Embodying Arithmetic: Counting on Your Hands and Feet

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Ontology Recapitulates Physiology.

- Epigraphy

The facts of the human body, evolutionary, physiological, social, and systemic, as experienced and symbolized (often differently) by every human, necessarily set the standards, and the limits, of human minds, and of theories about them. Following Turner (1994)[1], who conjures the ghost of Plato conjuring the ghost of Socrates conjuring the ghost of Protagoras, we may call (1) **Protagoras's Principle**:

1)a Panton metron anthropos.
  b Man is the measure of all things. (**Protagoras**)

Other linguists have come to this conclusion, as exemplified by what we will call **Ross's Rule**:

2) Me First. (**Ross**)

And so have other philosophers. Mark Johnson put it this way in *The Body in the Mind*:[2]

> "The body has been ignored because reason has been thought to be abstract and transcendent, that is, not tied to any of the bodily aspects of human understanding ... our bodily movements and interactions in various physical domains of experience are structured ... and that structure can be projected by metaphor onto abstract domains." (1987:xv)

One need not be a linguist or a philosopher to know this. Delmore Schwartz sings of

3)a "the withness of the body"
  b The heavy bear who goes with me (**Schwartz**)

That is, human sensorimotor images and programs (and, as Schwartz laments, their associated appetites) are universal and unavoidable in human meaning, and therefore we should not be *too* surprised when we encounter them, or their traces, in unexpected cognitive venues. This is one of the basic assumptions of cognitive linguistics (e.g, Lakoff 1987[3], Langacker 1987[4]).

One of the more unexpected cognitive venues in which we have encountered traces of the Heavy Bear is mathematics. Recent work by Lakoff and Núñez[5] (1997; hereinafter L&N) has examined the basic metaphor themes of arithmetic reasoning, which lie behind, or below, or at the center, or beginning, of Mathematics – depending on one's metaphoric viewpoint.
As they put it,

4) "Our mathematical conceptual system, like the rest of our conceptual systems, is grounded … in our sensorimotor functioning in the world, in our very bodily experiences." (L&N 1997:10)

What does mathematics have to do with linguistics? The number of linguists who began in mathematics shows that this question needs no answer. But what does linguistics have to do with mathematics? In this paper, we demonstrate, by first outlining briefly the structure of these metaphors, after L&N, then elaborating on certain problems that arise in applying them, sketching a solution relating these to human sensorimotor interaction with numbers and spatial perception; with Johnson, (1987:112; emphasis in original) we would argue that these do not simply represent the " 'application' of a model of independently existing knowledge," but rather that "such *models* ***constitute*** an individual's understanding of a phenomenon and thereby influence their acts of inference." We conclude with some remarks on the implications of all this for semantics, as well as for math appreciation.

"We claim that mathematics is no more part of the external, mind-free world than color is. Both are aspects — very different aspects — of the embodied mind." (L&N p.5)

♦ Overview of the Lakoff and Núñez account

The three basic metaphors of arithmetic

L&N (p.12-14) propose three image schemas to underlie the cognitive metaphors of arithmetic. These are (see appendix for details):

5) i Arithmetic Is Object Collection
ii Arithmetic Is Object Construction
iii Arithmetic is Motion

Examples of each are:

6) a There are 4 fives *in* 20. (i)
b 3 plus 4 *makes* 7. (ii)
c Count *backwards from* 20. (iii)

The first two themes (Object Collection and Construction, which we will call the "Object/Manipulation" metaphors) are strongly coherent, inheriting structure from a common, more schematic, theme, which L&N give as

- Numbers Are Physical Objects
- Arithmetic Is Object Manipulation
- Adding Is Putting Objects Together

We note in passing, though L&N do not, that the **hands** are strongly implicated in both these themes; i.e, *manipulation* is embodied. We will return to this topic presently.

The third ("Motion") theme, however, is quite different. It is **not** coherent with the first two, and requires different grounding. Furthermore, this theme is far more useful in other, more complex varieties of mathematics; as L&N point out (p.14),

> "In the Object Collection and Object Construction metaphors, zero is not the same kind of thing as a number. It represents the absence of attributes — the absence of a collection or constructed object. It is only in the Motion metaphor that zero is the same kind of thing as a number — it is a location in space. … It should be clear that the Collection and Construction metaphors also work just for the natural numbers. Multiplication by zero, for example, is not defined. Nor are negative numbers, rational numbers, and the reals."

