mathrm{NE}} - \sum_{r=1}^{R} C_r(s_r^{\mathrm{NE}})}{\sum_{r=1}^{R} s_r^{\mathrm{OPT}} - \sum_{r=1}^{R} C_r(s_r^{\mathrm{OPT}})} \tag{26}
$$

$$
\text{s.t.} \quad C_r'(s_r^{\mathrm{NE}})\left(1 + \frac{s_r^{\mathrm{NE}}}{R-1}\right) \le p \quad \forall r \text{ s.t. } 0 < s_r^{\mathrm{NE}} \le 1 \tag{27}
$$

$$
C_r'(s_r^{\mathrm{NE}})\left(1 + \frac{s_r^{\mathrm{NE}}}{R-1}\right) \ge p \quad \forall r \text{ s.t. } 0 \le s_r^{\mathrm{NE}} < 1 \tag{28}
$$

$$
C_r'(s_r^{\mathrm{OPT}}) \le 1 \quad \forall r \text{ s.t. } 0 < s_r^{\mathrm{OPT}} \le 1 \tag{29}
$$

$$
C_r'(s_r^{\mathrm{OPT}}) \ge 1 \quad \forall r \text{ s.t. } 0 \le s_r^{\mathrm{OPT}} < 1 \tag{30}
$$

$$
0 \le s_r^{\mathrm{NE}}, s_r^{\mathrm{OPT}} \le 1 \quad \forall r \tag{31}
$$

$$
0 \le p < 1 \tag{32}
$$

Hence, we have achieved our goal and completely eliminated the demand side of the optimization problem. Specifically, all the demand constraints have been replaced with an expression that is a function of the supply-side allocation. Now we must find the worst such allocation.

Step VI: Linear Marginal Cost Functions The next step is to show that, in searching for a worst case allocation, we can restrict our attention to linear marginal cost functions of the form $C_r'(s_r) = \beta_r s_r$ where $\beta_r > 0$. In this section, we briefly sketch the proof of this fact and defer the full treatment to the full version of the paper. Our proof technique is based on the work of Johari, Mannor and Tsitsiklis on demand-side markets with elastic supply [4], [6].

The proof consists in exhibiting, for any family of cost functions $C_r(s_r), r \in R$, two new families $\hat{C}_r()$ and $\bar{C}_r()$ with the property that the C_r have a better performance ratio than the $\bar{C}_r$ which, in turn, have a a better performance ratio than the $\hat{C}_r$. Furthermore, the $\hat{C}_r$ will be a family with linear marginal costs, as desired. The cost functions are defined as

$$\bar{C}'_r(s_r) = \begin{cases} C'_r(s_r) & \text{if } s_r < s_r^{\text{NE}} \\ \dfrac{C'_r(s_r^{\text{NE}})}{s_r^{\text{NE}}}\, s_r & \text{if } s_r \geq s_r^{\text{NE}} \end{cases} \qquad \text{and} \qquad \hat{C}'_r(s_r) = \frac{C'_r(s_r^{\text{NE}})}{s_r^{\text{NE}}}\, s_r$$

where s_r^{NE} is the Nash equilibrium allocation to supplier r when the cost functions are $C_r(s_r)$. Observe that the s_r^{NE} still satisfy the Nash equilibrium conditions (27) and (28) for both $\bar{C}_r$ and $\hat{C}_r$. Thus $\bar{s}_r^{\text{NE}} = \hat{s}_r^{\text{NE}} = s_r^{\text{NE}}$. The heart of the proof consists in showing that the optimal welfare can only improve when going from one family to the next.

Step VII: Eliminating the Supply Constraints. Assuming linear marginal cost functions, the optimization problem (26)-(32) becomes

$$\begin{aligned}
\min \quad & \frac{(1-p)^2 R + p\sum_{r=1}^R s_r^{\text{NE}} - \frac{1}{2}\sum_{r=1}^R \beta_r (s_r^{\text{NE}})^2}{\sum_{r=1}^R s_r^{\text{OPT}} - \frac{1}{2}\sum_{r=1}^R \beta_r (s_r^{\text{OPT}})^2} && (33)\\
\text{s.t.} \quad & \beta_r s_r^{\text{NE}}\left(1 + \frac{s_r^{\text{NE}}}{R-1}\right) \leq p \quad \forall r \text{ s.t. } 0 < s_r^{\text{NE}} \leq 1 && (34)\\
& \beta_r s_r^{\text{NE}}\left(1 + \frac{s_r^{\text{NE}}}{R-1}\right) \geq p \quad \forall r \text{ s.t. } 0 \leq s_r^{\text{NE}} < 1 && (35)\\
& \beta_r s_r^{\text{OPT}} \leq 1 \quad \forall r \text{ s.t. } 0 < s_r^{\text{OPT}} \leq 1 && (36)\\
& \beta_r s_r^{\text{OPT}} \geq 1 \quad \forall r \text{ s.t. } 0 \leq s_r^{\text{OPT}} < 1 && (37)\\
& 0 \leq s_r^{\text{NE}}, s_r^{\text{OPT}} \leq 1 \quad \forall r && (38)\\
& \beta_r > 0 \quad \forall r && (39)\\
& 0 \leq p < 1 && (40)
\end{aligned}$$

with the new variables β_r, $r = 1, ..., R$. From $s_r^{\text{OPT}} \geq s_r^{\text{NE}}$, we can then deduce that (34) and (35) hold with equality. Suppose they don't for some r. Then $s_r^{\text{NE}} = s_r^{\text{OPT}} = 1$. Constraint (34) is $\beta_r < \frac{p}{1+1/(R-1)} < p < 1$. Hence, $\beta_r = \frac{p}{1+1/(R-1)}$ will be a feasible solution (i.e. constraint (36) will still be satisfied). Furthermore, increasing β_r to $\frac{p}{1+1/(R-1)}$ will only decrease the objective function since this is equivalent to subtracting a positive number from the numerator and the denominator. We can further simplify the system by replacing constraints (36) and (37) with $s_r^{\text{OPT}} = \min(1/\beta_r, 1)$. It is easy to see that s_r^{OPT} and β_r satisfy the equation above if and only if they satisfy (36) and (37). The reduced optimization problem now becomes:

$$\begin{aligned}
\min \quad & \frac{(1-p)^2 R + p\sum_{r=1}^R s_r^{\text{NE}} - \frac{1}{2}\sum_{r=1}^R \beta_r (s_r^{\text{NE}})^2}{\sum_{r=1}^R s_r^{\text{OPT}} - \frac{1}{2}\sum_{r=1}^R \beta_r (s_r^{\text{OPT}})^2} && (41)\\
\text{s.t.} \quad & \beta_r s_r^{\text{NE}}\left(1 + \frac{s_r^{\text{NE}}}{R-1}\right) = p \quad \forall r && (42)\\
& s_r^{\text{OPT}} = \min(1/\beta_r, 1) \quad \forall r && (43)\\
& 0 < s_r^{\text{NE}}, s_r^{\text{OPT}} \leq 1 \quad \forall r && (44)\\
& \beta_r > 0 \quad \forall r && (45)\\
& 0 \leq p < 1 && (46)
\end{aligned}$$

We can insert the equality constraints (42) and (43) into the objective function (41) to obtain:

$$\begin{aligned}
\min \quad & \frac{(1-p)^2 R + p\sum_{r=1}^{R} s_r^{\mathrm{NE}} - \frac{p}{2}\sum_{r=1}^{R} \frac{s_r^{\mathrm{NE}}}{1+s_r^{\mathrm{NE}}/(R-1)}}{\sum_{r=1}^{R} \min(1/\beta_r, 1) - \frac{p}{2}\sum_{r=1}^{R} \frac{\min(1/\beta_r,1)^2}{s_r^{\mathrm{NE}}(1+s_r^{\mathrm{NE}}/(R-1))}} && (47)\\
\text{s.t.} \quad & 0 < s_r^{\mathrm{NE}} \leq 1 \quad \forall r && (48)\\
& \beta_r = \frac{p}{s_r^{\mathrm{NE}}\left(1+s_r^{\mathrm{NE}}/(R-1)\right)} \quad \forall r && (49)\\
& 0 \leq p < 1 && (50)
\end{aligned}$$

The objective function (47) can be rewritten as:

$$\frac{\sum_{r=1}^{R}\left((1-p)^2 + p s_r^{\mathrm{NE}} - \frac{p}{2}\frac{s_r^{\mathrm{NE}}}{1+s_r^{\mathrm{NE}}/(R-1)}\right)}{\sum_{r=1}^{R}\left(\min(1/\beta_r, 1) - \frac{p}{2}\frac{\min(1/\beta_r,1)^2}{s_r^{\mathrm{NE}}(1+s_r^{\mathrm{NE}}/(R-1))}\right)}$$

Consequently, the minimum of the optimization problem (47)-(50) is greater than or equal to

$$\begin{aligned}
\min \quad & \frac{(1-p)^2 + ps - \frac{p}{2}\frac{s}{1+s/(R-1)}}{\min(\frac{s(1+s/(R-1))}{p}, 1) - \frac{p}{2s(1+s/(R-1))}\min(\frac{s(1+s/(R-1))}{p}, 1)^2} && (51)\\
\text{s.t.} \quad & 0 < s \leq 1 && (52)\\
& 0 \leq p < 1 && (53)
\end{aligned}$$

We have now reduced the system (33)-(40) to a two-dimensional minimization problem. The next step is to try to explicitly find the minimum.

Step VIII: Computing the Worst Case Welfare Ratio. To obtain Theorem 3 we need to solve the optimization problem (51)-(53) with R as a parameter. We show how to do this in the full version of the paper. Thus we have proved our main result. It has several ramifications. Firstly, the worst case welfare ratio occurs with duopolies, that is when $R = 2$. There we obtain $s = 0.566812\cdots$ which gives a worst case welfare ratio of $0.588727\cdots$. Moreover, observe that this bound is tight. Our proof is essentially constructive; costs and valuations can be defined to to create an instance that produces the bound. Secondly, the welfare ratio improves as the number of supplies increases. Specifically as $R \to \infty$, the bound tends to $\frac{16}{25}$. Thus we obtain Corollaries 1 and 2.

So, as supply-side competition increases, the welfare ratio does improves. The opposite occurs as demand-side competition increases. Specifically, adapting our approach gives Corollary 3.

5 Concave Marginal Cost Functions.

The welfare ratio tends to zero if the cost function is linear, that is if the marginal cost function is a constant; for an example see the full version of the paper. We can get some idea of how the welfare ratio tends to zero for concave marginal cost functions by considering a class of polynomial cost functions with degree $1 + \frac{1}{d}$. These functions give a welfare ratio of $\Omega(\frac{1}{d^2})$, for any constant d. A proof of this (Corollary 4) is given in the full version of the paper. See Neumayer [10] for another example of inefficiency in the presence of linear cost functions.

6 Extensions to Networks and Arbitrary Markets.

We can generalize our results for bandwidth markets over a single network connection to the case where bandwidth is shared over an entire network. In that model, each consumer q is associated with a source-sink pair, and providers at associated with edges of the network at which they can offer bandwidth. A consumer's payoff is a function of the maximum (s_q, t_q)-flow it can obtain using the bandwidth it has purchased in the network.

The welfare guarantees for the network model are the same as for the single-link case. A formal description of the network model and a proof of Theorem 4 is given in the full version of the paper. Moreover, if we identify links $e \in E$ with arbitrary resources, then our results extend to a general class of markets with any number of resources. The exact definition of these markets and a proof of Theorem 5 are also given in the full version of the paper.

7 Smooth Market-Clearing Mechanisms

It was shown in [3] and [8] that in one-sided markets, the proportional allocation mechanism uniquely achieves the best possible welfare ratio within a broad class of so-called *smooth market-clearing mechanisms*. This family has a natural extension to the case of two-sided mechanisms, and we show that, given a symmetry condition, the two-sided proportional allocation mechanism is optimal amongst that class of single-parameter mechanisms. A description of smooth market-clearing mechanisms and a proof of Theorem 6 is given in the full version of the paper.

