Nonlinear Eigenvalue Localization for Damping Bounds

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The NEP Picture

\[ T(\lambda)v = 0, \quad v \neq 0. \]

where

- \( T : \Omega \to \mathbb{C}^{n \times n} \) analytic, \( \Omega \subset \mathbb{C} \) simply connected
- Regularity: \( \det(T) \neq 0 \)

Nonlinear spectrum: \( \Lambda(T) = \{z \in \Omega : T(z) \text{ singular}\} \).

What do we want?

- Qualitative information (e.g. no eigenvalues in RHP)
- Error bounds on computed/estimated eigenvalues
- Knowledge of all the eigenvalues in some region

Why? Because of dynamics connections!
Why Eigenvalues?

\[ y' - Ay = 0 \quad \text{\(\rightarrow\)} \quad y(t) = e^{\lambda t} v \]

\[ y_{k+1} - Ay_k = 0 \quad \text{\(\rightarrow\)} \quad y_k = \lambda^k v \]

\[ (\lambda I - A)v = 0 \]

One standard use: analyze dynamics of LTI systems

- Special solutions characterizing full system
- General case: linear combinations of special solutions
- Asymptotic stability analysis and decay rates
Why Nonlinear Eigenvalues?

We want special solutions and asymptotic decay rates for

\[
y'' + By' + Ky = 0 \quad \overset{y=e^{\lambda t}v}{\rightarrow} \quad (\lambda^2 I + \lambda B + K)v = 0
\]

\[
y' - Ay - By(t - 1) = 0 \quad \overset{y=e^{\lambda t}v}{\rightarrow} \quad (\lambda I - A - Be^{-\lambda})v = 0
\]

\[
T(d/dt)y = 0 \quad \overset{y=e^{\lambda t}v}{\rightarrow} \quad T(\lambda)v = 0
\]

- Higher-order ODEs
- Delay differential equations
- Boundary integral equation eigenproblems
- Radiation boundary conditions
- Dynamic element formulations
One Motivation

\[
T(\omega)v \equiv (K - \omega^2 M + G(\omega)) v = 0
\]

Note: Getting a handle on \( G(\omega) \) is not simple!
Linearization: Meaning 1

\[
(K - \omega^2 M + sG(\omega)) \mathbf{v} = 0
\]

Usual trick: Differentiate, pre-multiply by row eigenvector:

\[
0 = (K - \omega_0^2 M)\mathbf{v}_0 = 0
\]

\[
0 = \mathbf{v}_0^T(-2\omega_0 \omega M + G(\omega_0))\mathbf{v} + \mathbf{v}_0^T(K - \omega_0^2 M) \dot{\mathbf{v}}
\]

\[
= \mathbf{v}_0^T(-2\omega_0 \omega M + G(\omega_0))\mathbf{v}
\]

A little algebra yields:

\[
\dot{\omega} = \frac{1}{2\omega_0} \frac{\mathbf{v}_0^T G(\omega_0) \mathbf{v}_0}{\mathbf{v}_0^T M \mathbf{v}_0} = \frac{\omega_0}{2} \frac{\mathbf{v}_0^T G(\omega_0) \mathbf{v}_0}{\mathbf{v}_0^T K \mathbf{v}_0}
\]
For “weak” damping, write

\[ Q \equiv \frac{1}{2} \frac{|\omega|}{\text{Im} \omega} \approx \frac{1}{2} \frac{\omega_0}{\text{Im} \dot{\omega}} = \frac{v_0^T K v_0}{v_0^T \text{Im}[G(\Omega_0)] v_0} \]

Basically looks like (mode-dependent) viscous damping:

\[
\text{Quality factor} = \frac{\text{Modal stiffness}}{\text{Modal damping}}
\]
Beyond weak damping

But this long run is a misleading guide to current affairs. In the long run we are all dead.

— John Maynard Keynes (1923)

- First-order perturbation no longer a good approximation.
- Need to go beyond single mode solutions.
- May need to consider more than just asymptotics.

How to reason about a more complete dynamics picture?
Many real NEPs come from a decision to “hide” some state by dealing with it semi-analytically:

- Higher-order ODEs — hide extra derivatives
- Delay differential equations — hide lagged state (e.g. in delay lines)
- Boundary integral equation eigenproblems — hide domain unknowns
- Radiation boundary conditions — hide behavior outside computational domain

If variables were not hidden, we would have a linear problem!
Ex: Second-order ODE and quadratic eigenvalue problem

\[ y'' + Dy' + Ky = 0 \]

\[ \lambda^2 y + \lambda Dy + Ky = 0 \]

Trade nonlinearity vs size more generally:

\[ T \left( \frac{d}{dt} \right) y = 0 \]

\[ T(\lambda)y = 0 \]

... but \( u \) may be infinite dimensional (e.g. DDE case).
Laplace transforms:

\[ T \left( \frac{d}{dt} \right) y = f \quad \rightarrow \quad T(z)Y(z) = F(z) + \text{I.C. terms} \]

\[ y(t) = \mathcal{L}^{-1}[Y](t) = \frac{1}{2\pi i} \int_{\Gamma} Y(z)e^{zt} \, dz \]

or first-order connection:

\[ T \left( \frac{d}{dt} \right) y = f \quad \rightarrow \quad \frac{du}{dt} - Au = Bf, \quad y = Cu \]

\[ y(t) = C \exp(tA)u_0 + \int_0^t [C \exp((t - s)A)B] f(s) \, ds \]

But what do I do if I’m too lazy and ignorant to solve exactly?
Bounds by Resolvent

Approach: Exploit Laplace transform pairs

\[
\exp(tA) \xrightarrow{\mathcal{L}} (zI - A)^{-1}
\]

\[
\Psi(t) \xrightarrow{\mathcal{L}} T(z)^{-1}
\]

Here \(\Psi(t) = C\exp(tA)B\) and \(T(z)^{-1} = C(zI - A)^{-1}B\).