Rather than **manipulation**, the schematic description of this theme involves **ambulation**; i.e, it is not the **hands** but the **legs** and **feet** that are embodied here.

Collectors, constructors, and travelers

Among the theoretical apparatus that L&N provide for analyzing arithmetic with these three themes is the *mathematical agent*. As they put it (p.10; italics added – see handout Appendix),

> "A mathematical agent is a metaphorical idealized actor, that is, an idealized actor in the source domain of a metaphor characterizing some aspect of mathematics. For example, when addition is conceptualized as *putting* objects *in* a collection, the mathematical agent is the one *who* does the collecting. In this case, the agent does nothing but collect objects; we will call such an agent a Collector. Similarly, when addition is conceptualized as *taking steps* of a certain length in a certain direction, the one *who* does the moving is a metaphorical mathematical agent, and correspondingly, we call *him* a Traveler.
>
> "Mathematical agents, when they appear, have only the minimal essential features needed to perform the kind of action performed. Indeed, the properties of the agent are often so minimal that it is difficult or impossible to distinguish the agent from the action. For example, take a source domain when there is an agent that moves. All that can be mapped by a mathematical metaphor onto the target domain is that the agent moves. No particular qualities of an agent, like hair color or gender, can be mapped, because hair color and gender play no roles in the metaphors that ground our mathematical understanding."

Let us flesh out some of these "minimal essential features" of mathematical agents, which are fairly schematic as L&N give them. As L&N signal by the use of personal pronouns *who* and *him*, the prototypic mathematical agent is a human being. And such prototypic humans come equipped with prototypic bodies, which must have at least arms and legs in order "to perform the kind of action performed." Again, L&N have used metaphoric terminology that makes this clear: one *puts objects in a collection*, a job that requires hands, and one *takes steps*, which requires feet.

The Usual Suspects

On this view, the two manipulative themes (i) and (ii) and the ambulatory theme (iii) have parts that involve experientially-grounded actions of collecting, combining, and locomotion. These, in turn, involve several extremely basic metaphors, of which we will consider here the **Container** and the **Path** image schemas.[6] L&N explicitly claim movement requires a Path, and this is hardly surprising; as Turner (1994:11) points out, "A human being who lacked this image-schema of SOURCE-PATH-GOAL would be so incapacitated as to seem fundamentally deficient." Similarly, though less overtly, a collection requires a Container to delimit its bounds **as** a collection; this is especially true if it is a collection of real objects. These schemas are present in spirit in all concepts that deal with the continuous/discrete distinction; containers are by their nature discrete assemblages, while locomotion is the paradigm instance of continuity.

Linking and grounding metaphors

L&N distinguish two kinds of metaphoric links in their account of mathematics.

7) "While grounding metaphors allow us to ground our understanding of mathematics in familiar domains of experience, linking metaphors allow us to link one branch of mathematics to another. For example, when we metaphorically understand numbers as points on a line, we are linking arithmetic and geometry." (L&N 1997:12)

They refer to the Motion metaphor here, and we will return to that; but for now, we wish to move our attention to a different linking metaphor that L&N feel compelled to posit in all three of the basic themes, which will turn out to cause problems.

"Addition is commutative." (Tom Lehrer, *New Math*)

♦ Problems

Addition, according to any of these metaphors, is a fairly simple matter, not much more complex than counting; indeed, it is simple repetitive counting. In the Collection theme, one adds 3 and 4 by collecting 3 units, then collecting 4 more, and noting that one has accumulated 7 units. In the Combination theme, one accretes 3 and then accretes 4 more, and finds that there are 7 parts. In the Motion theme, one moves 3 units and then moves 4 units, and finds oneself 7 units from the starting point. In all of these cases, note that it can make no difference whether one does the 3-unit part before the 4-unit part or vice versa, as long as all the units are conceptually "the same". This follows naturally from the experiences of collecting, combining, or moving; and this natural property of addition, when it is present in a mathematical structure, is called **Commutativity**. Subtraction, by contrast with addition, is **not** commutative. What about multiplication? It turns out that multiplication **is** commutative, like addition. Does this also follow naturally? That depends on how it is conceived.

L&N define multiplication the same way in all three themes (pp 12-13):

8) "Multiplication Is The Repeated addition of collections / objects / quantities of the same size a given number of times."