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8 Appendices

8.1 Appendix A: Competitive Equilibria and Optimal Welfare

Here we prove Theorem 1; that is, we show that a competitive equilibrium exist and achieves maximum welfare. A pair $(\mathbf{b}, p)$ where $\mathbf{b} \geq 0$ and $p > 0$ is a called a *competitive equilibrium* if:

$$\begin{aligned} \Pi_k(b_k, p) &\geq \Pi_k(\bar{b}_k, p) \text{ for all } \bar{b}_k \geq 0, \text{ for every } k = 1, ..., R+Q \\ p(\mathbf{b}) &= \frac{\sum_q b_q + \sum_r b_r}{R} \end{aligned}$$

Thus, in a competitive equilibrium, the agents are maximizing their payoffs under the assumption that they are price-takers. We will show that the necessary and sufficient conditions under which the demand and supply allocations $(\mathbf{d}, \mathbf{s})$ induced by $(\mathbf{b}, p)$ form a competitive equilibrium are identical to the conditions under which $(\mathbf{d}, \mathbf{s})$ is an optimal solution to (OPT).

We have already found the optimality conditions in Section 4. So let's calculate the required conditions for a competitive equilibrium. For a supplier, the payoff as a function of b is:

$$\Pi_r(b_r) = p\left(1 - \frac{b_r}{p}\right) - C_r\left(1 - \frac{b_r}{p}\right)$$

Since this is differentiable and concave for all $b_r \geq 0$ (recall that we assume that the right derivative exists at $b_r = 0$), the maximum occurs when the following conditions are satisfied:

$$\begin{aligned} C_r'\left(1 - \frac{b_r}{p}\right) &\geq p \qquad \text{if } 0 < b_r \leq p \\ C_r'\left(1 - \frac{b_r}{p}\right) &\leq p \qquad \text{if } 0 \leq b_r < p \end{aligned}$$

To see this, note that when $0 < b_r \leq p$, we have $\Pi_r'(b_r) \geq 0$. If $\Pi_r'(b_r) < 0$ then we can increase Π_r by infinitesimally decreasing b_r (and on $(0, p]$ we can always choose a smaller b_r). The first condition then follows. The second condition is derived analogously. Also observe that we can let $b_r \leq p$, since at $b_r > 0$ supplier r will receive a negative payoff.

Similarly, we can write the payoff of consumer q as a function of b_q:

$$\Pi_q(b_q) = V_q(\frac{b_q}{p}) - p\frac{b_q}{p}$$

This is again concave by assumption, and the conditions are

$$\begin{aligned} V_q'\left(\frac{b_q}{p}\right) &= p \qquad \text{if } b_q > 0 \\ V_q'(0) &\leq p \qquad \text{if } b_q = 0 \end{aligned}$$

Next, define $b_r = (1 - s_r)\lambda$ and $b_q = d_q\lambda$. Then it can be checked that the equilibrium conditions above are satisfied with $p = \lambda$. We must have $\lambda \geq 0$ because $V_q'(d_q) \geq 0$ by assumption, and $\lambda \geq V_q'(d_q)$ for all q. Thus there exists a competitive equilibrium and it is an optimal solution to (OPT). Finally, suppose $(\mathbf{b}, p)$ is a competitive equilibrium. Then define $s_r = 1 - \frac{b_r}{p}$ and $d_q = \frac{b_q}{p}$. Then we can check that the KKT conditions are satisfied, and the allocation is an optimal solution to (OPT). This completes the proof of Theorem 1.

8.2 Appendix B: The Existence of Nash Equilibria

To prove Theorem 2 we show that the Nash equilibrium conditions obtained in Section 4 are exactly the necessary and sufficiency conditions for the following system.

$$
\begin{aligned}
\text{(NASH)} \quad \max \quad & \sum_{q=1}^{Q} \hat{V}_q(d_q) - \sum_{r=1}^{R} \hat{C}_r(s_r) \\
\text{s.t.} \quad & \sum_{q=1}^{Q} d_q = \sum_{r=1}^{R} s_r \\
& 0 \leq s_r \leq 1 \\
& d_q \geq 0
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{V}_q &= \left(1 - \frac{d_q}{R}\right) V'(d_q) + \frac{1}{R}\int_0^{d_q} V_q(z)dz \\
\hat{C}_r &= \left(1 + \frac{s_r}{R-1}\right) C_r'(s_r) - \frac{1}{R-1}\int_0^{s_r} C_r(z)dz
\end{aligned}
$$

By assumption, the $\hat{V}_q$ and $\hat{C}_r$ are differentiable and strictly concave and strictly convex, respectively. The KKT conditions for (NASH) then give:

$$
\begin{aligned}
\left(1 + \frac{s_r}{R-1}\right) C_r'(s_r) &\leq \lambda \qquad && \text{if } 0 < s_r \leq 1 \\
\left(1 + \frac{s_r}{R-1}\right) C_r'(s_r) &\geq \lambda \qquad && \text{if } 0 \leq s_r < 1 \\
\left(1 - \frac{d_q}{R}\right) V_q'(d_q) &\leq \lambda \\
\left(1 - \frac{d_q}{R}\right) V_q'(d_q) &\geq \lambda \qquad && \text{if } 0 < d_q
\end{aligned}
$$

where λ is the dual variable corresponding to the equality constraint. These are the necessary and sufficient conditions for an optimal solution. Since the objective function of (NASH) is strictly convex, the optimal solution is unique.

We now claim that an optimal solution to (NASH) is a Nash equilibrium for the two-side proportional allocation mechanism. Recall that for these price-anticipators, a Nash equilibrium is a vector $\mathbf{b} \geq 0$ such that for every agent $k = 1, ..., Q + R$,

$$
\Pi_k(b_k, \mathbf{b}_{-k}) \geq \Pi_k(\bar{b}_k, \mathbf{b}_{-k}), \text{ for all } \bar{b}_k \geq 0
$$

As before, define $b_q = d_q\lambda$ and $b_r = (1-s_r)\lambda$. Then one can check that the Nash equilibrium conditions are satisfied with $p = \lambda$. We must have $\lambda \geq 0$ because $V_q'(d_q) \geq 0$ by assumption and $\lambda \geq V_q'(d_q)$ for all q. There are at least two positive bids because otherwise the supply equals demand constraint of cannot be satisfied.

Conversely, we claims that any Nash equilibrium is an optimal solution to (NASH). To see this take any Nash equilibrium $(\mathbf{b}, p)$. Then define $s_r = 1 - \frac{b_r}{p}$ and $d_q = \frac{b_q}{p}$. Then we can easily check that the KKT conditions are satisfied, and the allocation is an optimal solution to (NASH). This completes the proof of Theorem 2.

So we have to solve the optimization problem (51)-(53). We will consider R as a parameter. At the minimum of (51), we will have either

$$\frac{s\,(1+s/(R-1))}{p} \leq 1 \qquad (54) \qquad \text{or} \qquad \frac{s\,(1+s/(R-1))}{p} > 1 \qquad (55)$$

When (54) holds, we solve the following system:

$$\min \quad \frac{(1-p)^2 + ps - \frac{p}{2}\frac{s}{1+s/(R-1)}}{\frac{s(1+s/(R-1))}{2p}} \tag{56}$$

$$\text{s. t.} \quad s(1+s/(R-1)) \leq p \tag{57}$$

$$0 < s \leq 1 \tag{58}$$

$$0 \leq p < 1 \tag{59}$$

Now set $W = R-1$ and let g be the objective function (60). Thus

$$g(s) = 2pW\left(\frac{(1-p)^2}{s(W+s)} + \frac{p}{(W+s)} - \frac{pW}{2(W+s)^2}\right)$$

Differentiating we obtain

$$\frac{\partial g}{\partial s} = -\frac{p^2 W}{s^2(W+s)^3}\left((1-p)^2(W+2s)(W+s) + ps^3\right)$$

and this is strictly negative for any W, s. Therefore, the optimal value of s must occur at one of the boundaries of the feasible region. Since g is strictly decreasing in s, we must have $s(1+s/W) = p$. Thus it is enough to consider a variant of the second case (55) in which the inequality is no longer strict. Observe that the point at which the minimum of (56-59) is achieved is also part of the feasible region of the following new optimization problem:

$$\min \quad \frac{(1-p)^2 + ps - \frac{ps}{2(1+s/W)}}{1 - \frac{p}{2s(1+s/W)}} \tag{60}$$

$$\text{s. t.} \quad s(1+s/W) \geq p \tag{61}$$

$$0 < s \leq 1 \tag{62}$$

$$0 \leq p < 1 \tag{63}$$

Let f denote the objective function (60). We have

$$\frac{\partial f}{\partial s} = \frac{2p\left(2s^4 - W^2((p-1)^2 + ps - s^2) + Ws(-2(p-1)^2 - 3ps + 4s^2)\right)}{\left(pW - 2s(W+s)\right)^2}$$

and at $s = 1$ this becomes

$$\frac{\partial f}{\partial s}(1,p) = -\frac{2p(-2 + (-2-p+2p^2)W + (p-1)pW^2)}{((p-2)W-2)^2}$$

which is always positive since each term in the numerator is negative. But by the KKT conditions, $\partial f/\partial s$ must be negative at $s = 1$ if that point is the minimum, and therefore we must rule out that possibility.

Similarly, when constraint (61) is tight, $\partial f/\partial s$ becomes

$$\frac{\partial f}{\partial s} = 2\left(2 - W - \frac{1}{s} + \frac{4s}{W} - \frac{4s^2}{W} - \frac{2s^3}{W^2} + \frac{W^2-1}{W+s}\right)$$

It can be checked that this is always negative. But, as before, if constraint (61) is tight at the minimum, $\partial f/\partial s$ must be positive, and we again have to rule out that possibility. Finally, we cannot have $s \to 0$ at the minimum, since the objective function would go to one in that case. Thus, at the minimum, we have

$$\frac{\partial f}{\partial s} = \frac{2p\left(2s^4 - (S-1)^2((p-1)^2 + ps - s^2) + (S-1)s(-2(p-1)^2 - 3ps + 4s^2)\right)}{\left(pS - 2s(S+s)\right)^2} = 0$$

Solving this equation for p, we find that

$$p^{\pm} = \frac{S(2-s) + (4-3s)s \pm \frac{\sqrt{s}}{\sqrt{S}}\sqrt{\gamma(s)}}{2(S+2s)}$$

where

$$\gamma(s) = 16s^4 + 10S^2 s(3s-2) + S^3(5s-4) + Ss^2(49s-24)$$

When we plug either p^+ or p^- into the objective (60), we find (quite surprisingly) that the objective function reduces to the same simple expression in both cases. The optimization problem (60)-(63) thus becomes:

$$\min \quad \frac{s^2(S^2 + 4Ss + 2s^2)}{S(S+2s)} \tag{64}$$

$$\text{s. t.} \quad 0 < s \leq 1 \tag{65}$$

$$0 \leq p^{\pm} < 1 \tag{66}$$

$$s(1 + s/S) \geq p^{\pm} \tag{67}$$

$$p^{\pm} \in \mathbb{R} \tag{68}$$

It is not hard to show that the objective function (64) is strictly increasing in s. The minimum value is thus achieved at the smallest feasible s. For condition (68) to be satisfied, we must have $\gamma(s) \geq 0$. One can check that $\gamma(s)$ has only one zero s^* on $[0,1]$, with $\gamma(s) \geq 0$ for $s \geq s^*$. Furthermore, at s^*,

$$p^{\pm} = \frac{S(2-s^*) + (4-3s^*)}{2(S+2s^*)} > 0$$

and constraint (66) holds. Finally, one can check using a computer that (67) holds for all s.

Since s^* is the smallest feasible point in our domain, we conclude that the minimum is achieved at that point. This finally proves the main claim of Theorem 3.

