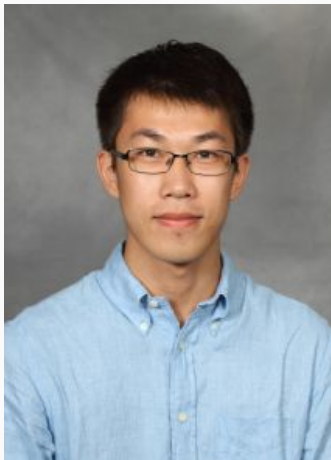


Understanding Graphs through Spectral Densities

13 Jun 2018

Department of Computer Science
Cornell University

Acknowledgements



Thanks **Kun Dong** (Cornell CAM), along with Anna Yesypenko, Moteleolu Onabajo, Jianqiu Wang.

Also: NSF DMS-1620038.

Can One Hear the Shape of a Drum?

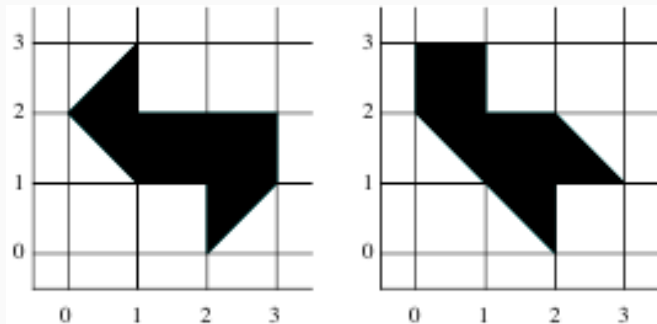
$$-\nabla^2 u = \lambda u \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

Assume that for each n the eigenvalue λ_n for Ω_1 is equal to the eigenvalue μ_n for Ω_2 . Question: Are the regions Ω_1 and Ω_2 congruent in the sense of Euclidean geometry?

I first heard the problem posed this way some ten years ago from Professor Bochner. Much more recently, when I mentioned it to Professor Bers, he said, almost at once: "You mean, if you had perfect pitch could you find the shape of a drum."

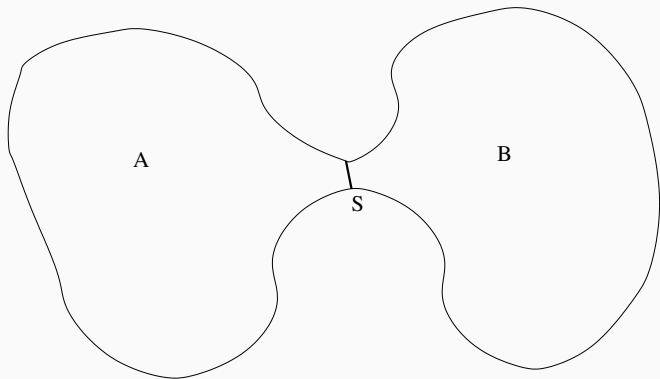
Mark Kac, American Math Monthly, 1966

Can One Hear the Shape of a Drum?



No in general (Gordon, Webb, Wolpert in 1992)
Yes with constraints (Zelditch in 2009)

What Do You Hear?



Size of bottlenecks (Cheeger inequality)

$$h \leq 2\sqrt{\lambda_2}$$

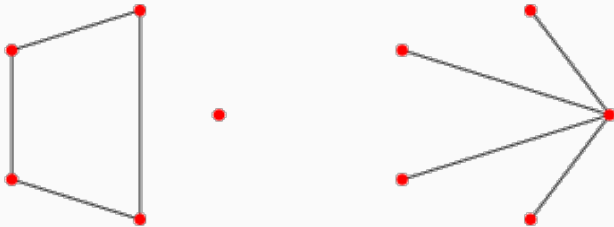
What Do You Hear?

Volume (Weyl law)

$$\lim_{x \rightarrow \infty} \frac{N(x)}{x^{d/2}} = (2\pi)^{-d} \omega_d \text{vol}(\Omega), \quad N(x) = \{\# \text{ eigenvalues } \leq x\}$$

Also: lengths of geodesics for a closed Riemannian manifold.

Can One Hear the Shape of a Graph?



From eigenvalues of adjacency, Laplacian, normalized Laplacian?

