To prove Stirling’s formula, \( N! \sim \sqrt{2\pi N} N^N e^{-N} \) for large \( N \), we first recall the method of steepest descent. An integral of the form,

\[
I = \int_a^b e^{Nf(x)} \, dx,
\]

when \( N \) is large, is dominated by the “critical points” \( x_0 \) of \( f \), at which \( f'(x_0) = 0 \). At any such point, we can approximate in the near vicinity of \( x_0 \),

\[
f(x) \approx f(x_0) - \frac{1}{2} |f''(x_0)| (x - x_0)^2.
\]

Assuming only a single critical point, we can then approximate

\[
I \approx e^{Nf(x_0)} \int_{-\infty}^{\infty} e^{-\frac{1}{2} N |f''(x_0)| x^2} \, dx = e^{Nf(x_0)} \sqrt{\frac{2\pi}{N|f''(x_0)|}}
\]

(where we have also extended the limits of integration from \(-\infty\) to \(\infty\), assuming that only the region near \( x_0 \) matters anyway, and then used \( \int_{-\infty}^{\infty} dx \, e^{-ax^2} = \sqrt{\pi/a} \)).

To apply to the factorial function, we first need an integral representation. The function \( \Gamma(N+1) \equiv \int_0^\infty e^{-x} x^N \, dx \) satisfies the recursion relation

\[
\Gamma(N+1) = -e^{-x} x^N \bigg|_{0}^{\infty} + N \int_0^\infty e^{-x} x^{N-1} \, dx = N \Gamma(N)
\]

(using integration by parts). Together with the boundary condition \( \Gamma(1) = \int_0^\infty e^{-x} \, dx = 1 \), we find that \( \Gamma(N+1) = N! \) for integer \( N \).

So we write

\[
N! = \Gamma(N+1) = \int_0^\infty e^{N \ln z - z} \, dz
\]

where \( z = x/N \). Hence \( f = \ln z - z, \ f' = 1/z - 1, \ f'' = -1/z^2 \), and \( z_0 = 1 \). Eqn. (1) gives

\[
N! \approx N^{N+1} e^{-N} \sqrt{\frac{2\pi}{N}} = \sqrt{2\pi N} N^N e^{-N}
\]

\[\text{†} \] If we write \( J = \int_{-\infty}^{\infty} dx \, e^{-x^2} \), then using \( r, \theta \) cylindrical coordinates its square can be written \( J^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, e^{-x^2-y^2} = \int_0^{2\pi} d\theta \int_0^{\infty} dr \, r \, e^{-r^2} = 2\pi \left( -\frac{1}{2} e^{-r^2} \bigg|_{0}^{\infty} \right) = \pi \). Thus \( J = \sqrt{\pi} \) and \( \int_{-\infty}^{\infty} dx \, e^{-ax^2} = \sqrt{\pi/a} \).