8.4 Appendix D: Positive Marginal Costs at Zero

Here we consider the case where $C_r'(0) > 0$ and show that the welfare ratio remains a constant provided $C_r'(0)$ is bounded below one. Specifically, assume $C_r(s_r) = \frac{1}{2}\beta_r s_r^2 + \gamma_r$ where $0 \le \gamma_r \le 1$. Then we can follow the original proof and find that the welfare ratio is the solution to the following problem:

$$\begin{aligned} \text{minimize} \quad & \frac{(1-p)^2 + ps - \frac{1}{2}(\frac{p}{1+s/(R-1)} - \gamma)s - \gamma s}{\min(\frac{1-\gamma}{\beta}, 1) - \frac{1}{2}\beta \min(\frac{1-\gamma}{\beta}, 1)^2 - \gamma \min(\frac{1-\gamma}{\beta}, 1)} \\ \text{s.t.} \quad & 0 \le s \le 1 \\ & 0 \le p \le 1 \\ & 0 \le \gamma \le 1 \end{aligned}$$

where

$$\beta = \left(\frac{p}{1+s/(R-1)} - \gamma\right)\frac{1}{s}$$

As before, we have two cases to consider. First suppose $1-\gamma \ge \beta$.

$$\begin{aligned} \text{minimize} \quad & \frac{(1-p)^2 + ps - \frac{1}{2}(\frac{p}{1+s/(R-1)} - \gamma)s - \gamma s}{1 - \frac{1}{2}\left(\frac{p}{1+s/(R-1)} - \gamma\right)\frac{1}{s} - \gamma} \\ \text{s.t.} \quad & 1-\gamma \ge \left(\frac{p}{1+s/(R-1)} - \gamma\right)\frac{1}{s} \ge 0 \\ & 0 \le s \le 1 \\ & 0 \le p \le 1 \\ & 0 \le \gamma \le 1 \end{aligned}$$

We claim that for any fixed $\gamma < 1$, the solution to this problem is strictly positive. First, we can drop the $\frac{1}{2}\left(\frac{p}{1+s/(R-1)} - \gamma\right)\frac{1}{s}$ term from the denominator, as this can only decrease the value of the objective function. We will also let $R \to \infty$ in the numerator. The objective function then becomes

$$\begin{aligned} \text{minimize} \quad & \frac{(1-p)^2 + \frac{ps}{2} - \frac{\gamma s}{2}}{1-\gamma} \\ \text{s.t.} \quad & 1-\gamma \ge \left(\frac{p}{1+s/(R-1)} - \gamma\right)\frac{1}{s} \ge 0 \\ & 0 \le s \le 1 \\ & 0 \le p \le 1 \\ & 0 \le \gamma \le 1 \end{aligned}$$

If there is a pair (p, s) that makes the objective function go to zero, it must satisfy

$$(1-p)^2 + \frac{ps}{2} - \frac{\gamma s}{2} \implies s = \frac{2(1-p)^2}{\gamma - p}$$

Observe that since $s \ge 0$, we must have $\gamma > p$. But by the first constraint, we have $p \ge \gamma$, a contradiction. So the point at which the objective function is outside our domain, and since that set is closed, we also cannot get arbitrarily close to that point.

Suppose now $\frac{1-\gamma}{\beta} < 1$. Then the optimization problem is

$$\begin{aligned} \text{minimize} \quad & \frac{(1-p)^2 + ps - \frac{1}{2}(\frac{p}{1+s/(R-1)} - \gamma)s - \gamma s}{\frac{1-\gamma}{\beta} - \frac{1}{2}(\frac{(1-\gamma)^2}{\beta}) - \gamma\frac{1-\gamma}{\beta}} \\ \text{s.t.} \quad & 1-\gamma < \beta \\ & 0 \le s \le 1 \\ & 0 \le p \le 1 \\ & 0 \le \gamma \le 1 \end{aligned}$$

and the denominator can be rewritten as

$$\left(1 - \gamma - \frac{(1-\gamma)^2}{2} - \gamma(1-\gamma)\right) \frac{s}{\frac{p}{1+s/(R-1)} - \gamma}$$

If the objective function goes to zero, then either the denominator goes to infinity, which can only happen when $\beta \downarrow 0$, or when the numerator goes to zero. The first case cannot occur because of the $1-\gamma < \beta$ constraint. The second case can be treated like above: if the numerator is zero, we must have

$$s = \frac{2(1-p)^2}{\gamma - p}$$

which implies $\gamma > p$, but from the $0 < 1-\gamma < \beta$ constraint we get that $p > \gamma$.

8.5 Appendix E: Constant Marginal Cost Functions and Total Welfare Loss

Here we show that if the marginal costs are constant then the welfare ratio tends to zero. Consider linear valuation and cost functions with slopes α_q and β_r, respectively,

$$V_q(d_q) = \alpha_q d_q$$
$$C_r(s_r) = \beta_r s_r$$

The bids corresponding to this allocation at a Nash equilibrium can then be defined according to the equilibrium conditions developed in Section 4. We will parametrize the allocations and the functions by a variable $0 < x < 1$ and then let $x \to 0$. So, let

$$\begin{aligned} s_r &= x && \forall r \\ \mu &= 1 - \frac{x}{Q} \\ \beta_r &= \frac{1 - \frac{x}{Q}}{1 + \frac{x}{R-1}} && \forall r \\ d_q &= R\frac{x}{Q} && \forall q \\ \alpha_q &= 1 && \forall q \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{q=1}^{Q} d_q &= \sum_{q=1}^{Q} R\frac{x}{Q} \\ &= Rx \\ &= \sum_{r=1}^{R} s_r \end{aligned}$$

and that $0 < s_r < 1$ and $d_q > 0$. This shows the above allocation is a feasible solution to (NASH). Next we will show that the KKT conditions hold at $(\mathbf{s}, \mathbf{d})$ with $\lambda = \mu$. This will imply that the allocation $(\mathbf{s}, \mathbf{d})$ corresponds to a Nash equilibrium. We have

$$\begin{aligned} V_q'(d_q)\left(1 - \frac{d_q}{R}\right) &= 1 \cdot \left(1 - \frac{d_q}{R}\right) \\ &= 1 - \frac{x}{Q} \\ &= p \end{aligned}$$

For the firms, we have

$$\begin{aligned} C_r'(s_r)\left(1 + \frac{s_r}{R-1}\right) &= \beta_r\left(1 + \frac{s_r}{R-1}\right) \\ &= \left(\frac{1 - \frac{x}{Q}}{1 + \frac{x}{R-1}}\right)\left(1 + \frac{x}{R-1}\right) \\ &= p \end{aligned}$$

Thus the KKT conditions are satisfied, and so these allocations form a Nash equilibrium.

The expression for the welfare ratio can be written as

$$\frac{\sum_{q=1}^{Q} V_q(d_q^{\mathrm{NE}}) - \sum_{r=1}^{R} C_r(s_r^{\mathrm{NE}})}{\sum_{q=1}^{Q} V_q(d_q^{\mathrm{OPT}}) - \sum_{r=1}^{R} C_r(s_r^{\mathrm{OPT}})} = \frac{\sum_{q=1}^{Q} \alpha_q(d_q^{\mathrm{NE}}) - \sum_{r=1}^{R} \beta_r(s_r^{\mathrm{NE}})}{R - \sum_{r=1}^{R} \beta_r}$$

This equality holds because every supplier r that has $s_r^{\mathrm{NE}} > 0$ has a slope $\beta_r < 1$. To see this, observe that by the KKT conditions, whenever $d_q > 0$,

$$V_q'(d_q^{\mathrm{NE}}) \;=\; \frac{\mu}{1-\frac{d_q}{R}} \;>\; \mu \;>\; \frac{\mu}{1+\frac{s_r}{R-1}} \;=\; C_r'(s_r^{\mathrm{NE}})$$

and recall that $\mu < 1$. Also, if $\beta_r \geq 1$, that supplier will not produce anything (neither at the Nash equilibrium nor at the optimal solution). So, without loss of generality, we can remove that supplier away and, thus, consider only the cases where $\beta_r < 1$. Since for any supplier the marginal cost of producing one unit is always less than one, in an optimal solution everyone produces $s_r = 1$ and gives it to the consumer with slope $\alpha_1 = 1$.

Now putting our definitions in the formula, we obtain:

$$\begin{aligned}
\frac{\sum_{q=1}^{Q} \alpha_q(d_q^{\mathrm{NE}}) - \sum_{r=1}^{R} \beta_r(s_r^{\mathrm{NE}})}{R - \sum_{r=1}^{R} \beta_r} &= \frac{\sum_{q=1}^{Q} 1 \cdot R\frac{x}{Q} - \sum_{r=1}^{R} \left(\frac{1-\frac{x}{Q}}{1+\frac{x}{R-1}}\right) x}{R - \sum_{r=1}^{R} \left(\frac{1-\frac{x}{Q}}{1+\frac{x}{R-1}}\right)} \\
&= \frac{Rx - R\left(\frac{1-\frac{x}{Q}}{1+\frac{x}{R-1}}\right) x}{R - R\left(\frac{1-\frac{x}{Q}}{1+\frac{x}{R-1}}\right)} \\
&= x
\end{aligned}$$

The result follows by simply letting $x \to 0$.

8.6 Appendix F: Concave Marginal Cost Functions

Suppose $C(s_r) = \frac{1}{1+1/d} s_r^{1+1/d}$ for some fixed d. The reduced optimization problem is:

$$\begin{aligned} \text{minimize} \quad & \frac{(1-p)^2 + ps - (\frac{p}{1+1/d})(\frac{s}{1+s/R-1})}{\min(1/\beta, 1) - (\frac{\beta}{1+1/d}) \min(1/\beta, 1)^{1+1/d}} \\ \text{s.t} \quad & 0 \le s \le 1 \\ & 0 \le p \le 1 \end{aligned}$$

where

$$\beta = \frac{p}{s^{1/d}(1 + s/(R-1))}$$

First suppose that $1/\beta > 1$. The optimization problem is:

$$\begin{aligned} \text{minimize} \quad & \frac{(1-p)^2 + ps - (\frac{p}{1+1/d})(\frac{s}{1+s/R-1})}{1 - (\frac{p}{1+1/d})(\frac{1}{s^{1/d}(1+s/R-1)})} \\ \text{s.t} \quad & s^{1/d}\left(1 + \frac{s}{R-1}\right) \ge p \\ & 0 \le s \le 1 \\ & 0 \le p \le 1 \end{aligned}$$

We can lower bound the objective function by setting the denominator to one. We can also let $R \to \infty$ in the objective function. Then the objective function becomes

$$(1-p)^2 + \left(1 - \frac{1}{1+1/d}\right) ps$$

and the only way this can go to zero is if $s \to 0, p \to 1$. But it's not hard to see that for any d, this is impossible by the first constraint.

Suppose now that $1/\beta < 1$. The optimization problem is:

$$\begin{aligned} \text{minimize} \quad & \frac{(1-p)^2 + ps - (\frac{p}{1+1/d})(\frac{s}{1+s/R-1})}{1/\beta - (\frac{1}{1+1/d})(1/\beta)^{1/d}} \\ \text{s.t} \quad & 1 < \beta \\ & 0 \le s \le 1 \\ & 0 \le p \le 1 \end{aligned}$$

This goes to zero either when the denominator goes to infinity, which can only happen when $\beta \downarrow 0$, or when the numerator goes to zero. The first case cannot occur because of the $1 < \beta$ constraint. The second case can be treated as above.

A welfare ratio of $\Omega(\frac{1}{d^2})$ can then easily be shown.

8.7 Appendix G: Extension to Networks and Arbitrary Markets

The case of consumers and suppliers simultaneously competing over a single link can be naturally extended to the case of a network of links. This network model is based upon the work of [3]. We will model the network as a graph $G = (V, E)$ with L edges, called *links*. There are Q users, each characterized by a source-target pair of nodes $s_q, t_q \in V$, $q = 1, ..., Q$, as well as R providers, each capable of providing service over a subset L_r of links $l_1, ..., l_{L_r}$, where $r = 1, ..., R$.

Each consumer wishes to send flow from its source to its target. For this reason, we define P to be the set of paths in the graph G, and we say that a path $p \in P$ *belongs* to $q \in Q$ (denoted $p \in q$) when p connects s_q, t_q. We assume without loss of generality that each path belongs to only one q. If some path p_k is shared by q_1 and q_2, then we simply define two paths p_{q_1}, p_{q_2} in P that contain the same links.

Furthermore, we let f_{qp} denote the flow that consumer q sends over the path p, and let $f_q = \sum_{p \in q} f_{qp} = \sum_{p \in q} f_p$ denote the total flow sent by q. In the last equality, f_p is the total flow on path p, and we have $f_p = f_{pq}$ since path p only belongs to consumer q. Each consumer has a valuation function V_q and receives a value of $V_q(f_q)$ for sending a flow of f_q.