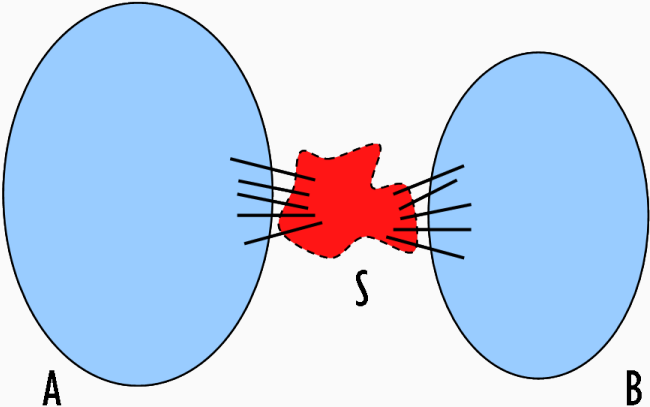
A Bestiary of Matrices

- Adjacency matrix: A
- Laplacian matrix: $L = D - A$
- Unsigned Laplacian: $L = D + A$
- **Random walk matrix:** $P = D^{-1}A$
- **Normalized adjacency:** $\bar{A} = D^{-1/2}AD^{-1/2}$
- **Normalized Laplacian:** $\bar{L} = I - \bar{A} = D^{-1/2}LD^{-1/2}$
- Modularity matrix: $B = A - \frac{dd^T}{2n}$
- Motif adjacency: $W = A^2 \odot A$

All have examples of co-spectral graphs

... through spectrum uniquely identifies *quantum graphs*

What Do You Hear?



Size of separators (Cheeger inequality) – L

What Do You Hear?

What information hides in the eigenvalue distribution?

1. Discretizations of Laplacian: something like Weyl's law
2. Sparse E-R random graphs: Wigner semicircular law
3. Some other random graphs: Wigner semicircle + a bit
(Farkas *et al*, Phys Rev E (64), 2001)
4. "Real" networks: less well understood

But computing all eigenvalues seems *expensive*!

Reminder: Spectral Mapping

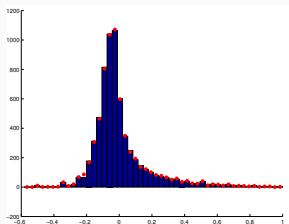
Consider a matrix H , and let f be analytic on the spectrum.

Then if $H = V\Lambda V^{-1}$,

$$f(H) = Vf(\Lambda)V^{-1}.$$

(generalizes to non-diagonalizable case)

Another Perspective: Density of States



Spectra define a *generalized function* (a *density*):

$$\mathrm{tr}(f(H)) = \int f(\lambda)\mu(\lambda) dx = \sum_{j=k}^N f(\lambda_k)$$

where f is an analytic test function. Smooth to get a picture: a *spectral histogram* or *kernel density estimate*.

Example: Estrada Index

Consider

$$\text{tr}(\exp(\alpha A)) = \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \cdot (\# \text{ closed random walks of length } k).$$

- Global measure of connectivity in a graph.
- Can clearly be computed via DoS.
- Generalizes to other weights.

DoS information equivalent to looking at the *heat kernel trace*:

$$h(s) = \text{tr}(\exp(-sH)) = \mathcal{L}[\mu](s)$$

where H is a positive semi-definite operator.

$H = L$ (continuous time random walk generator) \implies
 $h(s)/N = P(\text{self-return after time } s \text{ from uniform start}).$

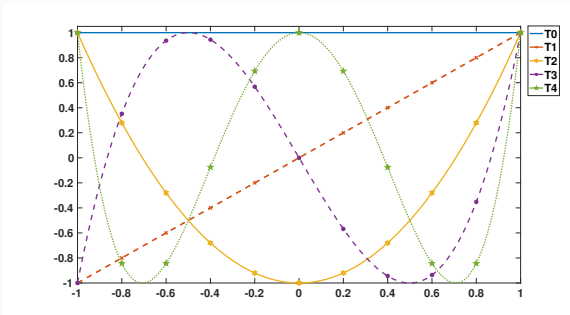
DoS information equivalent to looking at the *power moments*:

$$\text{tr}(H^j).$$

Natural interpretation for matrices associated with graphs:

- A : number of length k cycles.
- \bar{A} or P : return probability for k -step random walk (times N).
- L : ??

Chebyshev Moments



For numerics, prefer Chebyshev moments to power moments:

$$d_j = T_j(A)$$

where $T_j(z) = \cos(j \cos^{-1}(z))$ is the j th Chebyshev polynomial:

$$T_0(z) = 1, \quad T_1(z) = z, \quad T_{k+1}z = 2zT_k(z) - T_{k-1}(z).$$

Exploring Spectral Densities

Kernel polynomial method (see Weisse, Rev. Modern Phys.)

