Improving Upon Mode to Mode Comparisons in MDOF Systems: The Modal Assurance Fit (MAF)

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Abstract

Eigenmodes, known informally as modes, are the building blocks of any time-domain signal arising from a linear dynamical system, and are often used as a proxy for the system itself. Due to this fact, modal information is used in a wide variety of diagnostic applications. An equally wide variety of metrics comparing empirical modes (those extracted from data) and analytical modes (those extracted from a model) have been developed in tandem with their respective applications. Many of these metrics are derived from the Modal Assurance Criterion (MAC), a correlation metric originating in MDOF finite element model validation. We suggest an alternative to the MAC, which we dub the "Modal Assurance Fit" (MAF) —the MAF. While the MAC directly compares empirical modes to analytical modes, the MAF compares empirical modes to an analytical *model*. In this sense, the MAF is a mode-to-model comparison rather than a mode-to-mode comparison. The authors were inspired to develop the MAF after unsuccessfully applying the MAC to diagnose faults in power systems.

1 Introduction

We denote a Multiple Degrees of Freedom (MDOF) model by a control system containing a second order, linear ODE and a linear sensing matrix

$$\begin{aligned} Mu(t)'' + Cu(t)' + Ku(t) &= 0 \\ f(t) &= Hu(t) \end{aligned} \tag{1}$$

Where $M \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times n}$, and $K \in \mathbb{R}^{n\times n}$ are real, symmetric matrices representing the mass matrix, damping matrix, and stiffness matrix, respectively. $u(t)$ is the full system state and $f(t) = Hu(t)$ is the time-domain signal the sensing matrix H outputs. H will most commonly be a subset of the identity matrix. Common techniques of manipulating include diagonalization of C, inversion of M, or augmentation of into a larger, first order system.

The eigenvectors y and eigenvalues ω of this system (also known in the engineering literature as the vibration modes and vibrations frequencies, or just the modes and frequences) are important mathematical objects. The eigenvalues of (1) are the solution to the quadratic eigenvalue problem

$$(\omega^2 M + \omega C + K)y = 0$$

If K and M are symmetric and real, the modes of this system are M-orthogonal and real, see [1] for a comprehensive treatment of the quadratic eigenvalue problem. In the civil engineering

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literature, assumptions on M and C transform our our second order, linear ODE into a first order one, resulting in either a generalized or ordinary eigenvalue problem instead of a quadratic eigenvalue problem (see [2] for a brief survey). However, we will make no such assumptions in our presentation. Without loss of generality, the notation and set-up we will use for the rest of this paper follows

- The MDOF system $Mu''(t) + Cu'(t) + Ku(t) = 0,\ M, K \in \mathbb{R}^{n\times n},\ u(t) \in \mathbb{R}^n$
- The sensing matrix $H \in \mathbb{R}^{m\times n},\ m \leq n$
- Time domain sensor data $f(t) = Hu(t)$
- The *analytical* eigenvalue associated with the MDOF model ω_m
- The *analytical* mode y_m associated with the full order MDOF model, such that $(K + \omega_m M)y_m = 0$
- The part of the *analytical* mode seen by sensors $x_m \in \mathbb{R}^m,\ x_m = Hy_m$
- The *empirical* eigenvalue ω_e extracted from $f(t)$
- The *empirical* mode x_e seen by sensors, extracted from $f(t)$

We note that our notation differs slightly from papers in this area —generally, we will be using standard english letters for parameters associated with the MDOF system and greek letters for functions and constants (with ω being the exception).

2 Related Work

2.1 Modal Assurance Criterion

Engineers compute the analytical (extracted from a model) and experimental (extracted from f) modes and frequencies for a wide variety of tasks. But when doing so, one must sanity-check that the analytical and experimental modes and frequencies are "close together". This is most commonly quantified by computing the Modal Assurance Criterion.

The MAC was originally used in Civil Engineering to validate experimental modal vectors against some finite element model of a structure. Civil engineers would use the MAC to determine whether their MDOF model was an accurate representation of the actual machine or building. This was accomplished by perturbing the machine or building, measuring its transient response, and then comparing the modes of the response x_e to the modes of their model x_m. The MAC is defined as:

$$MAC(x_e, x_m) = \frac{x_e^T x_m}{\|x_e\|_2 \|x_m\|_2}$$

If x_e and x_m are unit vectors, the MAC takes an even more straightforward formulation.

$$MAC(x_e, x_m) = x_e^T x_m$$

In other words, the MAC is simply a normalized dot product between experimental and analytical modes. A MAC score of one indicates that the experimental and model modes are identical, while a MAC score of zero indicates that they are orthogonal. Modern usage of the MAC, at least in the literature, has expanded to a wide variety of applications in various engineering disciplines. Aside

from model validation, important applications of the MAC include fault diagnosis [3] [4] [5], model updating [6] [7] [8], and damage assessment [9] [10].

