First Note: Please make sure that you understand the proof of Theorem 34.1, especially the uniqueness part of the proof.

1. Prove or disprove the following statements.
   
   (a) For all integers $a, b$, we have $b \mid a$ iff $a \text{ div } b = \frac{a}{b}$.

   (b) For all integers $a, b$, we have $b \mid a$ iff $a \text{ mod } b = 0$.

2. Let $a, b, n \in \mathbb{Z}$ with $n > 0$. Prove that $a \equiv b \pmod{n}$ if and only if $a \text{ mod } n = b \text{ mod } n$.

3. Prove that the sum of any $k$ consecutive integers is divisible by $k$. 
4. Let $a$ and $b$ be positive integers. Find the sum of all the common divisors of $a$ and $b$.

5. If $n \in \mathbb{Z}^+$ and $n \geq 2$, prove that

$$\sum_{i=1}^{n-1} i \equiv \begin{cases} 0 \pmod{n}, & n \text{ odd} \\ \frac{n}{2} \pmod{n}, & n \text{ even} \end{cases}.$$

6. Prove that if $a$ and $b$ have a greatest common divisor, it is unique, i.e., they cannot have two (distinct) greatest common divisors.

7. Suppose $a, b \in \mathbb{Z}$ are relatively prime. Recall that there exist integers $x, y$ such that $ax + by = 1$. Prove that $\gcd(x, y) = 1$. 

8. (a) Let \( a, b, c \in \mathbb{Z} \). If \( a \mid bc \) and \( \gcd(a, b) = 1 \), prove that \( a \mid c \).

(b) Let \( p, q \in \mathbb{Z} \) be prime numbers and let \( a \in \mathbb{Z} \). Prove that \( p \mid a \) and \( q \mid a \) if and only if \( pq \mid a \).

(c) Let \( m, n \in \mathbb{Z} \) and \( p \) be a prime. Prove that if \( p \mid mn \), then \( p \mid m \) or \( p \mid n \). [Hint: Use Part 8(a).]

9. This problem is a continuation of Quiz 8. Let \( n \) be a positive integer and suppose \( a, b \in \mathbb{Z}_n \) are both invertible. Prove or disprove the following statements.

   (a) \( a \oplus b \) is invertible.

   (b) \( a \oplus b \) is invertible.
10. Find the multiplicative inverse of the following elements or state that none exists.

<table>
<thead>
<tr>
<th>(a) $2 \in \mathbb{Z}_{17}$</th>
<th>(c) $13 \in \mathbb{Z}_{1001}$</th>
<th>(e) $119 \in \mathbb{Z}_{1547}$</th>
<th>(g) $123 \in \mathbb{Z}_{4321}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b) $8 \in \mathbb{Z}_{17}$</td>
<td>(d) $101 \in \mathbb{Z}_{1001}$</td>
<td>(f) $121 \in \mathbb{Z}_{1547}$</td>
<td>(h) $447 \in \mathbb{Z}_{4321}$</td>
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</tbody>
</table>