- Spectral distribution on $[-1, 1]$ is a generalized function:

$$\int_{-1}^1 \mu(x) f(x) dx = \frac{1}{N} \sum_{k=1}^N f(\lambda_k)$$

- Write $f(x) = \sum_{j=1}^{\infty} c_j T_j(x)$ and $\mu(x) = \sum_{j=1}^{\infty} d_j \phi_j(x)$, where $\int_{-1}^1 \phi_j(x) T_k(x) dx = \delta_{jk}$
- Estimate $d_j = \text{tr}(T_j(H))$ by stochastic methods
- Truncate series for $\mu(x)$ and filter (avoid Gibbs)

Much cheaper than computing all eigenvalues!

Related methods

Golub and Meurant: Gauss quadrature for $\mu(x)$ via Lanczos

- Good: Approximately twice the convergence rate
- Bad: Gives quadrature rule vs a picture

Roder and Silver: Max entropy estimation

- Good: Better resolution from Chebyshev moments
- Bad: More complicated / expensive computation

Under investigation: Hybrid approaches.

Stochastic Trace and Diagonal Estimation

$Z \in \mathbb{R}^n$ with independent entries, mean 0 and variance 1.

$$E[(Z \odot HZ)_i] = \sum_j h_{ij} E[Z_i Z_j] = h_{ii}$$

$$\text{Var}[(Z \odot HZ)_i] = \sum_j h_{ij}^2.$$

Serves as the basis for stochastic estimation of

- Trace (Hutchinson, others; review by Toledo and Avron)
- Diagonal (Bekas, Kokiopoulou, and Saad)

Independent probes $\implies 1/\sqrt{N}$ convergence (usual MC).

Beyond Independent Probes

For probes $Z = [Z_1, \dots, Z_s]$, have *exact* diagonal

$$d = [(A \odot ZZ^T)e] \oslash [(Z \odot Z)e]$$

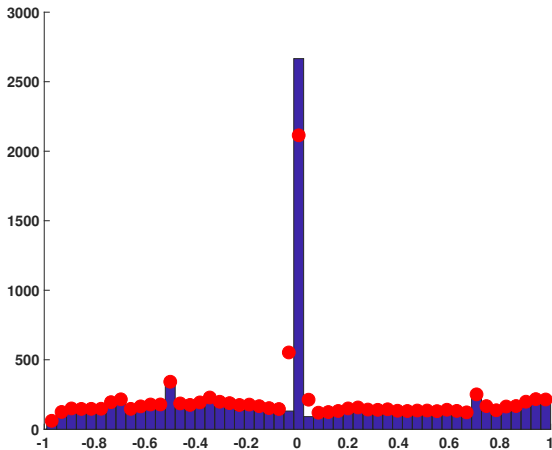
if $Z_{i,:} Z_{j,:}^T = 0$ whenever $A_{ij} \neq 0$.

Idea:

- Pick rows $\{Z_{i,:}\}$ such that $Z_{i,:} \perp Z_{j,:}$ whenever $A_{ij} \neq 0$
- A an adjacency matrix \implies graph coloring.

Combined with randomization, still gives unbiased estimates.

Example: PGP Network



Spike (non-smoothness) at eigenvalues of 0 leads to inaccurate approximation.

Motifs and Symmetry

Suppose $PH = HP$. Then

\mathcal{V} a max invariant subspace for $P \implies$

\mathcal{V} a max invariant subspace for H

So *local symmetry* \implies *localized eigenvectors*

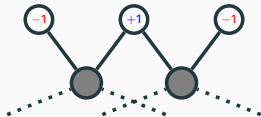
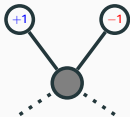
Suppose invariance under some symmetry group \mathcal{G} :

$$\forall P \in \mathcal{G}, \quad PH = HP$$

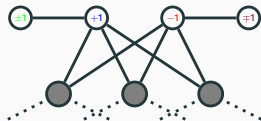
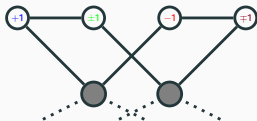
If \mathcal{G} non-Abelian ($P_1P_2 \neq P_2P_1$), must have multiple eigenvalues.