And surely this is correct, in the sense that we are all familiar with it, we all learned it in school, and use it frequently in the usual phrase for multiplication, e.g, "four ***times*** three."

However, there is a problem with it. While the numbers are defined experientially in all three themes, by collecting, combining, or moving, the "number of times" an operation is performed is **not** the same kind of thing at all as the result of doing the operation. In other words, like zero in the first two themes, the number of times an action is performed is **not** naturally a number in any of these themes. L&N recognize this, and deal with it by positing a linking metaphor, different for each theme, but essentially cognate: (pp 12-14; italics added)

9) "The number of times an action is performed Is the
   a) *collection* formed by adding a unit for each performance of the action."
   b) *object* formed by adding a unit for each performance of the action."
   c) *location* reached by starting at the origin and taking one step for each performance of the action."

Unlike addition, this schematization of multiplication is **not** naturally commutative. That is, it provides no natural reason why, say 14×31 should be equal to 31×14. As it turns out, collecting 31 units 14 times **does** result in a collection the same size as that resulting from collecting 14 units 31 times, but the process of multiplication itself, as presented in L&N, requires keeping track of two quite different variables: one counting the **size** of the collection or combination formed, or the numerical **position** of the location reached – depending on the action; and a completely different one counting the **number of times** that action is performed. These require totally different countings; in computing, we would say they are in different *registers*. There seems no *prima facie* reason to suppose that interchanging these two quite distinct variables **should** produce the same result, any more than interchanging the variables in subtraction or division would. This can, of course, be handled by positing still more "linking metaphors", but this is unsatisfactory, since commutativity follows naturally in the case of addition, and one would prefer for it to fall out as well in the closely related case of multiplication.

Our solution to this dilemma relies on embodiment. It is now time to investigate hands and feet.

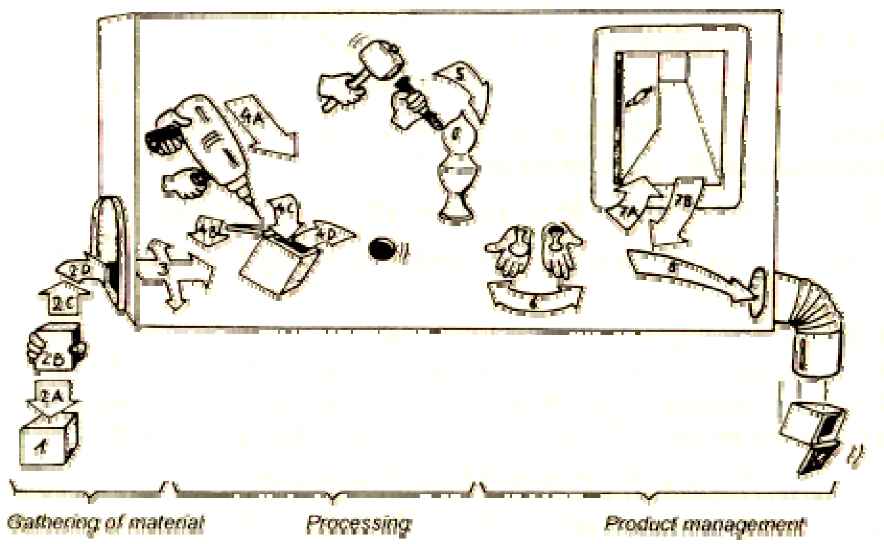


Figure 1. "The mental workshop": Overview of the ICM "Mental activity is manipulation"

First, hands. Figure 1, from Jäkel 1995,[7] exemplifies some of the parts of the metaphor themes involving symbolic manipulation, of which there are thousands of instantiations, e.g,

10) a He'll have to *come to grips with* that problem
b She's having trouble *grasping* the concept.
c Why did you *seize upon* that idea?
d Where did you *get hold of* such a silly solution?
e He *picked up* the idea that it was simple.
f I'm *collecting* examples of mathematical language.

This is a large and exceedingly complex schema; to put it in Blended Space terms, the hand is the Swiss Army Knife of body parts: when configured properly, it can hit, point, grasp, cup, move, catch, poke, break, etc. For our purposes, however, we need only open a few of the blades.