With different applications come different needs —the original MAC has spawned a wide variety of application-specific variants. See [11] for an older survey list of MAC variants. In the context of model validation, model updating and damage assessment, the MAC requires an *in-silico*, analytical model to draw modes from. However, when diagnosing faults, the MAC can draw its "analytical mode" from either an older empirical modal estimate or physical test model. In this case, a computer model is not needed.

Some of the drawbacks of the MAC, pulled from the literature, are enumerated below (as a disclaimer, we do not mean to imply that the MAC is a fundamentally problematic in any way).

2.1.1 Model Reduction

The degrees of freedom (DOF) on a finite element model are typically greater than the number of sensors measuring transient responses. Thus, engineers either reduce the mass matrix through various algorithms or pull out a subset of entries from the full finite element modes. To quote [12], "Comparison of modal vectors can be done at the reduced order or at the full order of the FEM. Reduction of the physical mass matrix or expansion of test modal vectors bring inherent approximations in the comparison criteria.". In the former case, accuracy is lost through approximation error, and in the latter case, some form of mode-to-mode matching is required.

2.1.2 Sensitivity to Invariant Subspaces

If there are eigenvalues of algebraic multiplicity greater than 1 (or eigenvalues spaced closely together), the MAC does not reveal any useful information [13], [14]. In the most degenerate case, one could obtain a MAC of 0 if the extracted mode is orthogonal to the the analytical mode computed but in the same invariant subspace.

2.1.3 Sufficiency but not Necessity

Ultimately, the all the points listed above hint at the main underlying weakness with the MAC: sufficiency but not necessity. A high MAC score is sufficient to show that analytical and experimental modes are similar, but a low MAC score does not prove that analytical and experimental modes are in conflict. The MAC is simply one metric correlating analytical and experimental data. We say "data" and not "modes" because the real question the MAC attempts to answer is "how likely is the experimental mode to have come from the analytical model?" The MAC changes a mode-to-model comparison with a mode-to-mode similarity measure.

2.2 Diagnostic Oberservers

Punch Line: Observers more difficult to construct, many are more concerned with specific modal behavior rather than

3 Modal Assurance Fit

The number of different MAC variants in the literature hints at inherent weaknesses in taking dot products between modes as a similarity comparison. We developed the MAF after applying the MAC to diagnose faults in control systems and discovering that it didn't perform that well. We propose direct mode-to-model comparison, which we dub the Modal Assurance Fit (MAF).

Definition *The modal assurance criterion is defined as:*

$$MAF(\omega_e, x_e) = \min_y \left\|(M\omega_e^2 + C\omega_e + K)y\right\|_2^2 - \left\|Hy - x_e\right\|_2^2$$

where x_e has unit length (i.e. any vector must be normalized before the MAF is applied)

The the first summand forces y to approximate a mode of the analytical model and second summand in the objective function forces Hy to approximate x_e. More intuitively, the first summand fits to the analytical model and the second summand fits to the experimental data. We can rewrite the MAF as

$$MAF(\omega_e, x_e) = \min_y \left\| \begin{pmatrix} M\omega_e^2 + C\omega_e + K \\ H \end{pmatrix} y - \begin{pmatrix} 0 \\ x_e \end{pmatrix} \right\|_2^2 \tag{2}$$

The MAF is thus the residual after fitting an empirical mode y of the MDOF system. One may choose to compute the MAF at either the full model order or at a reduced model order; the MAF is agnostic to the relative sizes of the empirical mode shape and the analytical model —so long as H is appropriately chosen to reflect the correct location of measurement sites.

Lemma 3.1 *The MAF is bounded between 0 and 1 i.e.*

$$0 \leq MAF(\omega, x) \leq 1$$

The lower bound on the MAF is 0 as it is the minimization of a non-negative function, and is achieved by setting y equal to y_m. By a similar argument, an upper bound on the MAF can be found by setting y equal to 0. This upper bound thus will be equal to $\|x_e\|_2^2 = 1$. Also, note that the best possible MAF score is a 0 and the worst possible MAF scores is a 1.