Motifs in Spectrum

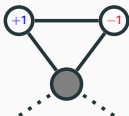
• $\lambda = 0$



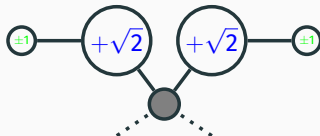
• $\lambda = \pm 1/2$



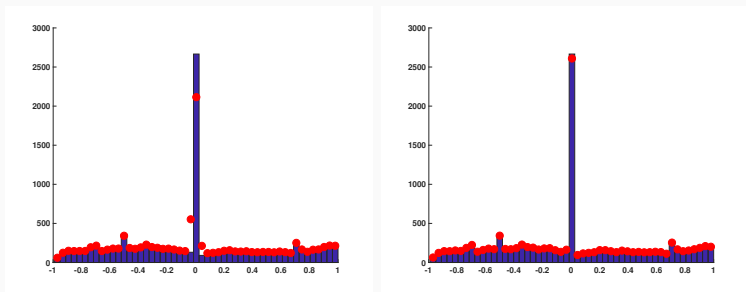
• $\lambda = -1/2$



$\lambda = \pm 1/\sqrt{2}$



Motif Filtering



Motif “spikes” slow convergence – deflate motif eigenvectors!
If $P \in \mathbb{R}^{n \times m}$ an orthonormal basis for the quotient space,

- Apply estimator to $P^T \bar{A} P$ to reduce size for $m \ll n$.
- or use $Proj_P(Z)$ to probe the desired subspace.

Diagonal Estimation and LDoS

Diagonal estimation also useful for *local* DoS $\nu_k(x)$;
in the symmetric case with $H = Q\Lambda Q^T$, have

$$\int f(x)\nu_k(x) dx = f(H)_{kk} = e_k^T Q f(\Lambda) Q^T e_k$$

$$\nu_k(x) = \sum_{j=1}^n q_{kj}^2 \delta(x - \lambda_j)$$

DoS is sum of local densities of states:

$$\mu(x) = \sum_{k=1}^n \nu_k(x)$$

- Write $f(x) = \sum_{j=1}^{\infty} c_j T_j(x)$ and $\nu_k(x) = \sum_{j=1}^{\infty} d_j \phi_j(x)$, where $\int_{-1}^1 \phi_j(x) T_k(x) dx = \delta_{jk}$
- Estimate $d_j = [T_j(H)]_{kk}$ by diag estimation
- Truncate series for $\mu(x)$ and filter (avoid Gibbs)

Diagonal estimator gives moments *for all k simultaneously!*

Other methods

Golub and Meurant: Gauss quadrature for $\nu_k(x)$ via Lanczos

- Good: No stochastic estimation error (vs KPM)
- Bad: Separate Lanczos *per node*

Roder and Silver: Max entropy estimation

- Good: Better resolution from Chebyshev moments
- Bad: More expensive computation per node

Under investigation: Hybrid approach.

Can compute common *centrality measures* with LDoS

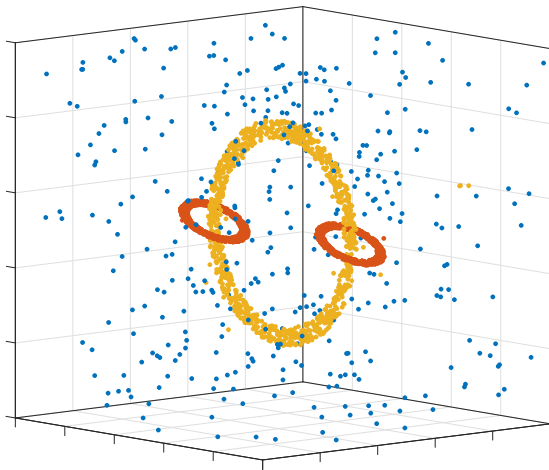
- Estrada centrality: $\exp(\gamma A)_{kk}$
- Resolvent centrality: $[(I - \gamma \bar{A})^{-1}]_{kk}$

Some motifs associated with localized eigenvectors:

- Chief example: Null vectors of \bar{A} supported on leaves.
- Use LDoS + topology to find motifs?

What else?

LDoS and Clustering



Phase Retrieval in Graph Reconstruction

Reconstruct graph from *fully resolved* LDoS at all nodes?

- Assume $H = Q\Lambda Q^T$
- No multiple eigenvalues \implies know $|Q|$ and Λ
- Can we recover signs in Q ?

Feels a little like phase retrieval...

Of course, we usually have noisy LDoS estimates!

