Validating Eigenvalues and Eigenvectors

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$Ax = \lambda x$?

- Consider system (A, H), $A \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{m\times n}$:

$$x'(t) = Ax(t)\ ,\ x(0) = x_0\ ,\ y(t) = Hx(t).$$

For eigenpairs (λ_i, x_i), we expect the solution

$$y(t) = \sum_{i=1}^{n} c_i\, e^{\lambda_i t}\, Hx_i.$$

- From data, extract candidate pairs $(\hat{\lambda}_i, u_i)$
- Did (A, H) generate each pair $(\hat{\lambda}_i, u_i)$?

Problem Statement

- More generally, we are given: an EVP $f(\lambda) : \mathbb{C} \to \mathbb{C}^{n\times n}$, a function $H : \mathbb{C}^n \to \mathbb{C}^m$, and a set $(U, \Lambda) = \{(\lambda_i, u_i) \mid u_i \in \mathbb{C}^m\}_{i=1}^{p}$.
- Measure similarity between (f, H) and (U, Λ).
- Applications in Control Systems, Structural Engineering, Inverse Problems.

Dot Products are Bad

- Currently: find eigenvectors $X = \{x_i, x_2, \ldots, x_p\}$.
- Then calculate p^2 normalized dot products

$$\frac{(Hx_i)^* u_j}{\|Hx_i\|_2 \|u_j\|_2} \text{ (a sensitive computation).}$$

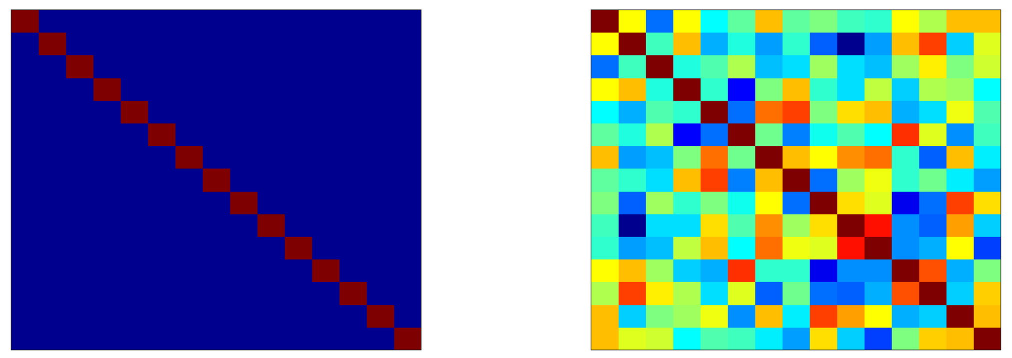

Figure 1: Dot Product is known as the Modal Assurance Criterion (MAC) in Structural Engineering.

Least Squares is Better

- Instead, solve the linear least squares problem for each pair (λ_i, u_i)

$$\min_z\ \gamma^2 \big\|f(\lambda)z\big\|_2^2 + \big\|Hz - u\big\|_2^2. \tag{1}$$

- We are fitting the eigenvector z by minimizing the sum of two terms. The first measures the error in the eigenvalue equation itself. The second measures the error between Hz and the observed vector u.

Calculation and Regularization

- In applications, both f and H might have some particular structure.
- E.g. in structural models, f an SPD matrix —LSQR + Incomplete Cholesky on $f(\lambda)$ as a preconditioner.
- More generally, if we have a fast solver for $f(\lambda)$, we may rewrite as:

$$\min_w\ \big\|\gamma w\big\|_2^2 + \big\|Hf(\lambda)^{-1}w - u\big\|_2^2\ ,\ w = f(\lambda)^{-1}z. \tag{2}$$

- Regularization Parameter γ? Something fixed or something calculated?

Variations

We present two variations.

- The **first** replaces 2-norms with weighted norms K_1 and K_2:

$$\min_z\ \big\|f(\lambda)z\big\|_{K_1}^2 + \big\|H(z) - u\big\|_{K_2}^2.$$

- Allows for partial validation, MLE.
- The **second** replaces the 2-norm with general loss functions $\mathcal{L}$ (e.g. 1-norm or Huber):

$$\min_z\ \big\|f(\lambda)z\big\|_{\mathcal{L}} + \big\|H(z) - u\big\|_{\mathcal{L}}.$$

- By calculating a sparse residual, we hope to isolate the subset of equations contributing most to the residual error.