The Collector agent, in L&N's sense, uses a repetitive reach-and-grasp motion, certainly a hard-wired gesture, used by all primates at least for grooming, and for eating. This gesture moves the grasped item to some container, as a variant on the ordinary primate hand-to-mouth feeding motion; the mouth is, of course, one prototype Container, but items placed there, particularly edible items, are less likely to wind up counted; a more likely prototype collection Container is, simply, the other hand. Then the cycle repeats, and the count of the Collection, as represented in that Container register, increments.

In real experience, with real objects being Collected, there is more to it, of course; each object must be located by some sense, and in typical instances (say, "gathering" activities in a "hunting-and-gathering" culture, where our ancestors – some might say principally our *female* ancestors, though this is extremely speculative – surely spent the vast majority of human existence adapting themselves to getting good at this task), it is locating the objects that is the tricky part, and the reaching part of the motion will vary from instance to instance, while the rest is almost stylized, and in its most simple case is likely to require only one hand (especially if the other hand is the locus of the Collection container), unless the object is large, embedded, or uncooperative.

This account accords nicely with the fact that the limit of the number of objects that can be collected in one hand hovers around George Miller's "Magic Number 7±2," which is roughly the limit of subitized integer perception. Note also that **either** the collected objects **or** the repetitive actions that collect them can be thought of as what is being counted, since they are in one-to-one correspondence in the prototype case. This is a key datum, since it allows the counting procedure to be linked with repetitive motion, of which walking is another example, thus simplifying the linking between hands and feet. To summarize the embodiment of the agent in the Collection schema:

11) a The Collecting movement has four parts: *reach, grasp, move, release*.
 b The movement is repetitive.
 c The movement requires only one hand.
 d The movement terminates in some Container (e.g, the other hand).

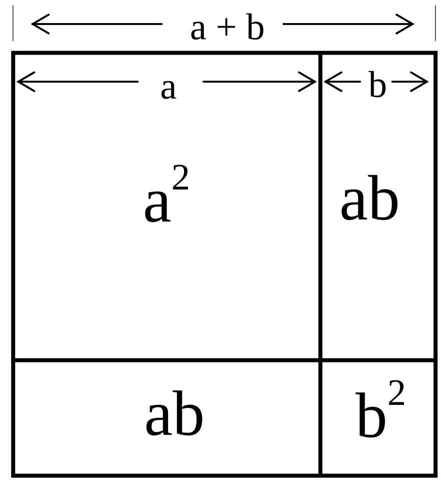
The Motion metaphor (iii), by contrast, does not use the hands, but rather the feet, and it uses two of them, which must not only be coordinated, but must also alternate, using the same stepping motion. That is, "steps" – which are what is to be counted – are of two types. What we might call $step_1$ refers to a single motion of **one** foot, while $step_2$ refers to one alternating cycle of **both** feet, each performing one $step_1$. Either kind of "step" can be what gets counted as the Traveler reaches a location on the Path, as long as it's done consistently.

The fact that hands and feet are evolutionarily homologous body parts contributes greatly to the linking metaphor between Object/Manipulation and Location/Ambulation. In both cases, it is a natural repetitive motion that is counted; in both cases, it is an extension of a limb, with flexure. However, the foot is considerably more limited than the hand in what it can do; it is no Swiss Army Knife, but a much more blunt instrument. It cannot collect or combine; it can barely grasp, and this marginal ability plays no part in its prototype motions. So, unlike Collection, in Ambulation there are no objects to be in 1-1 correspondence with the counted actions, and no Container to accumulate them in.

Instead, there is a series of successive discrete positions of the Traveler, as well as – crucially – a continuous Path that can be interpolated from them. It is abundantly clear when we walk that we are performing a series of alternating discrete motions, which are countable. But our personal perception of our locus on the Path traced by our footsteps is a continuous one, associated with our sense of perceived duration, which is of course also metaphorized quite commonly as motion, and **it** must be *measured*, approximately, rather than *counted*, exactly. The discrepancy between these two types of experiential locus, digital and analog, is the intuitive root of many ideas and concepts, from Zeno's paradox of Achilles and the Tortoise to the Continuum Hypothesis. In linguistics this is familiar as *count/mass*, or *punctual/continuous*, or a number of other distinctions.

The Motion metaphor, as L&N note, has a number of other features that befit it for mathematical concepts. Most relevantly for our purposes, it is **directed**. By cognizing subtraction as motion in a **reverse** direction, it licenses zero as a position (and therefore a number), and negative numbers as well, thereby closing the integers under subtraction in this metaphor, just as the manipulative metaphors allow the natural numbers to be closed under addition. By reference to the continuous interpolated Path, it also licenses fractions as intermediate positions on the Path, which form a rational number system closed under division.