Exploring Spectral Densities (with David Gleich)

- Compute spectrum of normalized Laplacian / RW matrix
- Compare KPM to full eigencomputation

Things we know

- Eigenvalues in $[-1, 1]$; nonsymmetric in general
- Stability: change d edges, have

$$\lambda_{j-d} \leq \hat{\lambda}_j \leq \lambda_{j+d}$$

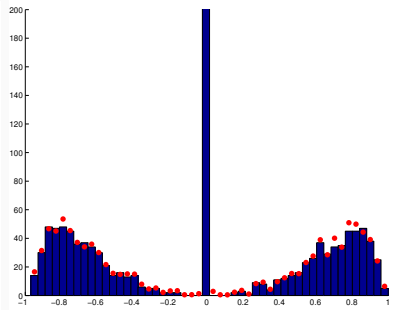
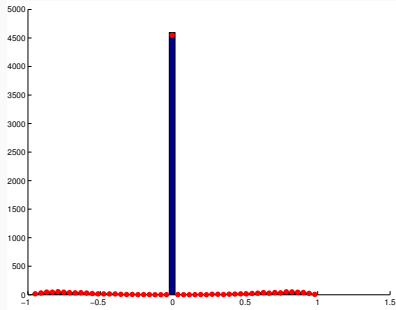
- k th moment = P (return after k -step random walk)
- Eigenvalue cluster near 1 \sim well-separated clusters
- Eigenvalue cluster near 0 \sim leaf clusters

What else can we “hear”?

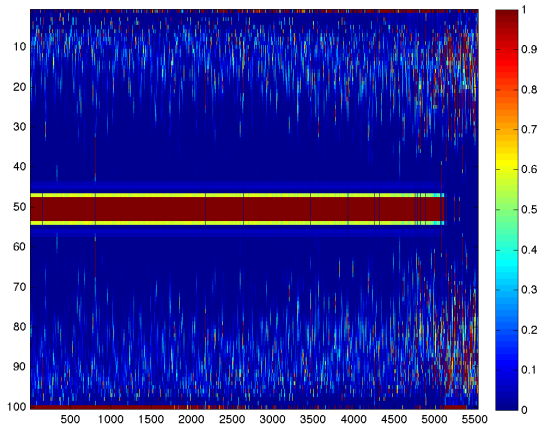
Experimental setup

- Global DoS
 - 1000 Chebyshev moments
 - 10 probe vectors (componentwise standard normal)
 - Histogram with 50 bins
- Local DoS
 - 100 Chebyshev moments
 - 10 probe vectors (componentwise standard normal)
 - Plot smoothed density on $[-1, 1]$
 - Spectrally order nodes by density plot

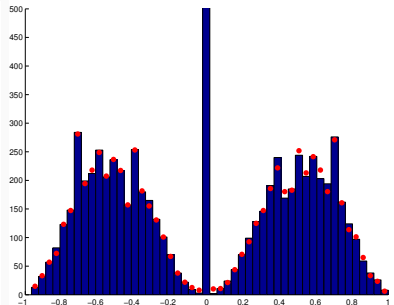
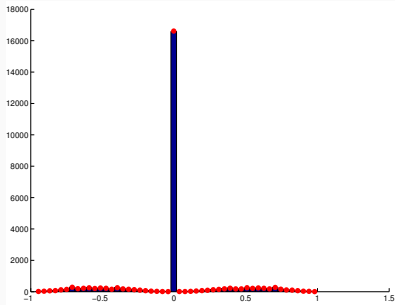
Suggestions for better pics are welcome!



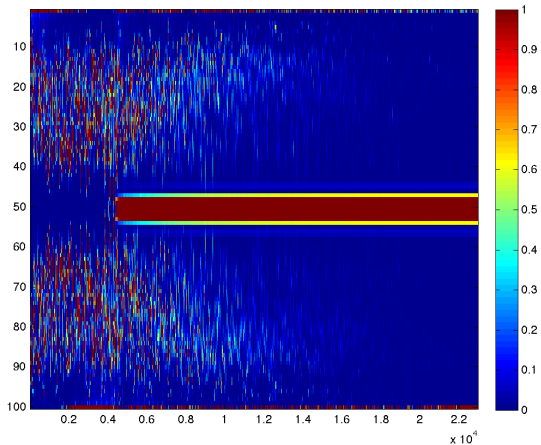
Erdos (local)



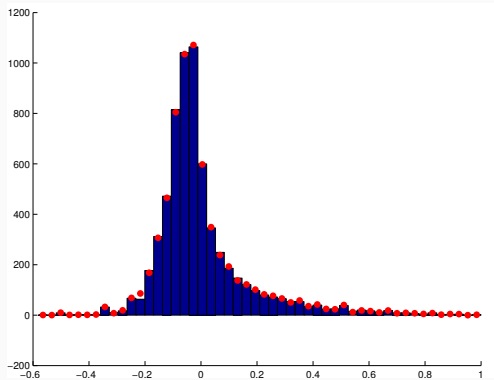
Internet topology



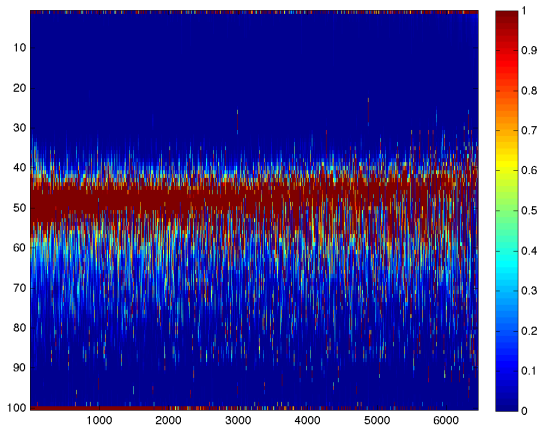
Internet topology (local)



