1 6-Coloring Theorem

Lemma 1. For any simple planar graph $G$, the average degree of $G$ is strictly less than 6.

Proof. The average degree of a graph is $2e/v$. Using $e \leq 3v - 6$ (for $v \geq 3$) we get $D \leq 2(3v - 6)/v$ or $D \leq 6 - 12/v$. So for $v \geq 3$, $D < 6$. For $v < 3$ we can check directly.

\[\square\]

Theorem. Any simple planar graph can be properly colored with six colors.

Proof. We prove the result by induction on the number of vertices.

(Base case) Suppose we have a graph such that $v \leq 6$. For $v \leq 6$, we can give each vertex a different color and use $\leq 6$ colors.

(Induction hypothesis) Now assume that any simple planar graph on $v = n$ vertices can be properly colored with six colors.

Let $G$ be any simple planar graph on $v = n + 1$ vertices. From our lemma above, we know that $G$ must have some vertex $w$ of degree $\leq 5$. Remove $w$ from $G$ to form $G'$. $G'$ has $v = n$ vertices and we may apply our induction hypothesis to know it can be properly colored in 6 colors. Properly color $G'$ with $\leq 6$ colors. Now, we can think of this as coloring all of $G$ except $w$. But, since $w$ has degree at most 5, one of the 6 colors will not be used for any of the neighbors of $w$ and we can finish coloring $G$.

\[\square\]