And it solves the problem of the commutativity of multiplication by reference to **orthogonal** directed motion, and to the resulting geometric concept of **area**. Recall that multiplication requires keeping track of **two** independent counts; with a Collector as agent, this causes difficulties, but Travelers have more options. When a Traveler is moving on the Path, the direction of motion is conceptually a straight line, like the direction of vision, with only one dimension, of length. However, it is equally possible to think of the traversed Path as having a nominal **width** of one unit, representing the "footprint" of the body. For counting purposes, or for addition or subtraction, this is irrelevant; for multiplication, it is not. One can, for example, turn at right angles to the direction of motion, and initiate a new motion in that dimension. That possibility allows for **two** registers instead of merely one, and therefore multiplication, construed not as length, but as **area**. Of course, one cannot move in two directions simultaneously, but that is not necessary.



$$(a + b)^2 = a^2 + 2ab + b^2$$

Figure 2.

The requisite concept is simple: the Traveler steps out the first length, once, producing "one times N", then steps one unit to the side, steps out the same length in the same direction as before ("two times N"), moves another unit sideways, repeats, and so on. The steps reached in each traversal are in fact unit squares, and their sum, the total area, is the product of the two orthogonal Traveler motions. And, since our perception of area is unoriented, this provides a natural explanation of the commutativity of multiplication. As Figure 2 demonstrates, it doesn't matter which dimension is construed first; neither one need be "first" in any sense, and the area is the same either way. This orthogonal/area interpretation of the Traveler is more natural and less abstract than the one-dimensional line interpretation that L&N give, and, once again, refers not only to the "withness" of the body, but also its **width**. Interestingly, one finds symbolic evidence of this interpretation in the use of a "cross" (×) to represent multiplication.

Figure 2 also shows, rather dramatically, just why we use the word "squared" to refer to the Second Power (i.e, repeated multiplication), and why algebra and geometry are in principle exactly the same, another result that follows naturally from this interpretation.

Summing up the embodiment of the Traveler agent in the Motion metaphor:

12) a The movement involves either a single $step_1$ or a dual $step_2$.
b The movement is repetitive.
c The movement can be construed either as discrete or continuous.
d The movement requires both feet, operating in coordination, and involves a **width**.
e The movement is directional, with two orthogonal dimensions.
f The orthogonal Motion maps the accumulative function of the Container in (i-ii)

The abstract Agents that L&N propose for these themes are a good start, but, as Háj Ross has been pointing out for quite some time now, when one is in the business of investigating and describing human language and thought, and their metaphoric and cognitive bases, one omits true human elements at considerable risk. Embodiment is a natural way, one might even say **the** natural way, of delineating human metaphors, and human linguistics. The body is with us, like it or not. Even in arithmetic.

Notes

[1] Turner, Mark. 1994. "Design for a Theory of Meaning." In W. Overton and D. Palermo (eds.), *The Nature and Ontogenesis of Meaning*. Hillsdale, NJ: Erlbaum, pp 91-107.

[2] Johnson, Mark. 1987. *The Body in the Mind: The Bodily Basis of Meaning, Reason and Imagination.* Chicago: University of Chicago Press.

[3] Lakoff, George. 1987. *Women, Fire, and Dangerous Things: What Categories Reveal about the Mind.* Chicago: University of Chicago Press.

[4] Langacker, Ronald W. 1987. *Foundations of Cognitive Grammar*. Volume 1. Theoretical Prerequisites. Stanford: Stanford University Press.

[5] Lakoff, George, and Rafael Núñez. 1997. "The Metaphorical Structure of Mathematics: Sketching Out Cognitive Foundations For a Mind-Based Mathematics." In Lyn English (ed.), *Mathematical Reasoning: Analogies, Metaphors, and Images*. Hillsdale, NJ: Erlbaum. pp 1-54.

[6] There is also the **Part/Whole** schema, which is implicated in theme (ii), but we do not consider this here.

[7] Jäkel, Olaf. 1995. "The metaphorical concept of mind: 'Mental activity is manipulation'." In Taylor, J. and R. MacLaury (eds.), *Language and the Cognitive Construal of the World*. Mouton de Gruyter. pp 197-230.