Dynamics via Nonlinear Pseudospectra

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The NEP Picture

$$T(\lambda)v = 0, \quad v \neq 0.$$

where

- $T : \Omega \to \mathbb{C}^{n \times n}$ analytic, $\Omega \subset \mathbb{C}$ simply connected
- Regularity: $\det(T) \not\equiv 0$

Nonlinear spectrum: $\Lambda(T) = \{z \in \Omega : T(z)$ singular$\}$.

What do we want?

- Qualitative information (e.g. no eigenvalues in right half plane)
- Error bounds on computed/estimated eigenvalues
- Assurances that we know all the eigenvalues in some region

Why? Because of dynamics connections!

Why Eigenvalues?

$$y' - Ay = 0 \xrightarrow{y(t)=e^{\lambda t}v} (\lambda I - A)v = 0$$

$$y_{k+1} - Ay_k = 0 \xrightarrow{y_k=\lambda^k v} (\lambda I - A)v = 0$$

One standard use: analyze dynamics of LTI systems

- Special solutions characterizing full system
- General solutions as linear combinations of special solutions
- *Asymptotic* stability analysis and decay rates

Why Nonlinear Eigenvalues?

We want special solutions and asymptotic decay rates for

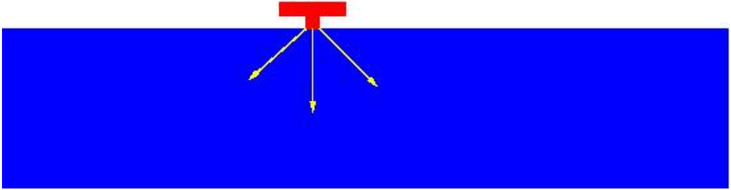
$$y'' + By' + Ky = 0 \xrightarrow{y=e^{\lambda t}v} (\lambda^2 I + \lambda B + K)v = 0$$

$$y' - Ay - By(t-1) = 0 \xrightarrow{y=e^{\lambda t}v} (\lambda I - A - Be^{-\lambda})v = 0$$

$$T(d/dt)y = 0 \xrightarrow{y=e^{\lambda t}v} T(\lambda)v = 0$$

- Higher-order ODEs
- Delay differential equations
- Boundary integral equation eigenproblems
- Radiation boundary conditions
- Dynamic element formulations

My Motivation



$$T(\omega)v \equiv \left(K - \omega^2 M + G(\omega)\right) v = 0$$

Hidden Variables

Many real NEPs come from a decision to "hide" some state by dealing with it semi-analytically:

- Higher-order ODEs —
  hide extra derivatives
- Delay differential equations —
  hide lagged state (e.g. in delay lines)
- Boundary integral equation eigenproblems —
  hide domain unknowns
- Radiation boundary conditions —
  hide behavior outside computational domain

Linearization

Ex: Second-order ODE and quadratic eigenvalue problem

$$y'' + Dy' + Ky = 0 \qquad \longrightarrow \qquad \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} - \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = 0$$

$$\lambda^2 y + \lambda D y + K y = 0 \qquad \longrightarrow \qquad \lambda\begin{bmatrix} y \\ \lambda y \end{bmatrix} - \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}\begin{bmatrix} y \\ \lambda y \end{bmatrix} = 0$$

Trade **nonlinearity vs size** more generally:

$$T\left(\frac{d}{dt}\right) y = 0 \qquad \longrightarrow \qquad \frac{du}{dt} - \mathcal{A}u = 0 \text{ and } y = Cu$$

$$T(\lambda) y = 0 \qquad \longrightarrow \qquad \lambda u - \mathcal{A}u = 0 \text{ and } y = Cu$$

... but u may be infinite dimensional (e.g. DDE case).

Exact Dynamics

Laplace transforms:

$$T\left(\frac{d}{dt}\right) y = f \quad \longrightarrow \quad T(z)Y(z) = F(z) + \text{I.C. terms}$$

$$y(t) = \mathcal{L}^{-1}[Y](t) = \frac{1}{2\pi i}\int_\Gamma Y(z)e^{zt}\,dz$$

or first-order connection:

$$T\left(\frac{d}{dt}\right) y = f \quad \longrightarrow \quad \frac{du}{dt} - \mathcal{A}u = Bf, \quad y = Cu$$

$$y(t) = C\exp(t\mathcal{A})u_0 + \int_0^t \left[C\exp((t-s)\mathcal{A})B\right] f(s)\,ds$$

But what do I do if I'm too lazy and ignorant to solve exactly?