We will make the following assumption on the valuation functions:

Assumption 3 *For all q, the valuation functions $V_q(f_q) : \mathbb{R}^+ \to \mathbb{R}^+$ are strictly increasing and concave. Over $f_q > 0$, the functions are differentiable. At $f_q = 0$, the right derivative exists, and is denoted $V_q'(0)$.*

In turn, each producer r can supply bandwidth at a subset L_r of the network links, and we denote this $l \in r$ for $l \in L_r$. We also assume that the costs of providing bandwidth at one link are independent of the costs at any other link. Formally, each producer r has $|L_r|$ cost functions C_{rl}, $l = 1, ..., |L_r|$ satisfying the following assumption:

Assumption 4 *For all r, l, there exists a continuous, convex, and strictly increasing function $p_{rl}(t)$: $\mathbb{R}^+ \to \mathbb{R}^+$ such that $p_{rl}(0) = 0$, and for all $s_{rl} \geq 0$ we have:*

$$C_{rl}(s_{rl}) = \int_0^{s_{rl}} p_{rl}(t)dt$$

and for $s_{rl} \in (-\infty, 0)$ we have $C_{rl}(s_{rl}) = 0$. The cost functions $C_{rl} : \mathbb{R} \to \mathbb{R}^+$ are thus continuous, strictly convex, strictly increasing, and differentiable over their entire domain.

Let s_{rl} be the bandwidth supplied by producer r at link $l \in L_r$, and let $\mathbf{s}_r = (s_{r1}, ..., s_{r|L_r|})$ be the corresponding vector. The total cost to r is then defined to be $C_r(\mathbf{s}_r) = \sum_{l \in L_r} C_{rl}(s_{rl})$.

The social objective in this model is again to allocate flow and supply allocation so as to maximize the aggregate surplus. Formally, we want to come as close as possible to the result of the following optimization problem:

$$\begin{aligned}
\text{maximize } & \sum_{q=1}^{Q} V_q(f_q) - \sum_{r=1}^{R} C_r(\mathbf{s}_r) \\
\text{such that } & f_q = \sum_{p \in q} f_p \ \forall q \\
& \sum_{r: l \in r} s_{rl} = \sum_{p: l \in p} f_p \ \forall l \\
& 0 \leq f_q \ \forall q \\
& 0 \leq s_{rl} \leq 1
\end{aligned}$$

Definition of the Mechanism. Consumers and suppliers now submit *vectors* of bids $\mathbf{w}_q$ and $\mathbf{w}_r$ (respectively). We define $\mathbf{w}_q := (w_{q1}, ..., w_{q|L|})$ where w_{ql} is the bid to link l. The vector $\mathbf{w}_r$ is defined in the same way. The collection of all the bids is denoted $\mathbf{w} = (\mathbf{w}_1, ..., \mathbf{w}_Q, \mathbf{w}_{Q+1}, ..., \mathbf{w}_{Q+R})$.

At each link l, a consumer q's bid selects a demand function of the form

$$c_{ql}(\mu_l) = \begin{cases} \frac{w_{ql}}{\mu_l} & \text{if } w_{ql} > 0 \\ 0 & \text{if } w_{ql} = 0 \end{cases} \tag{69}$$

and producer r's bid selects a supply function of the form

$$s_{rl}(\mu_l) = \begin{cases} 1 - \frac{w_{rl}}{\mu_l} & \text{if } w_{rl} > 0 \\ 1 & \text{if } w_{ql} = 0 \end{cases} \tag{70}$$

Then at each link, the mechanism then sets a price $\mu_l(\mathbf{w}_l)$ that clears the market by setting the price to

$$\mu_l(\mathbf{w}_l) = \frac{\sum_{q:l\in q} w_{ql} + \sum_{r:l\in r} w_{rl}}{R_l}$$

where R_l is the number of producers competing at link l.

Consumer q then gets allocated a capacity of c_{ql}, and pays for it $\mu_l c_{ql}$, while producer r provides s_{rl} units of bandwidth and receives a payment of $\mu_l s_{rl}$.

Once all capacities have been allocated, consumer q solves a max-flow problem on G with edge capacities set to c_{ql}, and with the source and target being s_q and t_q. Hence we have

$$f_q(\mathbf{w}) = \text{max-flow}(G, \mathbf{c}_q(\mathbf{w}), s_q, t_q)$$

This mechanism induces a game where the payoff to consumer q is

$$\pi_q(\mathbf{w}_q; \mathbf{w}_{-q}) = V_q\left(f_q\left(\mathbf{w}_q; \mathbf{w}_{-q}\right)\right) - \sum_{l\in q} w_{ql}$$

and the payoff to producer r is

$$\pi_r(\mathbf{w}_r; \mathbf{w}_{-r}) = \sum_{l\in r} \mu_l(\mathbf{w}) - \sum_{l\in r} w_{rl} - \sum_{l\in r} C_{rl}\left(1 - \frac{w_{rl}}{\sum_{q:l\in q} w_{ql} + \sum_{r:l\in r} w_{rl}}\right)$$

Existence of a Nash equilibrium.

Definition 1. *A game is said to be* concave *if the following holds:*

1. Every joint strategy, viewed as a point in the product space of the individual strategy spaces, lies in a convex, closed, and bounded region R in the product space.
2. Each player's payoff function is continuous and concave in its own strategy.

Theorem 7. *(Rosen [11]) Every concave n-person game has an equilibrium point.*

We now apply this to show that a Nash equilibrium exists in our game.

Theorem 8. *The extended resource allocation game has a Nash Equilibrium*

Proof. We have to show that our game is concave.

First let's look at condition 1. Fix a consumer q, and a response vector $\mathbf{w}_{-q}$. By definition, $\mathbf{w}_q \geq 0$. Also observe that $\pi_q(\mathbf{w}_q; \mathbf{w}_{-q}) \leq \pi_q(\mathbf{w}_q; \mathbf{0})$ for all $\mathbf{w}_q$. Also, since the allocation to q at link l is R_l for any $w_{ql} > 0$ as long as $\mathbf{w}_{-ql} = \mathbf{0}$, there is a W_q s.t. for $\mathbf{w}_q > W$, $\pi_q(\mathbf{w}_q; \mathbf{0}) < 0$. Thus we can restrict without loss of generality the consumer bids on $[0, W]$ where $W = \max_q W_q$.

The same thing can be shown for producers. Fix a producer r. Observe that

$$\begin{aligned}\pi_r(\mathbf{w}_r; \mathbf{w}_{-r}) &= \sum_{l\in r} \mu_l - \sum_{l\in r} w_{rl} - \sum_{l\in r} C_{rl}\left(s_{rl}\right) \\ &= \sum_{l\in r} \frac{\sum_{q\neq r} w_{ql} + \sum_q w_{ql}}{R_l} + \sum_{l\in r}\left(\frac{1}{R_l} - 1\right) w_{rl} - \sum_{l\in r} C_{rl}\left(s_{rl}\right)\end{aligned}$$

Suppose $w_{rl} > R_l C_{rl}(s_{rl})$ for all l. Then

$$\begin{aligned}
\pi_r(\mathbf{w}_r; \mathbf{w}_{-r}) &= \sum_{l \in r} \frac{\sum_{q \neq r} w_{ql} + \sum_q w_{ql}}{R_l} - \sum_{r \in l} R_l C_{rl}(s_{rl}) \\
&< \sum_{l \in r} \frac{\sum_{q \neq r} w_{ql} + \sum_q w_{ql}}{R_l} - \sum_{r \in l} C_{rl}(s_{rl}) \\
&= \pi_r(\mathbf{0}; \mathbf{w}_{-r})
\end{aligned}$$

and the producer could profitably deviate to $\mathbf{0}$. It's not hard to apply this reasoning separately to each link, and deduce that we must have $w_{rl} \leq R_l C_{rl}(s_{rl})\ \forall l$. This shows that we can assume without loss of generality that the payoffs are chosen within $[0, M]$ where $M = \max\left(\max_{r,l} RC_{rl}(1), W\right)$.

Next, we have to show that the payoffs are continuous and concave on our compact set. Observe that π_q and π_r are discontinuous when the response vectors $\mathbf{w}_{-q}, \mathbf{w}_{-r}$ equal $\mathbf{0}$. For this reason, pick two producers r_1 and r_2 arbitrarily and restrict their strategy space to $[\epsilon, M]$ while keeping the other strategy spaces at $[0, M]$. We will first show that the game resulting from this restriction is concave, and then we will explain why that restriction can be made without loss of generality.

Given the restriction, the payoffs π_q and π_r are continuous in w_q, w_r since the response vector is always non-zero. It is also straightforward to see that π_r is continuous and concave in w_r. To see that π_q is continuous, observe that every function in the composition $V_q(f_q(c_q(w_q)))$, particularly the max-flow function, is continuous in w_q. Also, since c_q is continuous and f_q is non-decreasing, we have that f_q is concave. Then it follows straightforwardly that V_q is concave in w_q.

By Rosen's theorem, we can then conclude that a Nash Equilibrium exists.

The Welfare Bound.

Theorem 9. *In the extended resource allocation game, we have*

$$\sum_{q=1}^{Q} V_q(f_q^{NE}) - \sum_{r=1}^{R} C_r(\mathbf{s}_r^{NE}) \geq \inf_S \frac{s^2(S^2 + 4Ss + 2s^2)}{S(S+2s)} \left(\sum_{q=1}^{Q} V_q(f_q^{NE}) - \sum_{r=1}^{R} C_r(\mathbf{s}_r^{NE}) \right)$$

where s is the unique positive root of the polynomial

$$\gamma(s) = 16s^4 + 10S^2 s(3s - 2) + S^3(5s - 4) + Ss^2(49s - 24)$$

This value can be numerically evaluated to approximately 0.588727.

Furthermore, this bound is tight.

The idea is to first characterize the Nash equilibrium in terms of capacity allocations to consumers and supply allocations to producers. Then we can linearize the utilities like in a single-link case, and essentially set for each consumer a separate valuation function for each link. Doing so will reduce the extended game to $|L|$ single-link games, to which we will apply our existing bound.

First, we need to show that given a response strategy $\mathbf{w}_{-q}$, for every capacity c_{ql} there exists a bid w_{ql} that ensures q gets that capacity.

Lemma 1. *The joint strategy $\mathbf{w}$ is a Nash equilibrium if and only if the following holds:*

1. At every link l, there are at least two suppliers r_1, r_2 that have positive bids w_{r_1}, w_{r_2}.

2. For each q,

$$\mathbf{c}_q(\mathbf{w}) \in \arg\max_{\bar{\mathbf{c}}} \left(V_q(f_q(\bar{\mathbf{c}}_q)) - \sum_{l \in q} w_{ql}^{-1}(c_{ql}, \mathbf{w}_{-q}) \right)$$

where $\mathbf{c}_q$ is defined according to (69) and $w^{-1}(c_{ql}, \mathbf{w}_{-1})$ is a function that satisfies

$$c_{ql}(y, \mathbf{w}_{-q}) = x \text{ if and only if } w_{ql}^{-1}(x, \mathbf{w}_{-q}) = y$$

3. For each r,

$$\mathbf{w}_r \in \arg\max_{\bar{\mathbf{w}}} \left(\sum_{l\in r} \mu_l(\mathbf{w}) - \sum_{l\in r} \bar{w}_{rl} - \sum_{l\in r} C_{rl}\left(1 - \frac{\bar{w}_{rl}}{\sum_{q:l\in q} w_{ql} + \sum_{r:l\in r} w_{rl}}\right)\right)$$

Proof. Suppose $\mathbf{w}$ is a Nash equilibrium. The first claim follows as before. To establish the existence of w^{-1}, we observe that since $\mathbf{w}_{-q} \neq \mathbf{0}$, c_{ql} (consumer q's capacity at link l) is a continuous function of w_{ql} (consumer q's bid at link l). Furthermore, $c_{ql}(x, \mathbf{w}_{-ql})$ is strictly increasing in x, $c_{ql}(0, \mathbf{w}_{-ql}) = 0$, and $\lim_{x\to\infty} c_{ql}(x, \mathbf{w}_{-ql}) = \infty$. It then follows that w^{-1} exists.

