Dynamics via Nonlinear Pseudospectra

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The NEP Picture

\[ T(\lambda)v = 0, \quad v \neq 0. \]

where

- \( T: \Omega \to \mathbb{C}^{n \times n} \) analytic, \( \Omega \subset \mathbb{C} \) simply connected
- Regularity: \( \det(T) \neq 0 \)

Nonlinear spectrum: \( \Lambda(T) = \{ z \in \Omega : T(z) \text{ singular} \} \).

What do we want?

- Qualitative information (e.g. no eigenvalues in RHP)
- Error bounds on computed/estimated eigenvalues
- Control on all eigenvalues in some region

Why? Because of dynamics connections!
Why Eigenvalues?

\[ y' - Ay = 0 \quad \Rightarrow \quad y(t) = e^{\lambda t}v \]
\[ (\lambda I - A)v = 0 \]

\[ y_{k+1} - Ay_k = 0 \quad \Rightarrow \quad y_k = \lambda^k v \]
\[ (\lambda I - A)v = 0 \]

One standard use: analyze dynamics of LTI systems

- Special solutions characterizing full system
- General: linear combinations of special solutions
- \textit{Asymptotic} stability analysis and decay rates
Why Nonlinear Eigenvalues?

We want special solutions and asymptotic decay rates for

$$y'' + By' + Ky = 0 \quad \overset{y = e^{\lambda t} v}{\rightarrow} \quad (\lambda^2 I + \lambda B + K)v = 0$$

$$y' - Ay - By(t - 1) = 0 \quad \overset{y = e^{\lambda t} v}{\rightarrow} \quad (\lambda I - A - Be^{-\lambda})v = 0$$

$$T(d/dt)y = 0 \quad \overset{y = e^{\lambda t} v}{\rightarrow} \quad T(\lambda)v = 0$$

- Higher-order ODEs
- Delay differential equations
- Boundary integral equation eigenproblems
- Radiation boundary conditions
- Dynamic element formulations
\[ T(\omega)v \equiv (K - \omega^2 M + G(\omega)) \, v = 0 \]
Many real NEPs come from a decision to “hide” some state by dealing with it semi-analytically:

- Higher-order ODEs — hide extra derivatives
- Delay differential equations — hide lagged state (e.g. in delay lines)
- Boundary integral equation eigenproblems — hide domain unknowns
- Radiation boundary conditions — hide behavior outside computational domain
Ex: Second-order ODE and quadratic eigenvalue problem

\[ y'' + Dy' + Ky = 0 \quad \rightarrow \quad \frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -K & -D \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = 0 \]

\[ \lambda^2 y + \lambda Dy + Ky = 0 \quad \rightarrow \quad \lambda \begin{bmatrix} y \\ \lambda y \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -K & -D \end{bmatrix} \begin{bmatrix} y \\ \lambda y \end{bmatrix} = 0 \]

Trade nonlinearity vs size more generally:

\[ T \left( \frac{d}{dt} \right) y = 0 \quad \rightarrow \quad \frac{du}{dt} - Au = 0 \text{ and } y = Cu \]

\[ T(\lambda)y = 0 \quad \rightarrow \quad \lambda u - Au = 0 \text{ and } y = Cu \]

... but \( u \) may be infinite dimensional (e.g. DDE case).
Laplace transforms:

\[ T \left( \frac{d}{dt} \right) y = f \quad \rightarrow \quad T(z)Y(z) = F(z) + \text{I.C. terms} \]

\[ y(t) = \mathcal{L}^{-1}[Y](t) = \frac{1}{2\pi i} \int_{\Gamma} Y(z)e^{zt} \, dz \]

or first-order connection:

\[ T \left( \frac{d}{dt} \right) y = f \quad \rightarrow \quad \frac{du}{dt} - Au = Bf, \quad y = Cu \]

\[ y(t) = C \exp(tA)u_0 + \int_0^t \left[ C \exp((t - s)A)B \right] f(s) \, ds \]

But what do I do if I’m too lazy and ignorant to solve exactly?
Asymptotics

First approach:

- Observe $y(t) \sim \exp(\alpha t)$ where $\alpha \equiv \max_{\lambda \in \Lambda(T)} \text{Re}(\lambda)$.
- Bound $\alpha$ somehow.
- Go explore Valencia.

But this approach hides too much...
Beyond (Before?) Asymptotics

But this long run is a misleading guide to current affairs. In the long run we are all dead.