3.1 Presentation of the MAF

The MAC is usually presented as in a color-coded matrix, where ij-th entry of the matrix is the MAC value between the i-th empirical and j-th analytical modes. The diagonal of the MAC is the key piece of interest —however, the off-diagonal entries also reveal whether any other modes are heavily correlated as well. This allows one to detect duplicate mode shapes arising from fitting error or nonlinearities.

In the case of the MAF, possessing k modes results in k scalar values, each of which measures a mode to model fit. As a result, there is no analagous color coded matrix presentation of the MAF, and no way to correlate different modes. Therefore, we define the cross modal assurance fit (CMAF) for two empirical mode shapes x_e and $\hat{x}_e$ as:

$$CMAF(x_e, \hat{x}_e) = CMAF(\omega_e, x_e, \hat{x}_e) = \min_y \left\| \begin{pmatrix} M\omega_e^2 + C\omega_e + K \\ H \end{pmatrix} y - \begin{pmatrix} 0 \\ \frac{x_e + \hat{x}_e}{\|x_e + \hat{x}_e\|_2} \end{pmatrix} \right\|_2^2 \tag{3}$$

The CMAF is simply the MAF, using the normalized average of two empirical modes. The logic behind the CMAF is that, if two empirical modes are rotations of each other, than their normalized average also is. Therefore, two heavily correlated modes will lead to a small CMAF. Note that the CMAF is not symmetric in its arguments i.e. $CMAF(x_e, \hat{x}_e) \neq CMAF(\hat{x}_e, x_e)$. The former uses ω_e, the latter uses $\hat{\omega_e}$. By definition, if $x_e = \hat{x}_e$, then the CMAF reduces to the MAF.

With the CMAF defined, we can now give an equivalent "MAC-esque" color coded matrix whose ij-th entry is the CMAF between the i-th and j-th empirical modes. This presentation is entirely optional and dependent upon the application and user preference; if the k scalars arising from the standard MAF computation suffice, there is no need to compute the CMAF matrix. Similar to how the MAF and MAC have correlation scores reversed, the CMAF matrix should be reversed from the MAC matrix, with diagonal entries small (lightly colored) and off-diagonal entries large (darkly colored).

3.2 Other Properties

Lemma 3.2 *The MAF is robust to duplicate or nearly duplicate eigenvalues.*

Eigenvectors associated with nearly duplicate eigenvalues are highly sensitive to perturbation; direct computation of these eigenvectors will lead to inaccurate results. Duplicate eigenvalues lead to an invariant subspace of dimension equal to the degree of duplicity; any vector pulled from this subspace will be a valid subspace. Thus, computing the MAC under these cases is problematic. However, the MAF is robust to duplicate eigenvalues. Because the MAF is a mode-to-model comparison rather than a mode-to-mode comparison, it does not care about sensitivity of specific analytic modes. Computing the MAF requires the straightforward construction of a sparse least squares problem, which may be solved using the LSQR method, a sparse least-squares solver [**?**]

By construction, the MAF does not involve the analytical frequencies ω_m or analytical modes x_m. Furthermore, the MAF uses both mode and frequency data. This is in contrast to the MAC, which only uses frequency data to perform model reduction or search for a set of candidate modes. In a simple illustration on this fact, a shifting the M, C, and K by some σM for constant σ changes only frequencies and not mode shapes; the MAC therefore stays unchanged while the MAF will detect this change. A few attempts have been made to alleviate this fact, see [15] and [16] for related MAC variants.

4 The Belovich-Popov-Hautus Condition

In the previous section, we introduced the Modal Assurance Fit. The Modal Assurance is closely related to the the Belovich-Popov-Hautus condition (or Hautus condition in short), sugggested independently by both Hautus and Popov [17] [18]. The Hautus condition formulates observability of a linear, time invariant system in terms of the modes of the system. In this light, the Hautus condition is particularly useful in the context of this paper because it looks exclusively at modal behavior.

Theorem 4.1 ***Belovich-Popov-Hautus condition:*** *The LTI system with state matrix A and output matrix C is observable if and only if the matrix valued function $\Gamma(s)$ is full rank for all s*

$$\Gamma(s) = \begin{pmatrix} A - sI \\ C \end{pmatrix}$$

We call $\Gamma(s)$ the *Hautus Matrix*. Rank deficiency of $\Gamma(s)$ is equivalent to the matrix C mapping an eigenvector of A to 0. This is because, given an eigenpair (ω, v), that $\Gamma(\omega)v = 0$ iff $Av = \omega v$ and $Cv = 0$. We note that it suffices to compute the rank of $\Gamma(s)$ for s restricted to to the eigenvalues of A.