If $\mathbf{w}_q$ is part of a Nash equilibrium, then by definition we must have

$$\mathbf{w}_q \in \arg\max_{\bar{\mathbf{w}}} \pi_q(\bar{\mathbf{w}}, \mathbf{w}_{-q}) = \arg\max_{\bar{\mathbf{w}}} \left(V_q\left(f_q\left(\mathbf{c}_q\left(\bar{\mathbf{w}}; \mathbf{w}_{-q}\right)\right)\right) - \sum_{l\in q} \bar{w}_{ql}\right)$$

The second claim follows from this fact. Suppose that it does not hold. Then, given $\mathbf{w}_{-q}$, there is a $\bar{\mathbf{c}}$ that results in a higher payoff. But then using the functions w_{ql}^{-1} we can find a vector of bids that will result in a higher value for π_q, which is a contradiction.

Finally the third claim holds by definition of a Nash equilibrium.

The other side of the implication can be established by reversing the argument above.

Since V_q is composed with the max-flow function, which may not be differentiable, we need to use the following tools from convex analysis.

Definition 2. *We say that γ is a* subgradient *of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ at x_0 if for all $x \in \mathbb{R}^n$ we have*

$$f(x) \geq f(x_0) + \gamma^\top (x - x_0)$$

The set of all subgradients of f at x_0 is called the subdifferential *and is denoted $\partial(f(x_0))$. The supergradient and the superdifferential are the equivalents of a subgradient and a subdifferential for a concave function.*

Lemma 2. *Let $\mathbf{w}$ be a Nash equilibrium. Then for every consumer q there exists a vector α_q such that the linearized valuation function*

$$\bar{V}_q(\bar{\mathbf{c}}) := \alpha_q^\top \left(\bar{\mathbf{c}} - \mathbf{c}_q(\mathbf{w})\right) + V_q(\mathbf{c}_q(\mathbf{w}))$$

satisfies the following relationship:

$$\mathbf{c}_q(\mathbf{w}) \in \arg\max_{\bar{\mathbf{c}}} \left(\bar{V}_q(\bar{\mathbf{c}}) - \sum_{l\in q} w_{ql}^{-1}(\bar{c}_{ql}, \mathbf{w}_{-ql})\right) \tag{71}$$

In other words, $\mathbf{w}_q$ is also a Nash equilibrium for the game where V_q has been replaced by $\bar{V}_q$. Moreover, the payoffs at $\mathbf{w}$ in the new game remain the same.

Proof. We can assume without loss of generality that $V_q(\mathbf{c}_q)$ is a convex function of $\mathbf{c}_q \in \mathbb{R}^{|L|}$ by extending it appropriately to the entire domain. Also observe that $c_{ql} = w_{ql}/\mu(\mathbf{w}_l)$ is concave in w_{ql}, which implies that $w_{ql}^{-1}(c_{ql}, \mathbf{w}_{-ql})$ is convex.

Recall that the payoff to q as a function of its allocated capacities is

$$\pi_q(\bar{\mathbf{c}}_q, \mathbf{w}_{-q}) = V_q(f_q(\bar{\mathbf{c}}_q)) - \sum_{l\in q} w_{ql}^{-1}(\bar{c}_{ql}, \mathbf{w}_{-q})$$

and it is concave. Then by the properties of subdifferentials we have

$$\partial(-\pi_q(\bar{\mathbf{c}}_q, \mathbf{w}_{-q})) = \partial(-V_q(\mathbf{c}_q)) + \sum_{l} \partial(w_{ql}^{-1}(c_{ql}, \mathbf{w}_{-ql}))$$

The addition here is defined over sets:

$$A + B = \{a + b | a \in A, b \in B\}$$

From condition 2 in Lemma 1, it follows that $\mathbf{0} \in \partial(-\pi_q(\bar{\mathbf{c}}_q, \mathbf{w}_{-q}))$, and from the property above we can deduce that there exist vectors $\alpha_q \in \partial(-V_q(\mathbf{c}_q))$ and $\beta_{ql} \in \partial(w_{ql}^{-1}(c_{ql}, \mathbf{w}_{-ql}))$ such that $\alpha_q = \sum_l \beta_{ql}$.

To show that (72) holds, it is enough to show that $\mathbf{0}$ is a supergradient of

$$\bar{\pi}_q(\bar{\mathbf{c}}_q, \mathbf{w}_{-q}) = \bar{V}_q(\bar{\mathbf{c}}_q) - \sum_{l \in q} w_{ql}^{-1}(\bar{c}_{ql}, \mathbf{w}_{-ql})$$

As before we have

$$\partial(-\bar{\pi}_q(\bar{\mathbf{c}}_q, \mathbf{w}_{-q})) = \partial(-\bar{V}_q(\bar{\mathbf{c}}_q)) + \sum_l \partial(w_{ql}^{-1}(\bar{c}_{ql}, \mathbf{w}_{-ql}))$$

Since we still have $\alpha_q \in \partial(-\bar{V}_q(\mathbf{c}_q))$ and $\beta_{ql} \in \partial(w_{ql}^{-1}(c_{ql}, \mathbf{w}_{-ql}))$, it must be that $\mathbf{0} \in \partial(-\bar{\pi}_q(\bar{\mathbf{c}}_q, \mathbf{w}_{-q}))$.

Now we can prove the main result

Proof. (of Theorem 9) We will first show that when each consumer has a linearized valuation function $\bar{V}_q(\bar{\mathbf{c}})$, the extended game reduces to $|L|$ single link games, to which the bound can be independently applied. Then we will prove that the price of anarchy of the game with linearized valuation functions is a lower bound for the price of anarchy with regular valuation functions, and thus establish our claim.

By Lemma 2, we know that if $\mathbf{w}$ is a Nash equilibrium, then

$$\mathbf{c}_q(\mathbf{w}) \in \arg\max_{\bar{\mathbf{c}}} \left(\bar{V}_q(\bar{\mathbf{c}}) - \sum_{l \in q} w_{ql}^{-1}(\bar{c}_{ql}, \mathbf{w}_{-ql}) \right)$$

But this is equivalent to

$$\mathbf{c}_q(\mathbf{w}) \in \arg\max_{\bar{\mathbf{c}}} \left(\sum_{l \in q} \alpha_{ql} \left(\bar{c}_{ql} - c_{ql}(\mathbf{w})\right) + V_q(c_{ql}(\mathbf{w}) - \sum_{l \in q} w_{ql}^{-1}(\bar{c}_{ql}, \mathbf{w}_{-ql}) \right)$$

and therefore

$$c_{ql} \in \arg\max_{\bar{c}_{ql}} \left(\alpha_{ql} \left(\bar{c}_{ql} - c_{ql}(\mathbf{w})\right) + V_q(c_{ql}(\mathbf{w})/|L_q| - w_{ql}^{-1}(\bar{c}_{ql}, \mathbf{w}_{-ql}) \right)$$

For all the suppliers serving link l, we have from part 3 of Lemma 1

$$w_{rl} \in \arg\max_{\bar{w}} \left(\mu_l(\mathbf{w}) - w_{rl} - C_{rl} \left(1 - \frac{w_{rl}}{\sum_{q:l \in q} w_{ql} + \sum_{r:l \in r} w_{rl}} \right) \right)$$

Thus the vector of bids $\mathbf{w}_l$ is a Nash equilibrium of a simple single-link game and

$$\begin{aligned} &\sum_{q:l \in q}^{Q} \left(\alpha_{ql} \left(c_{ql}^{\mathrm{NE}} - c_{ql}(\mathbf{w}) \right) + V_q(c_{ql}(\mathbf{w})/|L_q| \right) - \sum_{r:l \in r}^{R} C_{rl}(s_{rl}(w_{rl}^{\mathrm{NE}}; \mathbf{w}_{-rl}^{\mathrm{NE}})) \\ &\qquad \geq \varphi \left(\sum_{q:l \in q}^{Q} \left(\alpha_{ql} \left(c_{ql}^{\mathrm{NE}} - c_{ql}(\mathbf{w}) \right) + V_q(c_{ql}(\mathbf{w})/|L_q| \right) - \sum_{r:l \in r}^{R} C_{rl}(s_{rl}(w_{rl}^{\mathrm{OPT}}; \mathbf{w}_{-rl}^{\mathrm{OPT}})) \right) \end{aligned}$$

where φ is the bound defined in the theorem.

Now we can use this to bound the price of anarchy of the extended game. We have

$$
\begin{aligned}
&\frac{\sum_{q=1}^{Q} V_q(f_q(\mathbf{c}_q^{\mathrm{NE}})) - \sum_{r=1}^{R} C_r(\mathbf{s}_r^{\mathrm{NE}})}{\sum_{q=1}^{Q} V_q(f_q(\mathbf{c}_q^{\mathrm{OPT}}) - \sum_{r=1}^{R} C_r(\mathbf{s}_r^{\mathrm{OPT}})} \geq \frac{\sum_{q=1}^{Q} \bar{V}_q(\mathbf{c}_q^{\mathrm{NE}}) - \sum_{r=1}^{R} C_r(\mathbf{s}_r^{\mathrm{NE}})}{\sum_{q=1}^{Q} \bar{V}_q(\mathbf{c}_q^{\mathrm{OPT}}) - \sum_{r=1}^{R} C_r(\mathbf{s}_r^{\mathrm{OPT}})} \\
&\quad = \frac{\sum_{q=1}^{Q} \sum_{l\in q} \left(\alpha_{ql}\left(c_{ql}^{\mathrm{NE}} - c_{ql}(\mathbf{w})\right) + V_q(c_{ql}(\mathbf{w})/|L_q|\right) - \sum_{r=1}^{R} \sum_{l\in r} C_{rl}(s_{rl}(w_{rl}^{\mathrm{NE}}; \mathbf{w}_{-rl}^{\mathrm{NE}}))}{\sum_{q=1}^{Q} \sum_{l\in q} \left(\alpha_{ql}\left(c_{ql}^{\mathrm{NE}} - c_{ql}(\mathbf{w})\right) + V_q(c_{ql}(\mathbf{w})/|L_q|\right) - \sum_{r=1}^{R} \sum_{l\in r} C_{rl}(s_{rl}(w_{rl}^{\mathrm{OPT}}; \mathbf{w}_{-rl}^{\mathrm{OPT}}))} \\
&\quad = \frac{\sum_{l=1}^{L} \left[\sum_{q:l\in q} \left(\alpha_{ql}\left(c_{ql}^{\mathrm{NE}} - c_{ql}(\mathbf{w})\right) + V_q(c_{ql}(\mathbf{w})/|L_q|\right) - \sum_{r:l\in r} C_{rl}(s_{rl}(w_{rl}^{\mathrm{NE}}; \mathbf{w}_{-rl}^{\mathrm{NE}}))\right]}{\sum_{l=1}^{L} \left[\sum_{q:l\in q} \left(\alpha_{ql}\left(c_{ql}^{\mathrm{OPT}} - c_{ql}(\mathbf{w})\right) + V_q(c_{ql}(\mathbf{w})/|L_q|\right) - \sum_{r:l\in r} C_{rl}(s_{rl}(w_{rl}^{\mathrm{OPT}}; \mathbf{w}_{-rl}^{\mathrm{OPT}}))\right]} \\
&\quad \geq \varphi
\end{aligned}
$$

In the first inequality we used the fact that $\bar{V}_q(\mathbf{w}^{\mathrm{NE}}) = V_q(\mathbf{w}^{\mathrm{NE}})$ and $\bar{V}_q(\mathbf{w}^{\mathrm{OPT}}) \geq V_q(\mathbf{w}^{\mathrm{OPT}})$. In the following equality the utilities and cost were expanded according to their definitions, and on the third line we inversed the order of summation. Finally, in the last inequality we used our existing bound $|L|$ times.

8.8 Appendix H: An Optimal Smooth Market-Clearing Mechanisms

Here we modify the arguments of [3] and [8] to show the proportional allocation mechanism is optimal within a class of mechanisms. First, we will present a formal definition of the general type of mechanisms that we consider. We will again call the vector of bids $\mathbf{w}$.