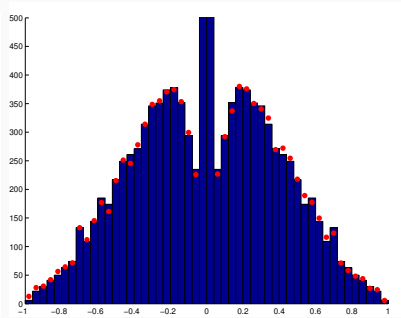
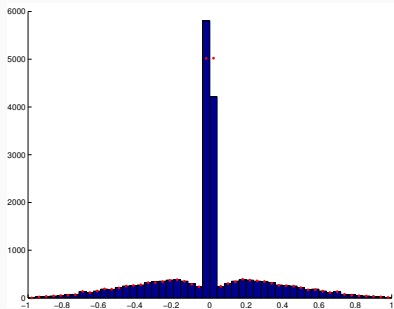
Marvel characters



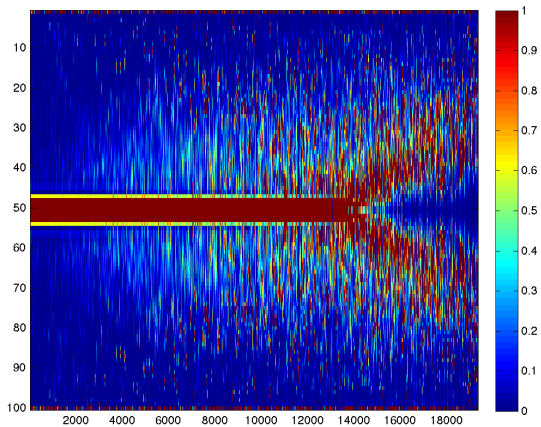
Marvel characters (local)

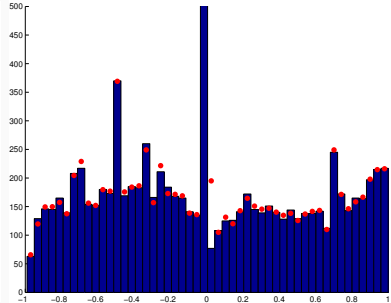
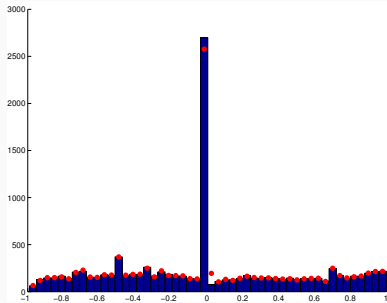


Marvel comics

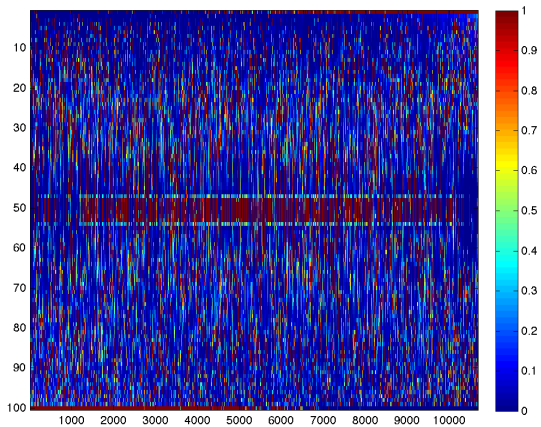


Marvel comics (local)

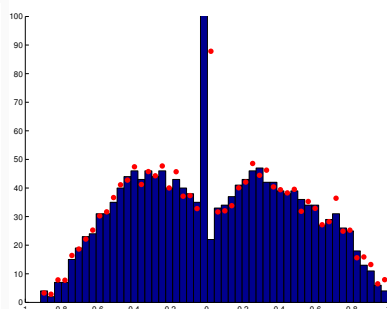
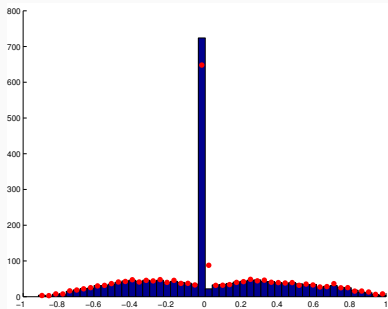