Lemma 4.2 *Given the eigenvalues $\{\omega_1, \omega_2, \ldots \omega_n\}$ of a matrix A, the Hautus Matrix defined as*

$$\Gamma(s) = \begin{pmatrix} A - sI \\ C \end{pmatrix}$$

is rank deficient if and only if $\Gamma(\omega_i)$ is rank deficient for some ω_i.

Intuitively, the any vector contained in the null space of $\Gamma(s)$ must by definition be an eigenvector of A. Therefore, checking rank deficiency of $\Gamma(\omega_1) \ldots \Gamma(\omega_n)$ is sufficient to check the rank deficiency of $\Gamma(s)$. Extending the Hautus condition to MDOF systems is quite straghtforward.

Theorem 4.3 ***Extended Belovich-Popov-Hautus condition:*** *The MDOF system with Mass, Damping, and Stiffness matrices M, C, and K as well as a sensing matrix H is observable if and only if the matrix valued function $\Gamma(s)$ is full rank*

$$\Gamma(s) = \begin{pmatrix} Ms^2 + Cs + K \\ H \end{pmatrix}$$

Following the same logic as before, our MDOF system is observable iff H does not annihilate any eigenvector solving the quadratic eigenvalue problem $M\omega^2 + C\omega + K$. We may view the Hautus matrix as a frequency-domain alternative to the observability matrix and gramian.

4.1 Numerical Rank and The Hautus Condition

Computing the *numeric* rank of a matrix rather than its *analytic* rank is preferable for a wide range of reasons, see [19] for a in-depth discussion on numeric rank. Traditionally, the numeric rank of a matrix is calculated by taking its Singular Value Decomposition and observing the decay of singular values; a matrix close to rank deficiency will have at least one singular value close to machine precision. Numeric rank, as applied to the Hautus Condition, takes the following form

Theorem 4.4 *Consider the matrix $\Gamma(s)$ for any fixed value of s and its singular values sorted in decreasing order as*

$$\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n$$

Then $\Gamma(s)$ is numerically rank deficient if σ_n is close to machine precision ϵ.

From the perpective of observability, the numeric rank of $\Gamma(s)$ informs us whether the underlying system is close to unobservable or not. Using numeric rank rather than analytical rank allows us to also characterize the energy of individual modes.

Theorem 4.5 *The energy of a mode associated with vibration frequency ω is defined as*

$$\frac{\sigma_1(\Gamma(\omega))}{\sigma_n(\Gamma(\omega))}$$

5 On Numerics

6 Conditioning of the MAF

The MAF's relationship to the Hautus condition becomes immediately apparent with the re-writing

$$MAF(\omega_e, x_e) = \min_y \left\| \Gamma(\omega_e) y - \begin{pmatrix} 0 \\ x_e \end{pmatrix} \right\|_2^2 \tag{4}$$

The MAF is a residual of a least squares problem. When considering the behavior of the MAF under perturbation, it makes to look at the *conditioning* of the MAF. The conditioning of the MAF is precisely the conditioning of its associated least squares problem. We say that a least squares problem is *ill-conditioned* if small changes in the input may lead to large changes in the output i.e. the numeric solution may be inaccurate. The conditioning of least-squares problem is estimated via the condition number of a matrix. The condition number of the matrix $\Gamma(\omega_e)$ is given by the formula

$$cond(\Gamma(\omega_e)) = \frac{|\sigma_1|}{|\sigma_n|}$$

If σ_n is small or zero, the condition number is large and we say the matrix is *ill-conditioned*. The condition number of our least squares problem is precisely the condition number of the Hautus Matrix.

Theorem 6.1 *The condition number of $MAF(\omega_e, x_e)$ is denoted as $MAFcond(\omega_e, x_e)$ and given by the formula*

$$MAFcond(\omega_e, x_e) = \frac{|\sigma_1|}{|\sigma_n|}$$

where σ_1 and σ_n are the largest and smallest singular values of $\Gamma(\omega_e)$, respectively.

By definition, the Hautus Matrix is only rank deficient if our MDOF system is unobservable. Equivalently, the Hautus Matrix will have a large condition number if and only if our system is unobservable or close to unobservable. Assuming our system is observable, we thus know the Hautus Matrix will be well-conditioned, implying that the MAF will similarly be well conditioned, leading to the following lemma.

