Network Spectral Densities

David Bindel

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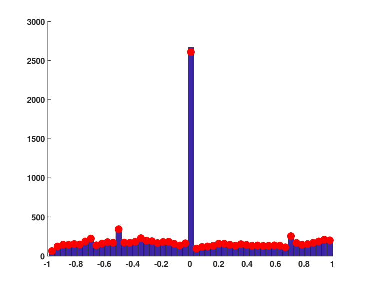
Department of Computer Science
Cornell University

Eigenvalues Two Ways

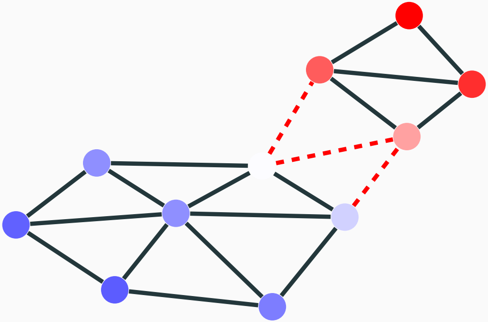
What can we tell from *partial* spectral information (eigenvalues and/or vectors)



Claim: Most spectral analyses involve one of two perspectives:

- Approximate something via a *few* (extreme) eigenvalues.
- Look at *all* the eigenvalues (or all in a range).

Spectral Graph Theory (and Algorithms)

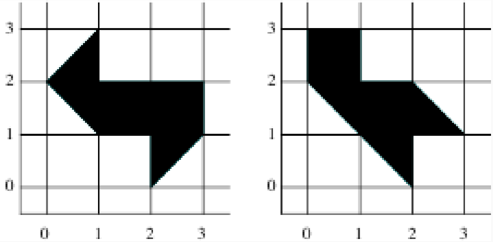


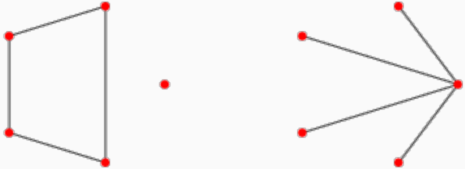
Lots about extreme eigenvalues:

- Cheeger's inequality
- Spectral partitioning and clustering
- Spectral embedding

What about distribution?

Can One Hear the Shape of a Drum?





"You mean, if you had perfect pitch could you find the shape of a drum." — Mark Kac (quoting Lipmann Bers)
American Math Monthly, *1966*

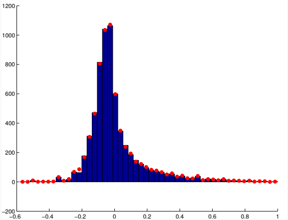
What Do You Hear?

What information hides in the eigenvalue distribution?

1. Discretizations of Laplacian: something like Weyl's law
2. Sparse E-R random graphs: Wigner semicircular law
3. Some other random graphs: Wigner semicircle + a bit (Farkas *et al*, Phys Rev E (64), 2001)
4. "Real" networks: less well understood

Goal: Explore by estimating eigenvalue distributions (fast).

Density of States

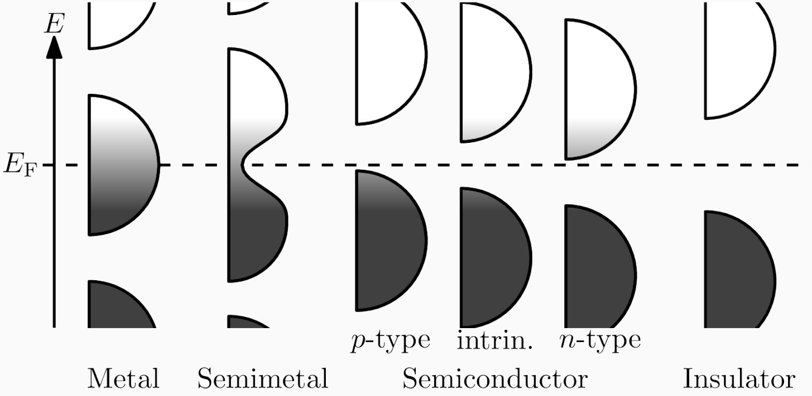


Spectra define a *generalized function* (a *density*):

$$\mathrm{tr}(f(H)) = \int f(\lambda)\mu(\lambda)\,dx = \sum_{k=1}^{N} f(\lambda_k)$$

where f is an analytic test function. Smooth to get a picture: a *spectral histogram* or *kernel density estimate*.

