Rounding Semidefinite Programming Hierarchies via Global Correlation

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joint work with

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Semidefinite Programming (SDP) Hierarchies

powerful algorithmic technique

Better approximations for combinatorial optimization problems?

Example: MAX CUT [Goemans-Williamson’94]

Unique Games Conjecture

strong implications for hardness of approximation

Is the conjecture true or false?

Constraint Satisfaction Problems (CSPs)

important class of optimization problems

What instances are hard / easy?

[e.g., works on dense or pseudo-dense instances]

[Sherali-Adams’90, Lovász-Schrijver’91, Lasserre’01]

[Khot’02,...]
**2-CSP**

*Input:* list of constraints on variable pairs \((x_i, x_j)\)

*Goal:* satisfy as many constraints as possible

**Example**

\[ x_i - x_j = c \mod k \]  
*(UNIQUE GAMES)*
2-Csp

**Input:** list of constraints on variable pairs \((x_i, x_j)\)

**Goal:** satisfy as many constraints as possible

**Main Result:** Algorithm for 2-Csp

**Approximation**

up to \(\varepsilon\) fraction satisfy maximum number of constraints

**Running Time**

exponential in \(\text{poly}(\varepsilon)\) and number of eigenvalues \(\geq \varepsilon'\)
in *constraint graph* (normalized adjacency matrix)

\textbf{Unique Games:} independent of alphabet
**Algorithm for 2-CSP**

- **approximation** up to $\varepsilon$ fraction satisfy maximum number of constraints
- **running time** exponential in $\text{poly}(\varepsilon)$ and number of eigenvalues $\geq \varepsilon'$ in *constraint graph* (normalized adjacency matrix)

- iterative rounding procedure for SDP hierarchies *(framework)*

- **QPTAS** for all *canonical gap instances* of *Max Cut* and *Unique Games*

- simpler *subexponential algorithm* for *Unique Games* [Arora, Barak, S.’10]

  *Do SDP hierarchies capture the true time complexity of *Unique Games*?*

- new *characterization* of higher eigenvalues of graphs (Poincaré-type inequality)
**Algorithm for 2-CSP**

*approximation* up to $\epsilon$ fraction satisfy maximum number of constraints

*running time* exponential in $\text{poly}(\epsilon)$ and number of eigenvalues $\geq \epsilon'$ in *constraint graph* (normalized adjacency matrix)

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Previous works

**2-CSP**  
dense or pseudo-dense instances  
[Arora-Karger-Karpinski’95, Fernandez de la Vega’96, …]

**Unique Games**  
alphabet-independent approximation, stronger assumptions about instances  
[Arora-Khot-Kolla-S.-Tulsiani-Vishnoi’08, Kolla-Tulsiani’07, Kolla’10]
Algorithm for 2-CSP

**approximation** up to $\varepsilon$ fraction satisfy maximum number of constraints

**running time** exponential in $\text{poly}(\varepsilon)$ and number of eigenvalues $\geq \varepsilon'$ in *constraint graph* (normalized adjacency matrix)

Subsequent works (using our framework)

better **3-COLORING** approximation on some graph families [Arora-Ge’11]

better approximations for **MAX-BISECTION** [Raghavendra-Tan’11]

Independent work [Guruswami-Sinop’11]

approximation schemes for quadratic integer programming with p.s.d. objective & few relevant eigenvalues
**Unique Games**

*Input:* list of constraints of form $x_i - x_j = c \mod k$

*Goal:* satisfy as many constraints as possible

*What we want*

$X_1, ..., X_n$ jointly distributed random variables over $[k]$

$\Pr(X_i - X_j \equiv c) \geq v_{OPT}$ for typical constraint $x_i - x_j \equiv c$
Goal: produce global random variables $X'_1, \ldots, X'_n$

$\{X'_i, X'_j\} \approx \{X_i, X_j\}$ for most constraints $x_i - x_j \equiv c$

$\Rightarrow$ iterative procedure
Components of iterative procedure

**Rounding**

sample variables independently according to their marginals

**Conditioning**

pick a vertex \( j \) and sample \( X_j \)
condition \( X_1, \ldots, X_n \) on sample for \( X_j \)

**Structure**

use / extract *structural* properties of instance
(Pairwise) Correlation

\[
\text{Corr}(X_i, X_j) \text{ measures how much the distribution of } X_i \\
\text{changes when conditioned on } X_j
\]

Examples

\[
\begin{align*}
\text{Corr}(X_i, X_j) &= 0 \iff X_i \text{ and } X_j \text{ independent} \\
\text{Corr}(X_i, X_j) &= 1 \iff X_i \text{ determined after fixing } X_j
\end{align*}
\]
Important fact

Can approximate $\text{Corr}(X_i, X_j)$ by inner products of unit vectors (tensoring trick [Khot-Vishnoi'05])