Lemma 6.2 *If the MDOF system is observable, then the MAF has a unique solution and is well-conditioned.*

Note that this lemma considers sufficiency and not necessity; just because the system is unobservable does not mean that the MAF is ill-conditioned. Indeed, if the system is close to unobservable along a particular mode, the sensors will probably not extract that mode from the signal, meaning that the mode will not be present in the empirical data set.

7 Weighted Norm MAF

One obvious change to the MAF is a modification of the 2-norm to some weighted norm. We introduce the WMAF as

$$WMAF(\omega_e, x_e) = \min_y \left\| \begin{pmatrix} M\omega_e^2 + C\omega_e + K \\ H \end{pmatrix} y - \begin{pmatrix} 0 \\ x_e \end{pmatrix} \right\|_W^2 \tag{5}$$

Where the W norm of a vector is defined as

$$\|x\|_W = \|x^T W x\|_2$$

For some positive semi-definite weight matrix W. While numerous weighted MACs have been defined, the weighing matrix by definition is *restricted to only the measured modal entries*; the WMAF does not have this restriction and allows for a weighing matrix modifying *the full order system model*.

So W might reflect some a priori knowledge of the application or MDOF system, such as modal sensitivity, heterogenous sensing, or a known error covariance matrix. Some quick algebra thus reveals that

$$WMAF(\omega_e, x_e) = \min_y \left\| \sqrt{W} \begin{pmatrix} M\omega_e^2 + C\omega_e + K \\ H \end{pmatrix} y - \sqrt{W} \begin{pmatrix} 0 \\ x_e \end{pmatrix} \right\|_2^2 \tag{6}$$

7.1 Weighting for Balancing

One simple example of a useful weighing matrix is

$$\sqrt{W} = \begin{pmatrix} \gamma I_n & 0 \\ 0 & I_m \end{pmatrix}$$

for some arbitrary scalar γ, where I_m and I_n are $m \times m$ and $n \times n$ identity matrices, respectively. This weighing matrix is equivalent to the regularized least squares problem

$$\min_y \ \|\gamma(M\omega_e^2 + C\omega_e + K)y\|_2^2 + \|Hy - x_e\|_2^2$$

Practically speaking,the entries of M, C, and K will likely be quite large relative to x_e, depending on the units used. M, C, and K might contain entries on the order of 10^9 while the entries of x_e will all be less than one. We would like perturbations of the MAF to be equally distributed between each summand in the objective function. More formally, assume that have a backward error of ϵ i.e. the difference between the true solution y and the computed solution $\hat{y}$ is

$$y - \hat{y} = \epsilon$$

If we propagate the error through the objective function, we obtain the formula

$$\|\gamma(M\omega_e^2 + C\omega_e + K)\epsilon\|_2^2 + \|H\epsilon\|_2^2$$

Clearly, the units of the first summand and the second are entire incomparable; a small error in the first summand will may greatly outweigh a large error in the second (or vice versa). We want to make sure that our ϵ solution error will lead to a comparable error when propagated through the objective function. In this respect, picking a γ to properly *balance* the summands will significantly increase the accuracy of the MAF. For generic ϵ, we would like both summands to be equally weighted. One straightforward choice is pick γ such that $\gamma(M\omega_e^2 + C\omega_e + K)$ has the same matrix norm as H i.e.

$$\|\gamma(M\omega_e^2 + C\omega_e + K)\|_2 = \|H\|_2$$

Which yields the solution

$$\gamma = \frac{\sigma_1(H)}{\sigma_1(M\omega_e^2 + C\omega_e + K)}$$

If H is a subset of the identity matrix, it will have unit norm and so we want to set γ equal to the reciprocal of the $\left\|\gamma(M\omega_e^2 + C\omega_e + K)\right\|_2$ i.e.

$$\gamma = \frac{1}{\sigma_1(M\omega_e^2 + C\omega_e + K)}$$

7.2 Weighing for Validation

In the context of model validation, one might only care if a particular subset of the full system model matches measured modes. For example modeling the truss of the bridge is far more delicate than modeling the base —the matching DOF of the truss should be more important than matching DOF of the base, especially if sensors are concentrated more heavily in the truss. However, both the truss and base are linked at numerous points; separation of the two either involves heavy approximation or proprietary calculation. Therefore, in the context of mode-to-mode based correlation, one cannot separately calculate correlation for the DOF associated with the truss and the DOF associated with the base; the dynamics both DOF will be linked together by the entire system. More plainly, the MAC and variants cannot validate a model which is only locally or approximately incorrect, as the global mode shape is affected by local perturbations.