— John Maynard Keynes
A Tract on Monetary Reform (1923)
Consider a first-order problem:

\[ y' = Ay + f, \quad y(0) = y_0 \]

\[ y(t) = \exp(tA)y_0 + \int_0^t \exp((t - s)A)f(s) \, ds \]

Bounds if \( A = V\Lambda V^{-1} \) and \( \|f(t)\| \leq \gamma \):

\[ \| \exp(tA) \| = \| V \exp(t\Lambda)V^{-1} \| \leq \kappa(V) \exp(t\alpha) \]

\[ \|y(t)\| \leq \kappa(V) \left( \exp(t\alpha)\|y_0\| + \frac{\gamma}{-\alpha} (1 - \exp(t\alpha)) \right) \]

where \( \alpha = \max \Re(\lambda) \) is the spectral abscissa.
Simple bounds if $A = V\Lambda V^{-1}$

$$\| \exp(tA) \| = \| V \exp(t\Lambda)V^{-1} \| \leq \kappa(V) \exp(t\alpha)$$

where $\alpha = \max \text{Re}(\lambda)$. Nothing says $V$ need be well-conditioned!
General solutions to LTI problems via Laplace transforms

\[
(zl - A)^{-1} = \mathcal{L} \left[ e^{tA} \right] = \int_0^\infty e^{-zt}e^{tA} \, dt
\]

\[
\exp(tA) = \mathcal{L}^{-1} \left[ (zl - A)^{-1} \right] = \frac{1}{2\pi i} \int_\Gamma (zl - A)^{-1}e^{zt} \, dz
\]

for large enough \( \text{Re}(z) \) and for appropriate \( \Gamma \), e.g.:

- \( \Gamma \) a closed contour surrounding spectrum.
- \( \Gamma \) a vertical line to the right of the spectrum.
Begin from the contour integral representation:

\[
\exp(tA) = \frac{1}{2\pi i} \int_\Gamma (zl - A)^{-1} e^{zt} \, dz
\]

Convert bounds on resolvent to bounds on \( \exp(tA) \)

\[
\| \exp(tA) \| \leq \frac{1}{2\pi} \int_\Gamma \| (zl - A)^{-1} \| \, |e^{zt}| \, d\Gamma.
\]

We need “only” summarize how \( \| (zl - A)^{-1} \| \) behaves.
Summarize $\|(zI - A)^{-1}\|$ with

$$\Lambda_\epsilon(A) \equiv \{ z \in \mathbb{C} : \|(zI - A)^{-1}\| > \epsilon^{-1} \}$$

$$= \bigcup \{ \Lambda(A + E) \mid \|E\| < \epsilon \}$$

Pseudospectral abscissa is

$$\alpha_\epsilon(A) \equiv \max_{z \in \Lambda_\epsilon(A)} \Re(z)$$

[Trefethen and Embree, 2005]
Pseudospectral Bounds

Set \( \Gamma = \partial \Lambda_\epsilon(A) \) and \( L_\epsilon \) the length of \( \Gamma \). Then:

\[
\| \exp(tA) \| \leq \frac{1}{2\pi} \int_{\Gamma} \|(zl - A)^{-1}\| |e^{zt}| \, d\Gamma \leq \frac{L_\epsilon}{2\pi\epsilon} \exp(t\alpha_\epsilon).
\]

NB: If eigenvectors (columns of \( V \)) are normalized,

\[
\kappa(V) \leq \lim_{\epsilon \to 0} \frac{L_\epsilon}{2\pi\epsilon} = \sum_j \|V^{-1}e_j\| \leq \sqrt{n} \kappa(V)
\]

Can also get a lower bound: for any \( \omega \in \mathbb{R} \) and \( \epsilon > 0 \),

\[
\sup_{t \geq 0} \| \exp(-\omega t) \exp(tA) \| \geq \frac{\alpha_\epsilon - \omega}{\epsilon}.
\]
Approach: Exploit same Laplace transform pairing as before

\[ \exp(tA) \xrightarrow{\mathcal{L}} (zI - A)^{-1} \]

\[ \Psi(t) \xrightarrow{\mathcal{L}} T(z)^{-1} \]

Here \( \Psi(t) = C \exp(tA)B \) and \( T(z)^{-1} = C(zI - A)^{-1}B \).

As before, to control behavior of \( \Psi(t) \):

- Asymptotic stability / decay: look at spectral abscissa
- Pre-asymptotic: consider “resolvent” norm \( \|T(z)^{-1}\| \)
Summarize $\|T(z)^{-1}\|$ with

$$\Lambda_\epsilon(T) \equiv \{z \in \mathbb{C} : \|T(z)^{-1}\| > \epsilon^{-1}\}$$

$$= \bigcup_{\|E\|<\epsilon} \Lambda(T+E)$$

Pseudospectral abscissa

$$\alpha_\epsilon(T) \equiv \max_{z \in \Lambda_\epsilon(T)} \text{Re}(z)$$

[Bindel and Hood, 2015]
Aside: Comparing Pseudospectra

Suppose $T, \hat{T} : \Omega \to \mathbb{C}^{n \times n}$ and

$$\|T(z) - \hat{T}(z)\| \leq \eta, \quad \forall z \in \Omega.$$  

Then

$$\Lambda_\epsilon(T) \subset \Lambda_{\epsilon+\eta}(\hat{T}).$$

Can approximate $T \approx \hat{T}$ polynomial locally and bound pseudospectra (for example)... but usually won’t get all of $\mathbb{C}$.