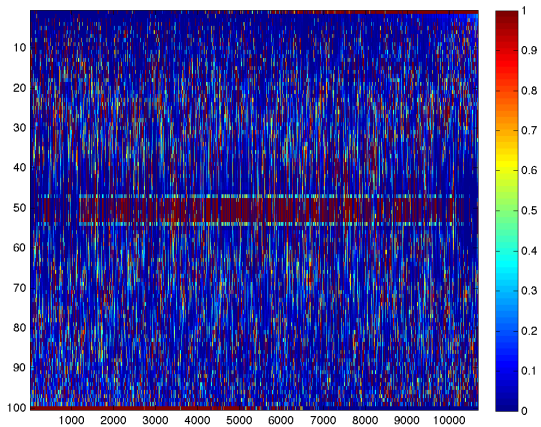
PGP (local)



Yeast

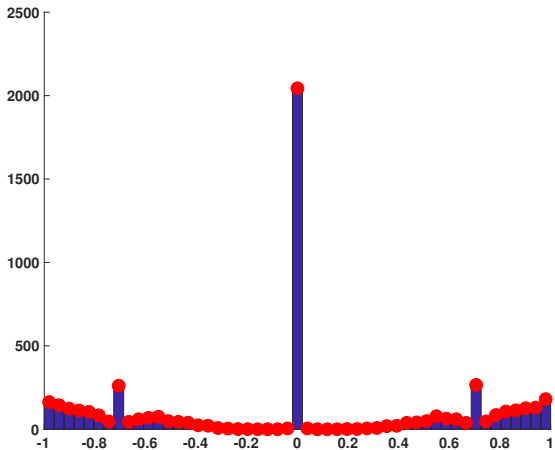


Yeast (local)



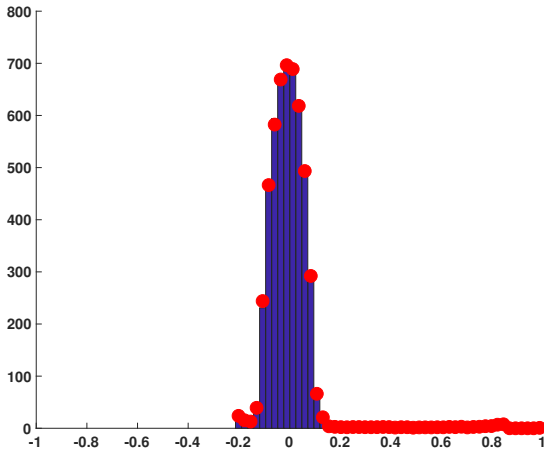
What about random graph models?

Barabási–Albert model



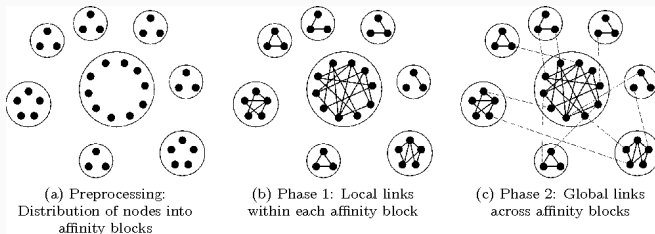
Scale-free network (5000 nodes, 4999 edges)

Watts–Strogatz



Small world network (5000 nodes, 260000 edges)

Model Verification: BTRE



Kolda et al, SISC (36), 2014

Block Two-Level Erdős-Rényi model (BTER)

- First Phase: Erdős-Rényi Blocks
- Second Phase: Using Chung-Lu Model to connect blocks with $p_{ij} = p(d_i, d_j)$

Model Verification: BTER

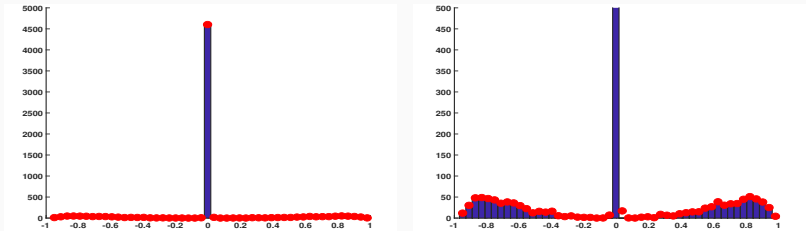


Figure 1: Erdos collaboration network.