Definition 3. *A* smooth two-sided market-clearing mechanism *is a tuple of functions* (D, S)*,* $D : (0, \infty) \times [0, \infty) \to \mathbb{R}$*,* $S : (0, \infty) \times [0, \infty) \to \mathbb{R}$ *such that for all* Q, R*, and for all* $w \in \mathbb{R}^{Q+R}, w \neq 0, w \geq 0$*, there exists a unique* $p > 0$ *that satisfies the following equation*

$$\sum_{q=1}^{Q} D_q(p, w_q) = \sum_{r=1}^{R} S_r(p, w_r)$$

We denote it as $p(w)$*. Each consumer receives an allocation of* $D(p, w_q)$ *and pays* $pD(p, w_q)$ *to the mechanism. Each supplier produces* $S(p, w_q)$ *and receives in return a payment* $pS(p, w_q)$*. The payoff to consumer* q *with valuation function* V_q *is*

$$\pi_q(w_q) = V_q(D(p, w_q)) - pD(p, w_q)$$

while the payoff to supplier r *with cost function* C_r *is*

$$\pi_r(w_r) = pS(p, w_r) - C_r(S(p, w_r))$$

Finally, if $w = 0$*, the payoff to any consumer and any supplier is* $-\infty$*.*

We will restrict valuation functions to the following class.

Definition 4. *The set* $\mathcal{U}$ *contains all valuation functions* $U(d) : \mathbb{R}^+ \to \mathbb{R}^+$ *that are strictly increasing and concave over* $d \geq 0$*. Over* $d > 0$*, the functions are differentiable, and at* $d = 0$*, the right derivative exists, and is denoted* $U(0)$*.*

Similarly, we restrict our attention to the following class of cost functions

Definition 5. *The set* $\mathcal{C}$ *contains all cost functions* $C(s) : \mathbb{R} \to \mathbb{R}^+$ *of the form*

$$C(s) = \begin{cases} \int_0^s p(t)dt & \text{if } s \geq 0 \\ 0 & \text{if } s \leq 0 \end{cases}$$

where $p(t) : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous and strictly increasing function, and* $p(0) \geq 0$*. The class* $\mathcal{C}_{conv} \subseteq \mathcal{C}$ *contains all cost functions such that* $p(t)$ *is also convex and* $p(0) = 0$*.*

Observe that the class $\mathcal{C}$ is more general than what we had before since we no longer require that $p()$ be convex and that $p(0) = 0$.

The next definition defines the class of mechanisms among which we will derive the optimality of our mechanism.

Definition 6. *The class* $\mathcal{M}$ *contains all smooth two-sided market-clearing mechanisms* $(D, S) \in \mathcal{M}$ *that satisfy the following criteria:*

(i) For all $U \in \mathcal{U}$, the payoff to a consumer with valuation function U is concave when the consumer is price taking. That is, for all $p > 0, U \in \mathcal{U}$, the function

$$\pi(w) = U(D(p, w)) - pD(p, w)$$

is concave for $w \geq 0$. Similarly, for all $C \in \mathcal{C}$, the payoff to a producer with cost function C is concave when the consumer is price taking. That is, for all $p > 0, C \in \mathcal{C}$, the function

$$\pi(w) = pS(p, w) - C(S(p, w))$$

is concave for $w \geq 0$.

(ii) For all $Q, R \geq 1$, and for any Q consumers with valuation functions $V_1, ..., V_Q \in \mathcal{U}$, and for any R suppliers with cost functions $C_1, ..., C_r \in \mathcal{C}$, each consumer's and each supplier's payoff when they are price anticipating is concave. That is, for each $i = 1, ..., Q$ and each response vector w_{-i}, the function

$$\pi_q(w_q, w_{-i}) = V_q(D(p(w), w_q)) - p(w)D(p(w), w_q)$$

is concave. (By w we denote the joint strategy vector (w_q, w_{-i}).) Also, for each $j = 1, ..., R$ and each response vector w_{-i}, the function

$$\pi_r(w_r, w_{-j}) = p(w)S(p(w), w_r) - C_r(S(p(w), w_r))$$

is concave.

(iii) The demand function D is non-negative: for all $p > 0$, and all $w \geq 0$, $D(p, w) \geq 0$. The supply functions are bounded from above: there exists a K s.t. for all $p > 0$ and all $w \geq 0$, $S(p, w) \leq K$.

(iv) The competitive equilibrium allocations are optimal.

(v) The demand and the supply functions are symmetric in w in the sense that for all $p > 0$, $w \geq 0$ we have

$$\frac{\partial D}{\partial w}(p, w) = -\frac{\partial S}{\partial w}(p, w)$$

The first four criteria are a straightforward generalization of the classes of mechanisms studied in the fixed demand and fixed supply cases. The fifth criterion is a symmetry condition. Roughly, it states that both sides of the market should use the same kind of demand and supply functions. Optimal demand and supply functions for one-sided markets are indeed symmetric, and it is reasonable to give both market sides the same power to express their preferences; hence this condition is not unnatural. However, we leave open the question of whether better efficiency guarantees exist for assymetric markets.

Finally, we define the optimal allocation to be the optimal solution to the following system:

$$\begin{aligned} \text{maximize } & \sum_{q=1}^{Q} V_q(d_q) - \sum_{r=1}^{R} C_r(s_r) \\ \text{such that } & \sum_{q=1}^{Q} d_q = \sum_{r=1}^{R} s_r \\ & 0 \leq s_r \leq 1 \\ & 0 \leq d_q \end{aligned} \tag{72}$$

Identifying the form that mechanisms in $\mathcal{M}$ can have. In this part, we will not use the symmetry assumption on $\mathcal{M}$. First, we start by establishing that the demand and supply functions must be affine.

Lemma 3. *Let $(D, S) \in \mathcal{M}$. Then there exist functions $a(p), b(p) : (0, \infty) \to \mathbb{R}^+$ such that for all $p > 0$ and all $w \geq 0$, $D(p, w) = a(p) + b(p)w$.*

Proof. We have to show that for a fixed $p > 0$, $D(p, w)$ is both a concave and a convex function of w. We will deduce this from criterion (i).

Let $U(d) = \alpha d$ for some $\alpha > 0$. Note that $U \in \mathcal{U}$. Then $\pi = (\alpha - p)D(p, w)$ must be concave in w by criterion (i). Thus if $\alpha > p$, $D(p, w)$ must be concave and if $\alpha < p$, $D(p, w)$ must be convex, so the function is actually affine. To show that $a(p), b(p) \geq 0$ for all p, recall that $D(p, w) \geq 0$ for all w, p by Def. 4.3, and consider the cases where $w = 0$ (to show $a(p) \geq 0$) and the limit as $w \to \infty$ (to show $b(p) \geq 0$).

Observe that by the symmetry assumption (v), we have $S(p, w) = c(p) - b(p)w$ for some $c(p) : (0, \infty) \to \mathbb{R}$. However, that assumption is not necessary to esablish that $S(p, w)$ is affine in w. The following lemma, whose detailed proof omit, formally states that.

Lemma 4. *Let $(D, S) \in \mathcal{M}$. Then there exist functions $c(p) : (0, \infty) \to \mathbb{R}$ and $d(p) : (0, \infty) \to \mathbb{R}^+$ such that for all $p > 0$ and all $w \geq 0$, $S(p, w) = c(p) - d(p)w$.*

Proof. (Sketch) This is again deduced from Definition 4. We consider a cost function $C(s) = \beta s^d$ where $d > 1$. We compute $d^2\pi/dw^2$ and show that unless $\partial^2 S/\partial w^2 = 0$ for all w, p, there is a choice of w, p, β, d such that $d^2\pi/dw^2 > 0$ at w, p. In one case, we have to take d sufficiently close to 1.

The next lemma uses criterion (iv) to further narrow down the possibilities for $D(p, w)$.

Lemma 5. *Let $(D, S) \in \mathcal{M}$, with $D(w, p) = a(p) + b(p)w$. Then for all p we have $a(p) = 0$.*

Proof. We will show that when $a(p) \neq 0$, we can construct a competitive equilibrium at which some bids must necessarily be negative.

Suppose $\exists p$ such that $a(p) \neq 0$. Let there be Q suppliers and R producers. Choose the Q allocations such that for all $i = 1, ..., Q$ we have $0 < d_q < a(p)$, and choose the R supply allocations so that supply equals demand.

Then choose the utilities and costs such that this allocation is optimal. The valuation functions should be strictly concave with $U_q'(d_q) = p$, and the cost functions should be chosen so that $C_r'(s_r) = p$ (recall that they are strictly convex by assumption).

Observe that these allocations are an optimal solution to (72) (in particular, they satisfy the optimality conditions given in a previous document with Lagrangian variable p). Because of the strict concavity and convexity assumptions we made, this solution is unique.

Now let w be a competitive equilibrium with the utilities and cost functions we just defined. Let μ be the price at that equilibrium. By (i), it exists for every p, and by #4 it must be optimal. Since the solution to (72) is unique, we must have $D(\mu, w) = d_q$. By our assumptions on the convexity, and the KKT theorems, the dual variables must also be unique and so $\mu = p$.

But then observe that for consumer q,

$$D(\mu, w_q) = a(\mu) + b(\mu)w_q = a(p) + b(p)w_q = d_q < a(p)$$

where the inequality follows from our initial definition of d_q. This can only hold when $w_q < 0$, but this is prohibited by assumption, and so we have the contradiction we wanted to obtain.

The next lemma shows we can similarly fix the value of $c(p)$ in the definition of $S(p, w)$ to a scalar.

Lemma 6. *Let $(D, S) \in \mathcal{M}$, with $S(w, p) = c(p) + d(p)w$. Then for all p we have $c(p) = K^* := \sup_{p,w} S(p, w)$.*

Proof. The proof in the same spirit as that of the previous lemma. We will show that if $c(p) < K^*$, then we can define a competitive equilibrium where some of the bids must be negative.

Suppose there is a p such that $c(p) < K^*$. Let $Q, R > 1$, let s_r be such that $c(p) < s_r < K^*$ and $s_r > 0$, and let d_q be such that demand matches supply. Let the V_q be strictly concave with $V_q'(d_q) = p$ and let C_r be such that $C_r'(s_r) = p$. Like before, this is an optimal solution to (72).

Then let w be a competitive equilibrium with the same utilities and costs. Like in the last proof, $S(\mu, w_r) = s_r$ and μ, the price at the competitive equilibrium, must equal p.

But then for consumer r we have

$$S(\mu, w_r) = c(\mu) - d(\mu)w_r = c(p) - d(p)w_r = s_r > c(p)$$

and again this can only hold if $w_r < 0$.

This concludes the results of this part. We have shown that all the mechanisms in $\mathcal{M}$ must have the form

$$D(p, w) = b(p)w$$
$$S(p, w) = K^* - d(p)w$$

and if we use our symmetry assumption, then $b(p) = d(p)$. Finally, observe that if we divide $S(p,w)$ by K^*, the resulting mechanism is still in $\mathcal{M}$, so from now on we will assume without loss of generality that $S(p,w)$ is of the form

$$S(p,w) = 1 - d(p)w$$

An Optimality Result. In this part we will adapt the optimality argument for Kelly's mechanism to our case. The main result is this theorem:

Theorem 10. *Out of all smooth two-sided market-clearing mechanisms $(D,S) \in \mathcal{M}$, the proportional allocation mechanism achieves the best possible welfare ratio in the game with $Q > 1$ consumers with utilities $V_1, ..., V_Q \in \mathcal{U}$ and $R > 1$ suppliers with costs $C_1, ..., C_r \in \mathcal{C}_{conv}$.*

The first step is to express $b(p(w)), d(p(w))$ as a function of w. When supply matches demand, we have

$$\begin{aligned}
\sum_{q=1}^{Q} D_q(p, w_q) &= \sum_{r=1}^{R} S_r(p, w_r) \\
b(p(w)) \sum_{q=1}^{Q} w_q &= R - d(p(w)) \sum_{r=1}^{R} w_r \\
b(p(w)) \sum_{q=1}^{Q} w_q &= R - b(p(w)) \sum_{r=1}^{R} w_r \\
b(p(w)) = d(p(w)) &= \frac{R}{\sum_{q=1}^{Q} w_q + \sum_{r=1}^{R} w_r}
\end{aligned}$$

On the third line we used our symmetry assumption. From that it follows that

$$\begin{aligned}
D(p(w), w_q) &= \frac{w_q}{\sum_{q=1}^{Q} w_q + \sum_{r=1}^{R} w_r} R \\
S(p(w), w_r) &= 1 - \frac{w_r}{\sum_{q=1}^{Q} w_q + \sum_{r=1}^{R} w_r} R
\end{aligned}$$

We will also use the following function:

$$1/b(p) := B(p) = \frac{\sum_{q=1}^{Q} w_q + \sum_{r=1}^{R} w_r}{R}$$

Since p is assumed to be uniquely defined for all $\boldsymbol{w}$, there must exist an inverse function $\phi : (0,\infty) \to (0,\infty)$ such that $B(x) = y$ if and only if $\phi(y) = x$. Thus

$$p(w) = \phi \left(\frac{\sum_{q=1}^{Q} w_q + \sum_{r=1}^{R} w_r}{R} \right) \tag{73}$$

The following lemma establishes that ϕ must be convex

Lemma 7. *The function $\phi : (0,\infty) \to (0,\infty)$ is surjective, convex and differentiable.*

Proof. Surjectivity holds because $p(w)$ can clearly take any value $p > 0$ (by the definition (73) and the fact that the bids can take any value).