However, the MAF allows one to do so. As mentioned before, the MAF is a mode-to-model comparison, and allows for a weighting at the full system order. Let us consider the index set of the full system model

$$\Omega = \{1, 2, \ldots, n\}$$

And with each DOF $i \in \Omega$, we give an importance assignment γ_i. The smaller γ_i is, the less important that DOF is. In the extreme case, if $\gamma_i = 0$, that DOF does not matter. Let

$$D = diag(\gamma_1, \gamma_2, \ldots \gamma_n) = \begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_n \end{pmatrix}$$

i.e. D is a diagonal whose i-th entry is γ_i. Assume we have some model error (aggregated into a single matrix) ΔJ. Then we note that

$$D(M\omega_e^2 + C\omega_e + K + \Delta J)v = D(M\omega_e^2 + C\omega_e + K)v + D\Delta Jv = D\Delta Jv = D\hat{v}$$

$$\hat{v} = \Delta Jv$$

Then the vector $D\hat{v}$ will be weighted by the prior importance assignment; DOF associated with larger γ_i will be weighted more heavily, and vice versa. In the extreme case, entries with γ_i equal to zero will not contribute at all. We can apply this idea to the MAF; when weighing the objective function with $\sqrt{W}$ where

$$\sqrt{W} = \begin{pmatrix} D & 0 \\ 0 & I_m \end{pmatrix}$$

D will effectively weight each degree of freedom separately with γ_i. Setting $\gamma_i = 0$ indicates that the i-th DOF is inconsequential when calculating the MAF.

7.3 Robust MAF for Outlier Detection

The residual of the MAF represents how well the fitted mode matches the equations associated with each DOF. The smaller the residual along that index, the better the match. In this light, the residual has *interpretability* that we seek to take advantage of. First, we note that the actual RMAF value no longer is meaningful as a correlation measure; instead, we hope that computation of the RMAF allows us to recover the precise error sites. We can make the MAF robust to outliers by using a nonlinear loss function. Two relevant loss functions are the Huber Loss $\phi(u)$ and the Tukey Biweight Loss $\rho(u)$, defined as

$$\phi(u) = \begin{cases} u^2 & |u| \leq M \\ M(2|u| - M) & |u| > M \end{cases} \tag{7}$$

$$\rho(u) = \begin{cases} M & |u| \leq M \\ (1 - M^2)^2 & |u| > M \end{cases} \tag{8}$$

The Huber Loss is essentially the $L2$ norm inside some region $[-M, M]$ and the $L1$ norm outside, which makes it more robust to outliers (which may occur in the context of MDOF systems if a particular subset of modal indices are very sensitive). Similarly, the Tukey Loss is essentially the $L2$ norm inside some region $[-M, M]$ and constant everywhere else. The Huber and Tukey loss functions weigh outliers "less" than the traditional $L2$ norm. Then we may define the RMAF as (without loss of generality, picking the Huber Loss):

$$\text{RMAF} = \min_{y} \ \phi(z) \tag{9}$$

$$z = \begin{pmatrix} M\omega_e^2 + C\omega_e + K \\ H \end{pmatrix} y - \begin{pmatrix} 0 \\ x_e \end{pmatrix}$$

Where $\phi(z)$ for some vector z is simply ϕ applied entry-wise to the vector. Each index of z represents how well y matches up with the equations associated with the i-th DOF. In the traditional $L2$ sense, the mismatch will be squared. However, using a robust loss function penalizes mismatches less heavily.

8 Numeric Experiments

In this section, we provide a set of simple numeric examples with finite element models pulled from Matrix Market [20], with no damping matrix provided. These numeric examples are meant to be a proof-of-concept rather than a set of exhaustive tests. As with before, keep in mind that the MAF scales in the opposite direction of the MAC; a lower score is better and and a higher score is worse.

In all subsequent examples, we will consider matrices drawn from the Harwell-Boeing test collection. For computational convenience, we consider matrices of "medium" size in MATLAB format. All the following plots may be reproduced from the code in [21]. We assume that sensors are placed randomly throughout the structure. In all examples, we have first balanced the objective function using the weighted norm described in Section (TODO)

8.1 Performance of the Basic MAF

The first thing we want to examine is the degregation of the MAF with respect to noise in the empirical mode (we assume that the frequency information is correct). We add gaussian noise of increasing variance to the empirical mode. The variance along each modal entry will be proportional to the magnitude of that entry.

We also look at performance of the MAF with respect to modeling error in the mass and stiffness matrices.