Approximate and control \(T(z)\) to control \(\Psi(t)\):

- Asymptotic stability / decay: look at spectral abscissa
- Pre-asymptotic: consider “resolvent” norm \(\|T(z)^{-1}\|\)
Nonlinear Pseudospectra

Summarize $\|T(z)^{-1}\|$ with

$$\Lambda_\epsilon(T) \equiv \{ z \in \mathbb{C} : \|T(z)^{-1}\| > \epsilon^{-1} \} = \bigcup_{\|E\| < \epsilon} \Lambda(T + E)$$

Pseudospectral abscissa is

$$\alpha_\epsilon(T) \equiv \max_{z \in \Lambda_\epsilon(T)} \text{Re}(z)$$

[Bindel and Hood, 2015]
Comparing Pseudospectra

Suppose $T, \hat{T} : \Omega \rightarrow \mathbb{C}^{n \times n}$ and

$$\|T(z) - \hat{T}(z)\| \leq \eta, \quad \forall z \in \Omega.$$ 

Then

$$\Lambda_\epsilon(T) \subset \Lambda_{\epsilon + \eta}(\hat{T}).$$

Can say locally $T \approx \hat{T}$ (polynomial) and bound pseudospectra (for example)... but we usually won’t get all of $\mathbb{C}$.

Can also work with easier sets (e.g. Gershgorin regions).
Set $\Gamma = \partial \Lambda_\epsilon(A)$ and $L_\epsilon$ the length of $\Gamma$. Then:

$$\|\psi(t)\| \leq \frac{1}{2\pi} \int_\Gamma \|T(z)^{-1}\| |e^{zt}| \, d\Gamma \leq \frac{L_\epsilon}{2\pi \epsilon} \exp(t\alpha_\epsilon).$$

But this may be useless (e.g. $L_\epsilon = \infty$) — need to be careful!

Can also get a lower bound: for any $\omega \in \mathbb{R}$ and $\epsilon > 0$,

$$\sup_{t \geq 0} \| \exp(-\omega t)\psi(t)\| \geq \frac{\alpha_\epsilon - \omega}{\epsilon}.$$
Example: Delay Differential Equation

DDE is

\[ u'(t) = Au(t) + Bu(t - \tau) \]

Characteristic function:

\[ T(z) = zI - A - Be^{-\tau z} \]

Assume \( A \) symmetric, \( \alpha(A) < 0 \), and \( \alpha(T) < 0 \).

**Problem:** Infinitely many eigenvalues! Have to be more clever.
• Seek a simpler reference problem ($\hat{u}' = A\hat{u}$).
• Split into reference + difference term.
• Choose a congenial contour right of both spectra.
• Bound contour integral involving difference term.
Define $R(z) = (zl - A)^{-1}$; for proper choices of $\Gamma$, 

$$\Psi(t) = \exp(tA) + \frac{1}{2\pi i} \int_{\Gamma} [T(z)^{-1} - R(z)]e^{zt} \, dz$$

Could choose difference reference (e.g. characteristic fn for PEP).

Now we need a contour on which we can control $\|T(z)^{-1} - R(z)\|$. 
Choose $\Gamma$ right of $\Lambda(T)$ and $\Lambda(A)$ but in LHP:

$$\Gamma = \Gamma_\infty \cup \Gamma_0 \quad \Gamma_\infty = \{x(y) + iy : |y| > y_0\}$$

$$x(y) = -\frac{1}{\tau} \log (|y|) \quad \Gamma_0 = \{x_0 + iy : |y| \leq y_0, x_0 = x(y_0)\}.$$
Let $E(z) = T(z)^{-1} - R(z)$, contour as before:

\[
\int_{\Gamma_0} \|E(z)\| |e^{zt}| \, d\Gamma \leq 2 \exp(x_0 t) \int_0^{y_0} \|E(x_0 + iy)\| \, dy
\]

\[
\int_{\Gamma_\infty} \|E(z)\| |e^{zt}| \, d\Gamma \leq \exp(x_0 t) \frac{C\tau}{t}
\]

using boundedness of $\|E(z)\|$ on $\Gamma$ together with curvature into RHP.

**Bound:**

\[
\|\Psi(t)\| \leq \|\exp(tA)\| + e^{x_0 t} \left( l_0 + \frac{C\tau}{t} \right)
\]
• Could choose vertical contour (lose $1/t$ factor in second term)
• Drop $R$ (bigger constants, but faster decay)
• Probably many more options!
For both first-order systems and more complex problems:

- Eigenvalues describe asymptotic dynamics
- Pre-asymptotic behavior requires more information:
  - Complete eigendecomposition: Nice if you can get it.
  - Conditioning of $V$: A blunt tool for blunt bounds.
  - Pseudospectra and company: A sharper tool for complex bounds.
- Pseudospectra alone don’t suffice — choices of contours, comparison problems, etc make a difference.
References