Differentiability of ϕ follows from the differentiability of b, which in turns follows from the differentiability of D and S.

Convexity is proved using #2 in Definition 4. First observe that since

$$\pi_q(w_q, w_{-i}) = V_q(D(p(w), w_q)) - p(w)D(p(w), w_q)$$

must be concave, $p(w)D(p(w), w_q)$ must be convex, since otherwise we can pick $V_q = \alpha_q d_q$ for a very small α_q and obtain a contradiction.

But

$$p(w)D(p(w), w_q) = \phi\left(\frac{\sum_{q=1}^{Q} w_q + \sum_{r=1}^{R} w_r}{R}\right)\frac{w_q}{\sum_{q=1}^{Q} w_q + \sum_{r=1}^{R} w_r}R$$

and when $w_{-i} = 0$, this becomes

$$p(w)D(p(w), w_q) = \phi\left(\frac{w_q}{R}\right)R$$

implying that ϕ is indeed convex.

Our next step is to establish necessary and sufficient conditions for a Nash equilibrium, given a mechanism $(D, S) \in \mathcal{M}$.

Lemma 8. *Let $(D, S) \in \mathcal{M}$, $R, Q > 1$ and $V_1, ..., V_Q \in \mathcal{U}$, $C_1, ..., C_r \in \mathcal{C}$. A vector w is a Nash equilibrium if and only if at least two components of w are non-zero, and there exist a non-zero vectors $d, s \geq 0$ and a scalar $\mu > 0$ such that $w_q = \mu d_q$, $w_r = \mu s_r$, $\sum_q d_q = \sum_r s_r$ and the following conditions hold for consumers:*

$$\begin{aligned} U_q'(d_q)\left(1 - \tfrac{d_q}{R}\right) = \phi(\mu)\left(1 - \tfrac{d_q}{R}\right) + \phi'(\mu)\mu\tfrac{d_q}{R} \quad &\textit{if } d_q > 0 \\ U_q'(0) \leq \phi(\mu) \quad &\textit{if } d_q = 0 \end{aligned}$$

and the following conditions hold for suppliers:

$$\begin{aligned} C_r'(s_r)\left(1 + \tfrac{s_r}{R-1}\right) \leq \phi(\mu)\left(1 + \tfrac{s_r}{R-1}\right) - \phi'(\mu)\mu\tfrac{s_r}{R-1} \quad &\textit{if } 0 < s_r \leq 1 \\ C_r'(s_r)\left(1 + \tfrac{s_r}{R-1}\right) \geq \phi(\mu)\left(1 + \tfrac{s_r}{R-1}\right) - \phi'(\mu)\mu\tfrac{s_r}{R-1} \quad &\textit{if } 0 \leq s_r < 1 \end{aligned}$$

Proof. Suppose w is a Nash equilibrium. Since the payoffs are all $-\infty$ when $w = 0$, we must have $w \neq 0$. If less than two components of w are non-zero, then, as before, we may show that there is no Nash equilibrium.

So suppose $w \neq 0$ and at least two componnents are positive. Then all the payoffs are differentiable in the user's bid, and so the following conditions are necessary and sufficient conditions for a Nash equilibrium. For a consumer:

$$\begin{aligned} \frac{\partial \pi_q}{\partial w_q} = V_q^{'}&\left(\frac{w_q}{\sum_q w_q + \sum_r w_r}R\right)\left(\frac{R}{\sum_q w_q + \sum_r w_r} - \frac{w_q R}{(\sum_q w_q + \sum_r w_r)^2}\right) \\ &- \phi\left(\frac{\sum_q w_q + \sum_r w_r}{R}\right)\left(\frac{R}{\sum_q w_q + \sum_r w_r} - \frac{w_q R}{(\sum_q w_q + \sum_r w_r)^2}\right) \\ &- \phi'\left(\frac{\sum_q w_q + \sum_r w_r}{R}\right)\left(\frac{w_q}{\sum_q w_q + \sum_r w_r}\right) = 0 \text{ if } w_q > 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq 0 \text{ if } w_q = 0 \end{aligned}$$

For a supplier:

$$\begin{aligned}\frac{\partial \pi_r}{\partial w_r} = \phi'\left(\frac{\sum_q w_q + \sum_r w_r}{R}\right)&\left(\frac{w_r}{\sum_q w_q + \sum_r w_r}\right)\\ +\phi\left(\frac{\sum_q w_q + \sum_r w_r}{R}\right)&\left(-\frac{R}{\sum_q w_q + \sum_r w_r} + \frac{w_r R}{(\sum_q w_q + \sum_r w_r)^2}\right)\\ -C_r'\left(\frac{w_r}{\sum_q w_q + \sum_r w_r}R\right)&\left(-\frac{R}{\sum_q w_q + \sum_r w_r} + \frac{w_r R}{(\sum_q w_q + \sum_r w_r)^2}\right)\\ &\geq 0 \text{ if } 0 < w_r \leq \frac{\sum_{k\neq j} w_k}{R-1}\\ &\qquad\qquad \leq 0 \text{ if } 0 \leq w_r < \frac{\sum_{k\neq j} w_k}{R-1}\end{aligned}$$

After substituting $\mu = (\sum_q w_q + \sum_r w_r)/R$, $d_q = w_q/\mu$, $s_r = w_r/\mu$ we get the conditions in the theorem.

For the other side of the implication, we simply reverse the argument above.

Now we can start proving the main theorem applying similar arguments to before. Let $(D, S) \in \mathcal{M}$ be fixed. This also fixes a function ϕ. As before, the worst case welfare ratio corresponds to

$$\inf \frac{\sum_{q=1}^{Q} V_q(d_q^{\mathrm{NE}}) - \sum_{r=1}^{R} C_r(s_r^{\mathrm{NE}})}{\sum_{q=1}^{Q} V_q(d_q^{\mathrm{OPT}}) - \sum_{r=1}^{R} C_r(s_r^{\mathrm{OPT}})}$$

subject to the conditions for a Nash equilibrium. Since we assume $V_q \in \mathcal{U}$ and $C_r \in \mathcal{C}_{conv}$ we can linearize the utilities and the marginal cost functions and instead consider the following optimization problem:

$$\begin{aligned}
\min\ & \frac{\sum_{q=1}^{Q} \alpha_q d_q^{\mathrm{NE}} - \frac{1}{2}\sum_{r=1}^{R} \beta_r (s_r^{\mathrm{NE}})^2}{\alpha_1 \sum_{r=1}^{R} s_r^{\mathrm{OPT}} - \frac{1}{2}\sum_{r=1}^{R} \beta_r (s_r^{\mathrm{OPT}})^2} && (74)\\
\text{s.t. } & \alpha_q\left(1 - \frac{d_q^{\mathrm{NE}}}{R}\right) \geq \phi(\mu)\left(1 - \frac{d_q^{\mathrm{NE}}}{R}\right) + \phi'(\mu)\mu\frac{d_q^{\mathrm{NE}}}{R}\ \forall q \text{ s.t. } d_q^{\mathrm{NE}} > 0 && (75)\\
& \alpha_q\left(1 - \frac{d_q^{\mathrm{NE}}}{R}\right) \leq \phi(\mu)\ \forall q && (76)\\
& \beta_r s_r^{\mathrm{NE}}\left(1 + \frac{s_r^{\mathrm{NE}}}{R-1}\right) \leq \phi(\mu)\left(1 + \frac{s_r^{\mathrm{NE}}}{R-1}\right) - \phi'(\mu)\mu\frac{s_r^{\mathrm{NE}}}{R-1}\ \forall r \text{ s.t. } 0 < s_r^{\mathrm{NE}} \leq 1 && (77)\\
& \beta_r s_r^{\mathrm{NE}}\left(1 + \frac{s_r^{\mathrm{NE}}}{R-1}\right) \geq \phi(\mu)\left(1 + \frac{s_r^{\mathrm{NE}}}{R-1}\right) - \phi'(\mu)\mu\frac{s_r^{\mathrm{NE}}}{R-1}\ \forall r \text{ s.t. } 0 \leq s_r^{\mathrm{NE}} < 1 && (78)\\
& \sum_{q=1}^{Q} d_q^{\mathrm{NE}} = \sum_{r=1}^{R} s_r^{\mathrm{NE}} && (79)\\
& \beta_r s_r^{\mathrm{OPT}} \leq \alpha_1\ \forall r \text{ s.t. } 0 < s_r^{\mathrm{OPT}} \leq 1 && (80)\\
& \beta_r s_r^{\mathrm{OPT}} \geq \alpha_1\ \forall r \text{ s.t. } 0 \leq s_r^{\mathrm{OPT}} < 1 && (81)\\
& d_q^{\mathrm{NE}} \geq 0\ \forall q\\
& 0 \leq s_r^{\mathrm{NE}}, s_r^{\mathrm{OPT}}, \alpha_q \leq 1\ \forall q, j\\
& 0 \leq \mu, \beta_r
\end{aligned}$$

The Nash equilibrium conditions are those that we derived in Lemma 11.

We now use the same technique as before. It is possible to show that (75-76) hold with equality, and so we rewrite these equalities as

$$\alpha_q = \frac{\phi(\mu)R + (\mu\phi'(\mu) - \phi(\mu))d_q}{R - d_q} \ \forall q$$

and

$$d_1 = \frac{(\alpha_1 - \phi(\mu))R}{\alpha_1 - \phi(\mu) + \mu\phi'(\mu)}$$

Inserting these two expressions in the objective function (74), we find that (74) is convex and symmetric in d_q, $i > 2$. Thus we must have

$$\forall q \ d_q = x := \frac{1}{Q-1}\left(\sum_{r=1}^{R} s_r - \frac{(\alpha_1 - \phi(\mu))R}{\alpha_1 - \phi(\mu) + \mu\phi'(\mu)}\right)$$