8.2 Weighted MAF

For experimentation, we look at a model locally perturbed by along the dominant entries of the stiffness matrix, representing a mismodeling that one would like to ignore. Our problem size was 420 by 420, and 50 dominant DOF sites of K were perturbed by gaussian noise of variance 0.5 (normalized to entry size). We assume that the location of the DOF error sites is known a-priori. To compensate for this mismodeling, the γ values associated with each perturbed entry is set to 0.05, and the γ values associated with nonperturbed entries is set to 1.

We then apply the WMAF and the MAF and plot each residual. We note that the value of the WMAF as calculated was 0.0263 indicating high correlation, whereas the MAF had a much larger value of 0.6471 indicating low correlation.

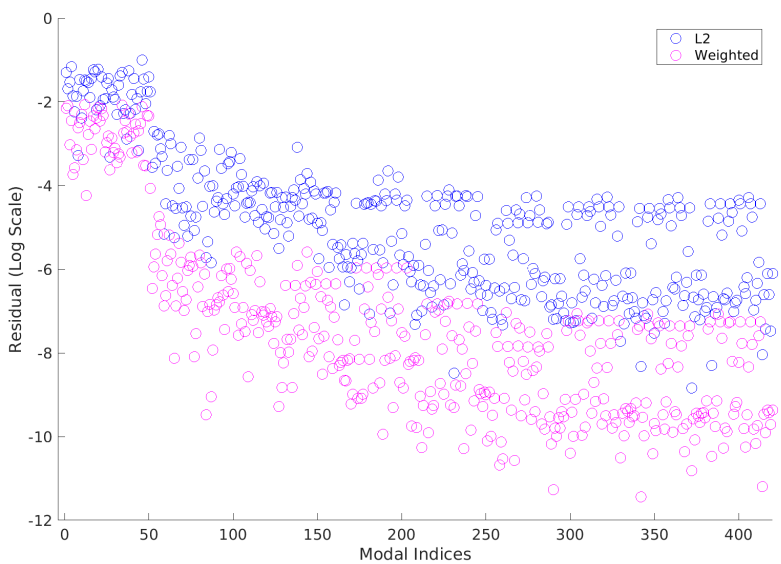

Figure 1: The true DOF error sites are plotted in red crosses (the residual of the non-error sites is close to machine precision). Using the Tukey Loss successfully recovers the error sites, but using the Huber Loss does not. The L2 loss naturally is also unable to recover the error sites.

8.3 Robust MAF

For experimentation, we look at a model locally perturbed by along the dominant entries of the stiffness matrix. Our problem size was 420 by 420, and 5 dominant DOF sites of K were perturbed by gaussian noise of variance 0.5 (normalized to entry size). However, unlike the previous case, the DOF error sites are not known a-priori. Using MALTAB's robust regression toolbox, we then run the RMAF using both the Huber and Tukey loss functions, with default parameters, to see if these localized model errors may be recovered. For comparison, the L2 loss is also used.

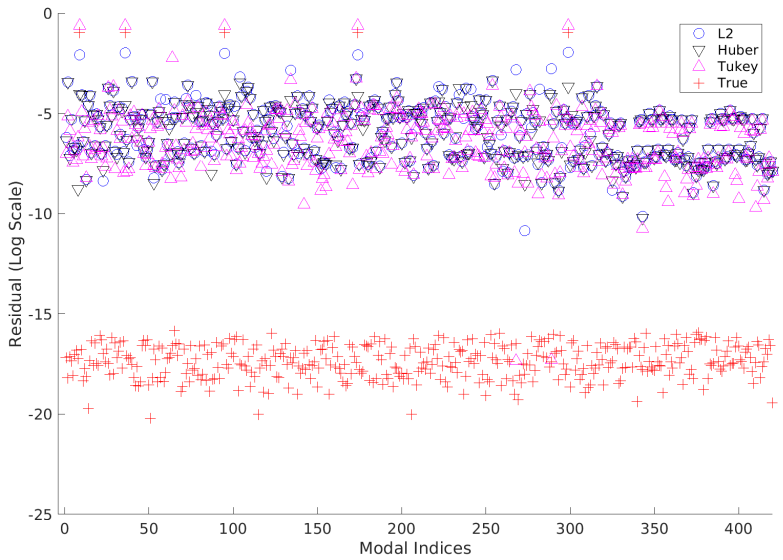

Figure 2: The true DOF error sites are plotted in red crosses (the residual of the non-error sites is close to machine precision). Using the Tukey Loss successfully recovers the error sites, but using the Huber Loss does not. The L2 loss naturally is also unable to recover the error sites.