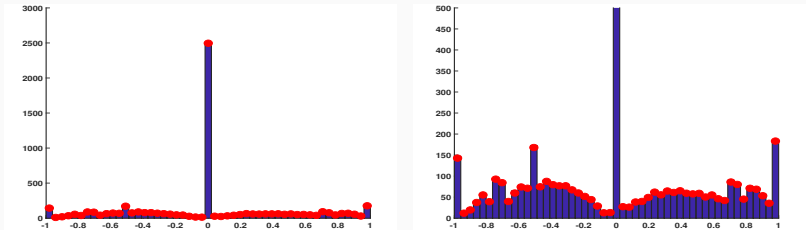
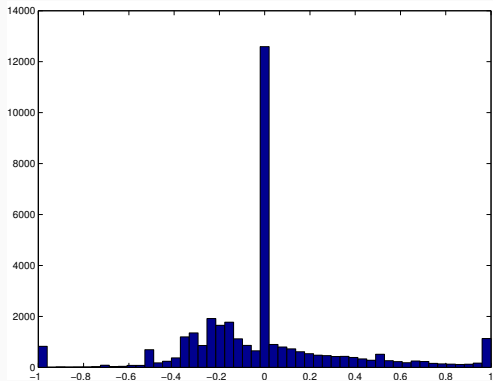


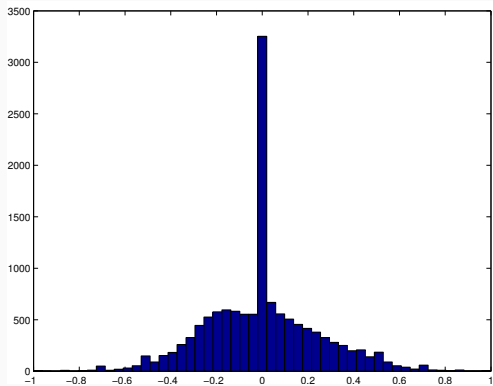
Figure 2: BTER model for Erdos collaboration network.

And a few more...

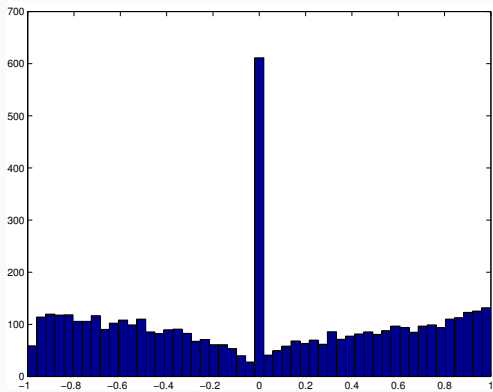
Enron emails (SNAP)

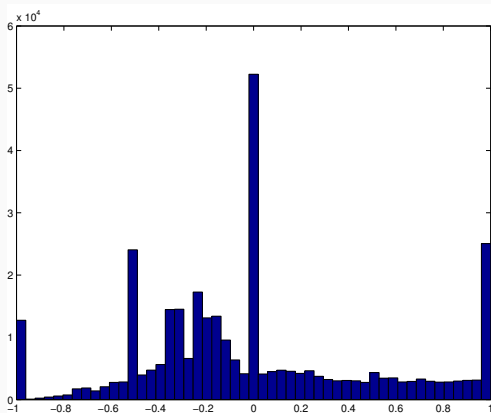


Reuters911 (Pajek)



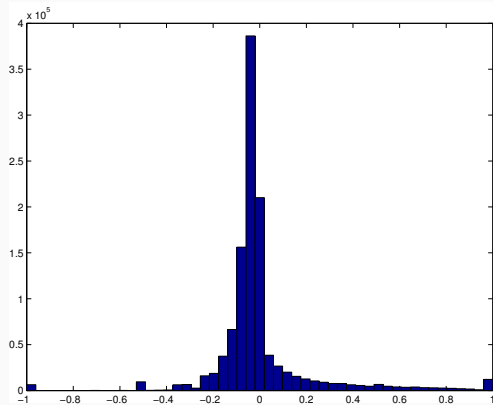
US power grid (Pajek)





$N = 326186$, $nnz = 1615400$, 80 s (1000 moments, 10 probes)

Hollywood 2009 (LAW)



$N = 1139905$, $nnz = 113891327$, 2093 s (1000 moments, 10 probes)

Questions for You?

Here until July 7, would love to talk!

- Any isospectral graphs for multiple matrices?
- Can we recover topology from (exact) LDoS?
- Variance reduction in diagonal estimators?
- Random graphs with spectra that look “real”?
- Compression of moment information for diag estimators?
- More applications?

What Do You Hear?