Perturbation Analysis

Consider perturbation in either (f, H) or in (λ, u), assuming a residual error of zero.

- Given a perturbation $(\delta f, \delta H)$,

$$\frac{\|x - z\|_2}{\|x\|_2} \le \epsilon\left(\kappa(M) + \frac{1}{\|M\|_2}\right)\ ,\ M = \begin{bmatrix} f(\lambda) \\ H \end{bmatrix},$$

$$\delta M = \begin{bmatrix} \delta f(\lambda) \\ \delta H \end{bmatrix}\ ,\ \epsilon = \frac{\|\delta M\|_2}{\|M\|_2}.$$

- Given a perturbation $(\delta\lambda, \delta u)$,

$$\frac{\|x - z\|_2}{\|x\|_2} \le \left(\|\delta u\|_2 + \frac{\|f(\delta\lambda)\|_2}{\|M\|_2}\right)\left(1 + \frac{\|M\|_2}{\tau}\right),$$

$$\tau = \frac{\|Hx\|_2}{\|x\|_2}.$$

- τ is an important quantity! Measures information extracted from x.

Numeric Experiments

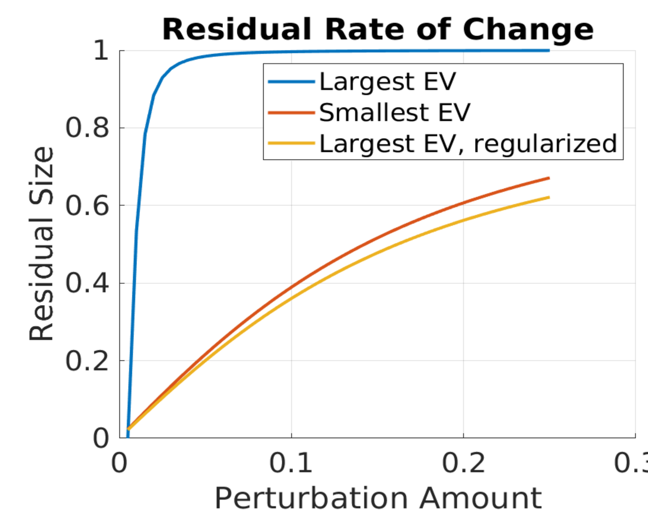

Figure 2: The unregularized problem i.e. $\gamma = 1$ has an unfavorable growth profile; thus, the need for regularization.

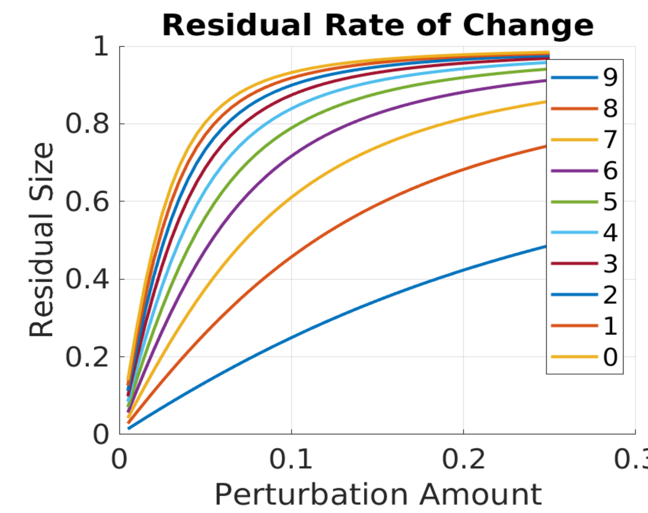

Figure 3: Numbers in legend indicate levels of regularization.

Applications

- A power system is the ODE

$$Ex'(t) = Ax(t)\ ,\ y(t) = Hx(t).$$

Given a set of fault models, we run our method:

	IEEE 39	IEEE 57	IEEE 145
single	.95(1)	.975(1)	.5(.8)
sparse	1(1)	1(1)	.9(1)
full	1(1)	1(1)	n/a

- A structural model is the ODE

$$Mx(t)'' + Cx(t)' + Kx(t) = 0$$
$$y(t) = Hu(t).$$

We add a sparse perturbation to ODE—and successfully recover perturbation locations.