First, we note that the actual RMAF value no longer is meaningful as a correlation measure; instead, we hope that computation of the RMAF allows us to recover the precise error sites (in this case, the RMAF values for both Huber and Tukey was greater than one).

In red crosses, the "true" residual is plotted using the unperturbed model; this isolates the five DOF error sites, which are the locations where the true residual is far greater than machine precision. The residuals of the L2, Huber, and Tukey solutions are also plotted. Like the true residual, the locations where the residual is far greater than its neighbors represents candidate DOF error locations. We see that using Tukey loss allows us to successfully locate the error sites. However, both the Huber and L2 functions are unsuccessful. In practice, one must be careful when choosing the Huber and Tukey parameters.

9 Conclusions

We have introduced the Modal Assurance Fit (MAF) as an alternative to the Modal Assurance Criterion (MAC). The MAF is a mode-to-model comparison characterized by a least-squares fit

of an observed mode to an analytical MDOF model. Not only does the MAF circumvent MDOF model reduction and mode matching, but it is also is simple and cheap to compute, strictly more powerful than the MAC, and easily modifiable.

As a final note, we have successfully applied the MAF in non-MDOF application involving model validation of Power Systems; A Power System yields the same formulation as many MDOF systems, but with nonsymmetric M and K matrices.

References

[1] Françoise Tisseur and Karl Meerbergen. The quadratic eigenvalue problem. *SIAM review*, 43(2):235–286, 2001.

[2] University of Colorado ASEN 3112 Lecture 22. Modal analysis of mdof forced damped systems.

[3] Keith Worden and AP Burrows. Optimal sensor placement for fault detection. *Engineering structures*, 23(8):885–901, 2001.

[4] Michael I Friswell and Daniel J Inman. Sensor validation for smart structures. *Journal of intelligent material systems and structures*, 10(12):973–982, 1999.

[5] Marek Krawczuk and Wiesław Ostachowicz. Damage indicators for diagnostic of fatigue cracks in structures by vibration measurements–a survey. *Journal of Theoretical and Applied Mechanics*, 34(2):307–326, 1996.

[6] Bijaya Jaishi and Wei-Xin Ren. Structural finite element model updating using ambient vibration test results. *Journal of Structural Engineering*, 131(4):617–628, 2005.

[7] JE Mottershead and MI Friswell. Model updating in structural dynamics: a survey. *Journal of sound and vibration*, 167(2):347–375, 1993.

[8] Michael Friswell and John E Mottershead. *Finite element model updating in structural dynamics*, volume 38. Springer Science & Business Media, 1995.

[9] AK Pandey, M Biswas, and MM Samman. Damage detection from changes in curvature mode shapes. *Journal of sound and vibration*, 145(2):321–332, 1991.

[10] Walter M West. Illustration of the use of modal assurance criterion to detect structural changes in an orbiter test specimen. *NASA*, 1986.

[11] Randall J Allemang. The modal assurance criterion–twenty years of use and abuse. *Sound and vibration*, 37(8):14–23, 2003.

[12] Miroslav Pastor, Michal Binda, and Tomáš Harčarik. Modal assurance criterion. *Procedia Engineering*, 48:543–548, 2012.

[13] Rune Brincker and Manuel Lopez-Aenlle. Mode shape sensitivity of two closely spaced eigenvalues. *Journal of Sound and Vibration*, 334:377–387, 2015.

[14] W D'AMBROGIO and A Fregolent. Higher-order mac for the correlation of close and multiple modes. *Mechanical systems and signal processing*, 17(3):599–610, 2003.

[15] D Fotsch and DJ Ewins. Further applications of the fmac. *ROLLS ROYCE PLC-REPORT-PNR*, 2001.

[16] Rodrigo Pascual, Jean-Claude Golinval, and Mario Razeto. A frequency domain correlation technique for model correlation and updating. *Sound and vibration*, 1997.

[17] MLJ Hautus. Stabilization controllability and observability of linear autonomous systems. In *Indagationes mathematicae (proceedings)*, volume 73, pages 448–455. Elsevier, 1970.

[18] Vasile Mihai Popov and Radu Georgescu. Hyperstability of control systems. 1973.

[19] Gene H Golub and Charles F Van Loan. Matrix computations. 3, 2012.

[20] Ronald F Boisvert, Roldan Pozo, Karin Remington, Richard F Barrett, and Jack J Dongarra. Matrix market: a web resource for test matrix collections. In *Quality of Numerical Software*, pages 125–137. Springer, 1997.

[21] https://github.com/ericlee0803/mafandmac.