### Conditioning

- Statistical distance between $\{X_i, X_j\}$ and $\{X_i\}\{X_j\}$

$$\max_c \sum_a |\text{Cov}(X_{ia}, X_{j(a+c)})|$$

- Decrease in variance when conditioning on $X_j$

$$\sum_a (\text{Var}(X_{ia}) - \text{Var}(X_{ia} | X_j))$$

### Structure

Similar to mutual information

$$I(X; Y) = H(X) - H(X | Y)$$
<table>
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<th>Rounding</th>
<th>Conditioning</th>
<th>Structure</th>
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</table>
sample variables independently according to their marginals

If $\text{Corr}(X_i, X_j) \leq \epsilon$ then *independent sampling*
satisfies constraint with probability $\geq v_{\text{OPT}} - \epsilon$

$\text{Rounding}$ fails $\Rightarrow \mathbb{E}_{i \sim j} \text{Corr}(X_i, X_j) > \epsilon$

$\text{Corr}(X_i, X_j) \approx$ statistical distance between independent and correlated sampling

Local Correlation (over edges of constraint graph)
pick a vertex \( j \) and sample \( X_j \)
condition \( X_1, \ldots, X_n \) on sample for \( X_j \)

**Issue:**

*computationally expensive*

\( r \)-local \( \rightarrow (r - 1) \)-local

**Idea:**

condition on vertex \( j \) only if \( \mathbf{E}_{i} \text{Corr}(X_i, X_j) > \frac{1}{r} \)
\( \Rightarrow \) can condition at most \( r \) times on such vertices

**Conditioning fails** \( \Rightarrow \) \( \mathbf{E}_{i,j} \text{Corr}(X_i, X_j) < \frac{1}{r} \)

**Global Correlation**
(over random vertex pairs)
Local vs Global Correlation and Higher Eigenvalues

If constraint graph has less than \( \delta \cdot r \) eigenvalues \( \geq \varepsilon - \delta \), then always either \textit{rounding} or \textit{conditioning} succeeds

\[ E_{i,j} \text{Corr}(X_i, X_j) > \varepsilon \Rightarrow \text{constraint graph has at least } \delta \cdot r \text{ eigenvalues larger than } \varepsilon - \delta \]

\[ E_{i,j} \text{Corr}(X_i, X_j) < 1/r \]

\Rightarrow \text{ good approximation for UNIQUE GAMES (and 2-CSP) on such graphs}
Local vs Global Correlation and Higher Eigenvalues

\[ E_{i \sim j} \text{Corr}(X_i, X_j) > \varepsilon \]
\[ E_{i,j} \text{Corr}(X_i, X_j) < 1/r \]

⇒ constraint graph has at least \( \delta \cdot r \) eigenvalues larger than \( \varepsilon - \delta \)

“Proof”

approximate correlations by *inner products* of unit vectors

*high local correlation* ⇒ vectors correlated with top eigenvectors

*low global correlation* ⇒ “vectors live in many dimensions”

⇒ many top eigenvectors necessary to accommodate vectors
Rounding

fails only if *local correlation high*
\[ \mathbb{E}_{i \sim j} \text{Corr}(X_i, X_j) > \varepsilon \]

Conditioning

fails only if *global correlation low*
\[ \mathbb{E}_{i, j} \text{Corr}(X_i, X_j) < 1/r \]

Small-Set Expansion

Local vs Global Correlation and Higher Eigenvalues

[uses Arora-Barak-S.'10, S.'11]

Small-Set Expansion

\[ \mathbb{E}_{i \sim j} \text{Corr}(X_i, X_j) > \varepsilon \]
\[ \mathbb{E}_{i, j} \text{Corr}(X_i, X_j) < n^{-\beta} \]

\[ \Rightarrow \]

constraint graph contains set \( S \) with \( \leq n^{-\beta/2} \) fraction of vertices and \( \geq \varepsilon^{1/\beta} \) fraction of edges

Idea: round \( X_S \) and \( X_{V \setminus S} \) separately

\[ \Rightarrow \]
subexponential algorithm for UNIQUE GAMES
Open Questions

What else can be done in subexponential time?
Better approximations for \textsc{Max Cut}, \textsc{Vertex Cover} on general instances?

\textit{Example:} $f(\varepsilon)$-approximation for \textsc{Sparsest Cut} in time $\exp(n^\varepsilon)$?

Towards understanding the Unique Games Conjecture

integrality gap instances with $\text{poly}(n)$ large eigenvalues?


\exists \text{ gap instances with } \text{qpoly}(n) = 2^{(\log n)^{\Omega(1)}} \text{ large eigenvalues}

\textit{Also:} gap remains for $\text{qqpoly}(n)$ levels of a (weak) SDP hierarchy

\textbf{Thank you! Questions?}