Or use easier-to-compute sets (e.g. Gershgorin regions).
Set $\Gamma = \partial \Lambda_{\epsilon}(A)$ and $L_{\epsilon}$ the length of $\Gamma$. Then:

$$\|\psi(t)\| \leq \frac{1}{2\pi} \int_{\Gamma} \|T(z)^{-1}\| |e^{zt}| \, d\Gamma \leq \frac{L_{\epsilon}}{2\pi \epsilon} \exp(t\alpha_{\epsilon}).$$

But this may be useless (e.g. $L_{\epsilon} = \infty$) — need to be careful!

Can also get a lower bound: for any $\omega \in \mathbb{R}$ and $\epsilon > 0$,

$$\sup_{t \geq 0} \| \exp(-\omega t)\psi(t)\| \geq \frac{\alpha_{\epsilon} - \omega}{\epsilon}.$$
Example: Delay Differential Equation

DDE is

\[ u'(t) = Au(t) + Bu(t - \tau) \]

Characteristic function:

\[ T(z) = zI - A - Be^{-\tau z} \]

Assume \( A \) symmetric, \( \alpha(A) < 0 \), and \( \alpha(T) < 0 \).

Problem: Infinitely many eigenvalues! Have to be more clever.
Sketch of Approach

- Seek a simpler reference problem ($\hat{u}' = A\hat{u}$).
- Split into reference + difference term.
- Choose a congenial contour right of both spectra.
- Bound contour integral involving difference term.
Define $R(z) = (zI - A)^{-1}$; for proper choices of $\Gamma$,

$$\Psi(t) = \exp(tA) + \frac{1}{2\pi i} \int_{\Gamma} [T(z)^{-1} - R(z)] e^{zt} \, dz$$

Could choose difference reference (e.g. from a PEP).

Still need: Control of $\|T(z)^{-1} - R(z)\|$ on a contour.
Choose \( \Gamma \) right of \( \Lambda(T) \) and \( \Lambda(A) \) but in LHP:

\[
\Gamma = \Gamma_\infty \cup \Gamma_0 \\
\Gamma_\infty = \{x(y) + iy : |y| > y_0\} \\
x(y) = \frac{-1}{\tau} \log(|y|) \\
\Gamma_0 = \{x_0 + iy : |y| \leq y_0, x_0 = x(y_0)\}.
\]
Let $E(z) = T(z)^{-1} - R(z)$, contour as before:

$$\int_{\Gamma_0} \|E(z)\| |e^{zt}| d\Gamma \leq 2 \exp(x_0 t) \int_0^{y_0} \|E(x_0 + iy)\| dy$$

$$\int_{\Gamma_\infty} \|E(z)\| |e^{zt}| d\Gamma \leq \exp(x_0 t) \frac{C\tau}{t}$$

using boundedness of $\|E(z)\|$ on $\Gamma$ + curvature into RHP.

Bound:

$$\|\Psi(t)\| \leq \|\exp(tA)\| + e^{x_0 t} \left(l_0 + \frac{C\tau}{t}\right)$$
• Vertical contour loses $1/t$ factor in second term
• Drop $R$ (bigger constants, but faster decay)
• Probably many more options!
The other type of nonlinearity

Slightly nonlinear / time-varying problems? Simple case:

\[ \dot{x} = (A + E(x, t))x \]

where \( \|E\| \leq \epsilon \). Standard (?) approach:

1. Find \( M \) associated with quadratic Lyapunov function for \( A \):
   \[ AM + MA = -I. \]
2. Look at dynamics of \( x^T M x \) for \( A + E \) (pessimize w.r.t. \( E \)):
   \[ 2x^T M \dot{x} = -\|x\|^2 + 2x^T (ME) x \]
   \[ \leq -\|x\|^2 + 2\epsilon \|Mx\| \|x\| \]
3. Gronwall-type bound
   \[ \|x(t)\|_M \leq \exp \left( -\frac{t}{2} M^{-1} \|M^{-1}\| (1 - 2\epsilon \|M\|) \right) \|x(0)\|_M \]
Stability of slightly nonlinear / time-varying DDE, damped, etc:

- Consider structured real perturbations $E$
- Replace Lyapunov-style bounds with $\ell^2$ bounds via NLPS (or be more clever about RHS of Lyapunov equation?)

Still figuring this out — pointers welcome!
For both first-order systems and more complex problems:

- Eigenvalues describe asymptotic dynamics
- Pre-asymptotic behavior requires more information:
  - Complete eigendecomposition: Nice if you can get it.
  - Conditioning of $V$: A blunt tool for blunt bounds.
  - Pseudospectra, etc: A sharper tool for complex bounds.
- Pseudospectra alone don’t suffice — choices of contours, comparison problems, etc make a difference.