As before, we can argue that the worst case occurs when $Q \to \infty$. Therefore, after inserting the expression for d_q into (74) we can take the limit as $Q \to \infty$ and finally obtain the following reduced optimization problem:

$$\min \ \frac{\frac{(\alpha_1-\phi(\mu))R}{1-\phi(\mu)+\mu\phi'(\mu)}(\alpha_1 - \phi(\mu)) + \phi(\mu)\sum_{r=1}^{R} s_r^{\mathrm{NE}} - \frac{1}{2}\sum_{r=1}^{R}\beta_r(s_r^{\mathrm{NE}})^2}{\alpha_1\sum_{r=1}^{R} s_r^{\mathrm{OPT}} - \frac{1}{2}\sum_{r=1}^{R}\beta_r(s_r^{\mathrm{OPT}})^2} \tag{82}$$

$$\text{s.t.} \ \beta_r s_r^{\mathrm{NE}}\left(1 + \frac{s_r^{\mathrm{NE}}}{R-1}\right) \le \phi(\mu)\left(1 + \frac{s_r^{\mathrm{NE}}}{R-1}\right) - \phi'(\mu)\mu\frac{s_r^{\mathrm{NE}}}{R-1} \ \forall r \text{ s.t. } 0 < s_r^{\mathrm{NE}} \le 1 \tag{83}$$

$$\beta_r s_r^{\mathrm{NE}}\left(1 + \frac{s_r^{\mathrm{NE}}}{R-1}\right) \ge \phi(\mu)\left(1 + \frac{s_r^{\mathrm{NE}}}{R-1}\right) - \phi'(\mu)\mu\frac{s_r^{\mathrm{NE}}}{R-1} \ \forall r \text{ s.t. } 0 \le s_r^{\mathrm{NE}} < 1 \tag{84}$$

$$\beta_r s_r^{\mathrm{OPT}} \le \alpha_1 \ \forall r \text{ s.t. } 0 < s_r^{\mathrm{OPT}} \le 1$$

$$\beta_r s_r^{\mathrm{OPT}} \ge \alpha_1 \ \forall r \text{ s.t. } 0 \le s_r^{\mathrm{OPT}} < 1$$

$$0 < \phi(\mu) \le \alpha_1 \tag{85}$$

$$0 \le s_r^{\mathrm{NE}}, s_r^{\mathrm{OPT}} \le 1 \ \forall r$$

$$0 \le \mu, \beta_r$$

Constraint (85) can be derived as before (if $\phi(\mu) > \alpha_1$, then $d_q = 0 \ \forall q$). Also observe that when we let $\phi(\mu) = \mu$, this reduces to one of the optimization problems before. Now arguing as in that proof, we can show that constraints (83-84) must hold with equality, and so we have

$$\beta_r = \frac{\phi(\mu)\left(1 + \frac{s_r^{\mathrm{NE}}}{R-1}\right) - \phi'(\mu)\frac{\mu s_r^{\mathrm{NE}}}{R-1}}{s_r^{\mathrm{NE}}\left(1 + \frac{s_r^{\mathrm{NE}}}{R-1}\right)}$$

Then we can also show that

$$s_r^{\mathrm{OPT}} = \min(\alpha_1/\beta_r, 1)$$

Substituting these values into (82) we find that the system further reduces to

$$\min \frac{\frac{(\alpha_1-\phi(\mu))R}{\alpha_1-\phi(\mu)+\mu\phi'(\mu)}(\alpha_1-\phi(\mu))+\phi(\mu)\sum_{r=1}^R s_r^{\mathrm{NE}}-\frac{1}{2}\sum_{r=1}^R \frac{\phi(\mu)\left(1+\frac{s_r^{\mathrm{NE}}}{R-1}\right)-\phi'(\mu)\frac{\mu s_r^{\mathrm{NE}}}{R-1}}{1+\frac{s_r^{\mathrm{NE}}}{R-1}} s_r^{\mathrm{NE}}}{\alpha_1\sum_{r=1}^R \min(\alpha_1/\beta_r,1)-\sum_{r=1}^R \frac{\phi(\mu)\left(1+\frac{s_r^{\mathrm{NE}}}{R-1}\right)-\phi'(\mu)\frac{\mu s_r^{\mathrm{NE}}}{R-1}}{s_r^{\mathrm{NE}}\left(1+\frac{s_r^{\mathrm{NE}}}{R-1}\right)}(\min(\alpha_1/\beta_r,1))^2} \tag{86}$$

$$\text{s.t.}\quad \beta_r=\frac{\phi(\mu)\left(1+\frac{s_r^{\mathrm{NE}}}{R-1}\right)-\phi'(\mu)\frac{\mu s_r^{\mathrm{NE}}}{R-1}}{s_r^{\mathrm{NE}}\left(1+\frac{s_r^{\mathrm{NE}}}{R-1}\right)}\ \forall r$$

$$0<\phi(\mu)\le\alpha_1$$
$$0\le s_r^{\mathrm{NE}}\le 1\ \forall r$$
$$0\le\mu$$

and we can further reduce this to

$$\min \frac{\frac{(\alpha_1-\phi(\mu))}{\alpha_1-\phi(\mu)+\mu\phi'(\mu)}(\alpha_1-\phi(\mu))+\phi(\mu)s-\frac{1}{2}\frac{\phi(\mu)\left(1+\frac{s}{R-1}\right)-\phi'(\mu)\frac{\mu s}{R-1}}{1+\frac{s}{R-1}}s}{\alpha_1\min(\alpha_1/\beta,1)-\frac{1}{2}\frac{\phi(\mu)\left(1+\frac{s}{R-1}\right)-\phi'(\mu)\frac{\mu s}{R-1}}{s\left(1+\frac{s}{R-1}\right)}(\min(\alpha_1/\beta,1))^2} \tag{87}$$

$$\text{s.t.}\quad \beta=\frac{\phi(\mu)\left(1+\frac{s}{R-1}\right)-\phi'(\mu)\frac{\mu s}{R-1}}{s\left(1+\frac{s}{R-1}\right)}$$

$$0<\phi(\mu)\le\alpha_1$$
$$0\le s\le 1$$
$$0\le\mu$$

With a little algebra, the objective function (87) can be rewritten as

$$\frac{\frac{\alpha_1-\phi(\mu)}{\alpha_1-\phi(\mu)(1-\mu\phi'(\mu)/\phi(\mu))}(\alpha_1-\phi(\mu))+\phi(\mu)s-\frac{1}{2}\phi(\mu)\frac{\left(1+\frac{s}{R-1}\right)-\frac{\phi'(\mu)\mu s}{\phi(\mu)(R-1)}}{1+\frac{s}{R-1}}s}{\alpha_1\min(\alpha_1/\beta,1)-\frac{1}{2}\phi(\mu)\frac{\left(1+\frac{s}{R-1}\right)-\frac{\phi'(\mu)\mu s}{\phi(\mu)(R-1)}}{s\left(1+\frac{s}{R-1}\right)}(\min(\alpha_1/\beta,1))^2}$$

Now if we let $\psi:=\phi'(\mu)\mu/\phi(\mu)$ and $\phi:=\phi(\mu)/\alpha_1$, we can rewrite the system (87) as

$$\min \frac{\frac{1-\phi}{1-\phi(1-\psi(\mu))}(1-\phi)+\phi s-\frac{1}{2}\phi\frac{\left(1+\frac{s}{R-1}\right)-\psi(\mu)\frac{s}{R-1}}{1+\frac{s}{R-1}}s}{\min(1/\beta,1)-\frac{1}{2}\phi\frac{\left(1+\frac{s}{R-1}\right)-\psi(\mu)\frac{s}{R-1}}{s\left(1+\frac{s}{R-1}\right)}(\min(1/\beta,1))^2} \tag{88}$$

$$\text{s.t.}\quad \beta=\phi\frac{\left(1+\frac{s}{R-1}\right)-\psi(\mu)\frac{s}{R-1}}{s\left(1+\frac{s}{R-1}\right)}$$

$$0<\phi\le 1$$
$$0\le s\le 1$$
$$0\le\mu$$

In (88), we consider $\psi(\mu)$ to be a fixed function of μ, and ϕ is a variable over which we are minimizing the objective function.

Observe also that since $\phi(\mu)$ is convex, $\psi(\mu)\ge 1$. When $\phi(\mu)$ is linear, $\psi(\mu)=1$ for all μ, and the (88) reduces to the function we had when deriving the bound. Recall that in that case $s^{\mathrm{OPT}}=1$, because the objective function was strictly increasing in s when $s^{\mathrm{OPT}} = 1/\beta$. The same argument can be applied here, so let's assume without loss of generality that $s^{\mathrm{OPT}} = 1$.

Then the system becomes

$$\begin{aligned}
\min \quad & \frac{\frac{(1-\phi)^2}{1-\phi(1-\psi(\mu))} + \phi s - \frac{1}{2}\phi \frac{\left(1+\frac{s}{R-1}\right)-\psi(\mu)\frac{s}{R-1}}{1+\frac{s}{R-1}} s}{1 - \frac{1}{2}\phi \frac{\left(1+\frac{s}{R-1}\right)-\psi(\mu)\frac{s}{R-1}}{s\left(1+\frac{s}{R-1}\right)}} \\
\text{s.t.} \quad & s\left(1 + \frac{s}{R-1}\right) \geq \phi\left(1 + \frac{s}{R-1} - \psi(\mu)\frac{s}{R-1}\right) \\
& 0 < \phi \leq 1 \\
& 0 \leq s \leq 1 \\
& 0 \leq \mu
\end{aligned}$$

The result then follows numerically. $\square$

8.9 Appendix J: Unbounded Supply Mechanisms

Here we see why the supply functions must be bounded above. Specifically, we have the following result.

Theorem 11. *Let $S(w_r, w_{-j})$ be a continuous scalar-based market mechanism for the supply side of a two-sided market, and let r denote a supplier. Suppose that for all r we have: (1) for all $s > 0$, w_{-j}, there is a bid w_s such that $S(w_s, w_{-j}) = s$; (2) the payoff to r is*

$$\pi_r = ps_r - C_r(s_r)$$

(3) the efficiency is calculated for a class of functions $\mathcal{C}$ such that $C \in \mathcal{C} \implies \alpha C \in \mathcal{C}$ for all $\alpha > 0$, and that every $C \in \mathcal{C}$ is bijective, continuous, and $\lim_{s\to\infty} C_r(s) = \infty$; (4) There exists a competitive equilibrium $\mathbf{s} > \mathbf{0}$ for some $p > 0$; (5) the efficiency at a competitive equilibrium is not worse than the efficiency at a Nash equilibrium. Then the welfare ratio (price of anarchy) has to be zero.

Proof. Let $\mathbf{s} > \mathbf{0}$ be a vector of competitive equilibrium allocations with price $p > 0$, let be r denote a supplier with cost function C_r. Since $s_r \neq 0$ for all r, we must have

$$\frac{d\pi_r}{ds_r} = p - C_r'(s_r^{CE}) = 0$$

At the optimal solution we must have $C_r'(s_r^{\text{OPT}}) = 1$ (assuming, as usual, that valuation functions are linear with max slope 1). Observe that $\mathbf{s}$ is also a competitive equilibrium with cost functions αC_r and price αp for any $\alpha > 0$ since

$$\alpha C_r'(s_r^{CE}) = \alpha p$$

holds and by concavity of payoffs, this is a local max of π_r as a function of s_r. Then by continuity of $s_r > 0$ this must also be a competitive equilibrium. The optimal solution is now

$$s_r^{\text{OPT}} = C_r^{-1}(1/\alpha)$$

and the efficiency is

$$\frac{d_1^{CE} + \sum_{q=1}^{Q} d_q^{CE} - \sum_{r=1}^{R} \alpha C_r(s_r^{CE})}{\sum_r s_r^{\text{OPT}} - \sum_r C_r(s_r^{\text{OPT}})} = \frac{d_1^{CE} + \sum_{q=1}^{Q} d_q^{CE} - \sum_{r=1}^{R} \alpha C_r(s_r^{CE})}{\sum_r C_r^{-1}(1/\alpha) - \sum_r 1/\alpha}$$

Now if we let $\alpha \to 0$, $C_r^{-1}(1/\alpha) \to \infty$, $1/\alpha \to \infty$ and the efficiency at the competitive equilibrium goes to zero. Since the efficiency at a Nash equilibrium was assumed to be at least as bad, it also goes to zero.

There are many very natural and elegant mechanisms that fall into this category. One is the usual Cournot mechanism: suppliers' bids are their outputs, and consumers' bids are their payment. Then each consumer receives a fraction of the total supply that is equal to the fraction of money he contributed, and each supplier's payment is handled similarly. This has zero efficiency.

Another example is when consumers choose parametrized demand functions $D(p, \theta) = \theta/p$ and suppliers choose supply functions $S(\theta, p) = \theta p$. Then $p = \sqrt{\sum_q \theta_q / \sqrt{\sum_r \theta_r}}$, the total amount produced is $\sqrt{\sum_q \theta_q \sqrt{\sum_r \theta_r}}$, and each consumer receives $\theta_q / \sum_q \theta_q$ of the total quantity produced (and pays θ_q), while each supplier produces $\theta_r / \sum_r \theta_r$ of the total amount.