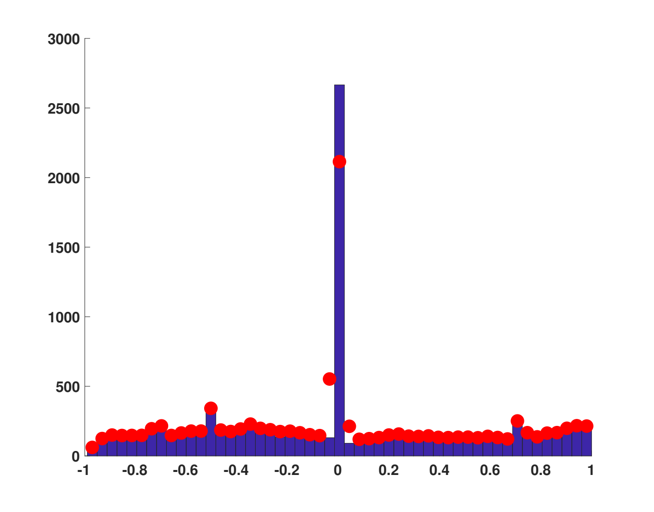
Stealing from Physicists



Kernel polynomial method (see Weisse, Rev. Modern Phys.):

What's different in the graph case?

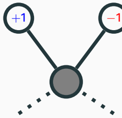
Example: PGP Network (Random Walk Matrix)

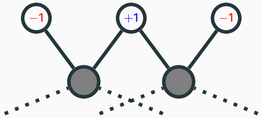


Spike (non-smoothness) at eigenvalues of 0 leads to inaccurate approximation.

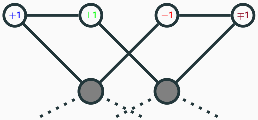
Spectral Spikes and Symmetry Groups

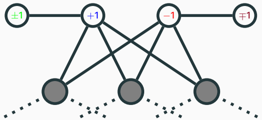
- $\lambda = 0$



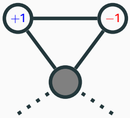


- $\lambda = \pm 1/2$

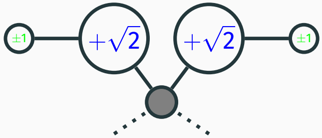




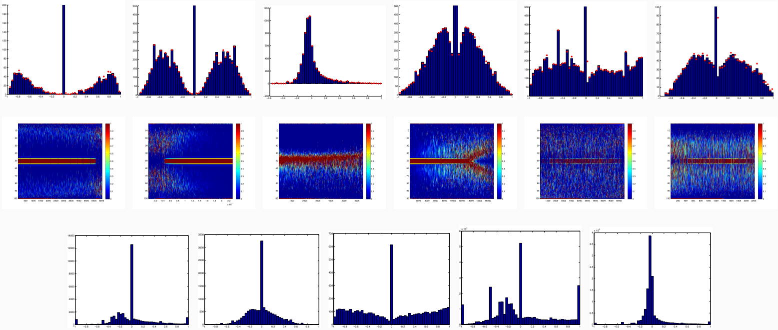
- $\lambda = -1/2$



$\lambda = \pm 1/\sqrt{2}$



What Do You Hear?



Latest:

- Dong, Benson, Bindel (KDD 2019).
- Longer talk at ILAS 2019 (slides online)