Discrete subgroups G of the three dimensional rotation group $SO(3)$

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A rotation in three dimensions is characterized by a unit vector $\hat{n}$ (polar angle and azimuth, equivalently lattitude and longitude), and an angle of rotation φ about that axis. It is thus a three parameter continuous group, where the nomenclature $SO(3)$ designates 3x3 orthogonal matrices of unit determinant. We wish to consider the discrete subgroups $G \subset SO(3)$.

Recall a group is a set of elements $g_i \in G$ together with a (multiplication) rule that associates to any two elements $g_1, g_2 \in G$ a third element $g_3 \in G$, usually written $g_3 = g_1 \cdot g_2$. The multiplication rule must be associative $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$, but is not necessarily commutative ($g_1 \cdot g_2 \neq g_2 \cdot g_1$ in general). A group also has an identity element $e \in G$ (the "trivial" element) that satisfies $e \cdot g = g \cdot e = g$ for all $g \in G$, and each element $g \in G$ has an inverse $g^{-1} \in G$, satisfying $g \cdot g^{-1} = g^{-1} \cdot g = e$. We take $N = |G|$ to be the number of elements in the group G.

When $G \subset SO(3)$, any non-trivial element $g \in G$ is a rotation and hence acting on the surface of a sphere (S^2) leaves fixed exactly two points, called poles (just as the north and south poles of the earth are fixed under the earth's rotation). The set of all poles left fixed by elements $g \in G$ partitions into equivalence classes C_i under the action of G: we say two poles p, p' are G-equivalent if there exists some element $g \in G$ that rotates p into p', i.e., $g \circ p = p'$ (where $g \circ p$ denotes the action of group elements on points of S^2). Suppose there are M of these equivalence classes, $C_1, C_2, \ldots, C_M$.

Note that the element $g \in G$ that rotates a given p into a given p' is not necessarily unique — it is instead arbitrary up to right multiplication ($g \to gh$) by any element $h \in H_p$, where $H_p \subset G$ is the subgroup of elements that leave the pole p fixed, $h \circ p = p$ for $h \in H_p$.* (H_p in general will consist of rotations by integer multiples of some $2\pi/n$ about an axis through p.) Note that any two poles p, p' in the same equivalence class will have isomorphic invariance groups $H_p \approx H_{p'}$, since if $p' = g \circ p$ then group elements of the form ghg^{-1}, $h \in H_p$, will leave p' fixed: $ghg^{-1} \circ p' = gh \circ p = g \circ p = p'$.** The number of elements $|C_i|$ in a given equivalence class of poles containing p therefore satisfies

$$|C_i| = N/n_i \ ,$$

where $N = |G|$ and $n_i = |H_p|$ is the number of elements in the subgroup that leaves the pole p fixed, and depends only on the equivalence class C_i of p.

The sum over equivalence classes $\sum_{i=1}^{M} |C_i|(n_i - 1)$ counts each pole multiplied by the number of non-trivial group elements that leave it fixed, and hence satisfies

$$\sum_{i=1}^{M} |C_i|(n_i - 1) = 2(N - 1) \ ,$$

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* H_p is also known as the *invariance group* of the pole p, and the elements of G identified under right multiplication by elements of H are known as the left cosets of H in G, denoted G/H. The poles in the equivalence class containing p thus correspond to the left cosets G/H_p.

** The group gHg^{-1} is known as the *conjugate* of H in G.

since each non-trivial group element leaves fixed exactly two poles. Dividing both sides by N, and using $|C_i| = N/n_i$ for each equivalence class, gives the group theoretic constraint:

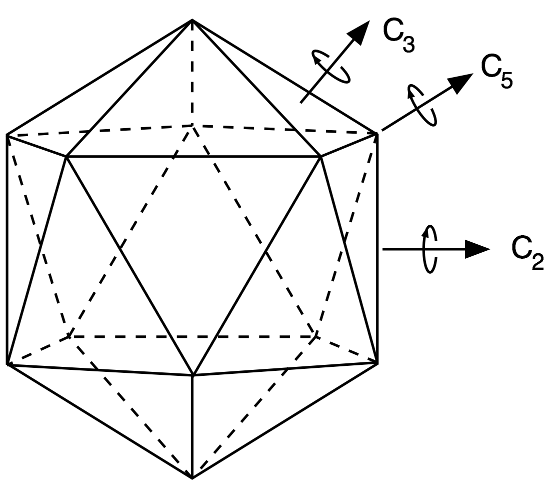
$$\sum_{i=1}^{M}\Big(1 - \frac{1}{n_i}\Big) = 2\Big(1 - \frac{1}{N}\Big) \ .$$

The right hand side above is always less than 2, and since each of the n_i is an integer in the range $2 \le n_i \le N$, each term on the left hand side is at least 1/2, so M can be at most 3. This makes it simple to enumerate all possibilities.

There is one solution with $M = 2$, i.e., two equivalence classes of poles: $n_1 = n_2 = N$. This is an N element group known as $\mathbf{C}_N$, the cyclic group of rotations by $2\pi m/N$, $m = 0, \ldots N-1$. The first solution for $M = 3$ is also an infinite class of groups, with $n_1 = n_2 = 2$, and $n_3 = N/2$, where $N \ge 4$ is even. These are known as the dihedral groups $\mathbf{D}_{N/2}$ and consist of a $\mathbf{C}_{N/2}$ together with an additional $N/2$ $\mathbf{C}_2$'s acting on axes symmetrically placed in a plane orthogonal to the $\mathbf{C}_{N/2}$ symmetry axis.

The three remaining $M = 3$ solutions are the polyhedral groups:

- $n_1 = 2$, $n_2 = n_3 = 3$ specifies the tetrahedral group $\mathbf{T}$, with $N = 12$. (Acting on a tetrahedron, it has four 4 $\mathbf{C}_3$ axes, one through each vertex to center of the opposite face, and 3 $\mathbf{C}_2$ axes through the centers of pairs of opposite sides.)
- $n_1 = 2$, $n_2 = 3$, $n_3 = 4$ specifies the octahedral group $\mathbf{O}$, with $N = 24$, the symmetry group of the cube and its dual, the octahedron. (Acting on a cube, it has 3 $\mathbf{C}_4$ axes through the centers of opposite faces, 4 $\mathbf{C}_3$ axes through opposite vertices, and 6 $\mathbf{C}_2$ axes through the centers of pairs of opposite edges.)
- $n_1 = 2$, $n_2 = 3$, $n_3 = 5$ specifies the icosahedral group $\mathbf{I}$, with $N = 60$, the symmetry group of the icosahedron and its dual, the dodecahedron. (Acting on the icosahedron, it has 6 $\mathbf{C}_5$ axes through opposite vertices, 10 $\mathbf{C}_3$ axes through the centers of opposite faces, and 15 $\mathbf{C}_2$ axes through the centers of opposite edges.)



An icosahedron has 20 equilateral faces, 30 edges, and 12 vertices. The figure shows three of the axes of symmetry. The total number N of group elements of $\mathbf{I}$, adding all the non-trivial $\mathbf{C}_5$, $\mathbf{C}_3$, and $\mathbf{C}_2$ group elements plus the identity, is $6(5-1) + 10(3-1) + 15(2-1) + 1 = 60$.