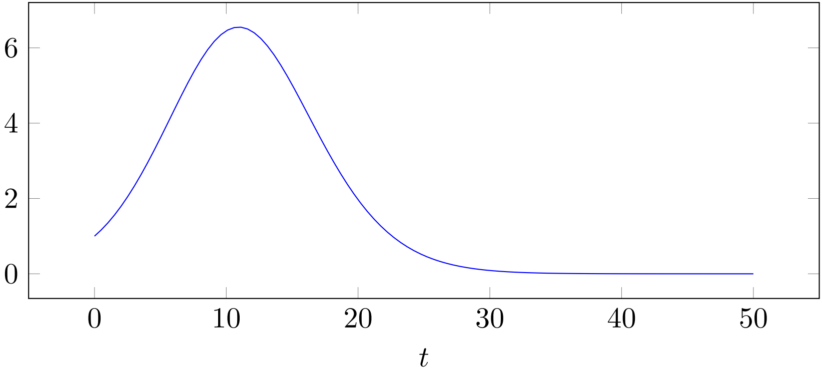
Asymptotics

First approach:

- Observe $y(t) \sim \exp(\alpha t)$ where $\alpha \equiv \max_{\lambda \in \Lambda(T)} \operatorname{Re}(\lambda)$.
- Bound α somehow.
- Go explore Barcelona.

But this approach hides too much...

Beyond (Before?) Asymptotics



But this **long run** *is a misleading guide to current affairs.*
In the long run *we are all dead.*

— *John Maynard Keynes*
A Tract on Monetary Reform *(1923)*

Asymptotic Behavior and First-Order IVPs

Consider a first-order problem:

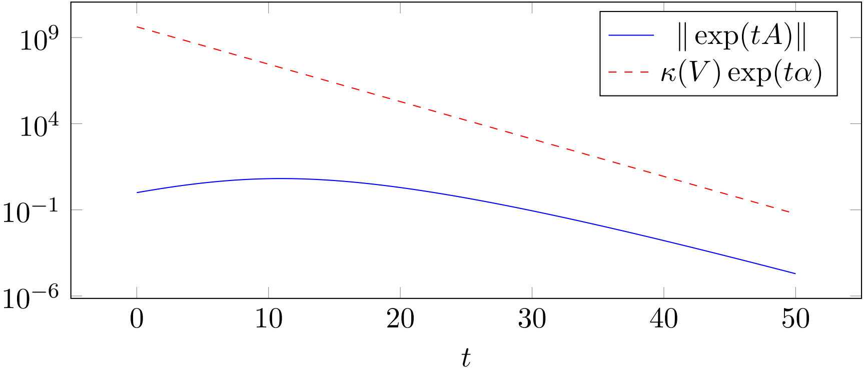
$$y' = Ay + f, \quad y(0) = y_0$$
$$y(t) = \exp(tA)y_0 + \int_0^t \exp((t-s)A)f(s)\,ds$$

Bounds if $A = V\Lambda V^{-1}$ and $\|f(t)\| \leq \gamma$:

$$\|\exp(tA)\| = \|V\exp(t\Lambda)V^{-1}\| \leq \kappa(V)\exp(t\alpha)$$
$$\|y(t)\| \leq \kappa(V)\left(\exp(t\alpha)\|y_0\| + \frac{\gamma}{-\alpha}(1-\exp(t\alpha))\right)$$

where $\alpha = \max\mathrm{Re}(\lambda)$ is the spectral abscissa.

Pre-Asymptotic Behavior for IVP *aka* the Hump



Simple bounds if $A = V\Lambda V^{-1}$

$$\|\exp(tA)\| = \|V\exp(t\Lambda)V^{-1}\| \leq \kappa(V)\exp(t\alpha)$$

where $\alpha = \max \mathrm{Re}(\lambda)$. Nothing says V need be well-conditioned!

The Complex Connection

General solutions to LTI problems via Laplace transforms

$$(zI - A)^{-1} = \mathcal{L}\left[e^{tA}\right] = \int_0^\infty e^{-zt} e^{tA}\, dt$$
$$\exp(tA) = \mathcal{L}^{-1}\left[(zI - A)^{-1}\right] = \frac{1}{2\pi i}\int_\Gamma (zI - A)^{-1} e^{zt}\, dz$$

for large enough $\mathrm{Re}(z)$ and for appropriate Γ, e.g.:

- Γ a closed contour surrounding spectrum.
- Γ a vertical line to the right of the spectrum.

A Basic Tool

Begin from the contour integral representation:

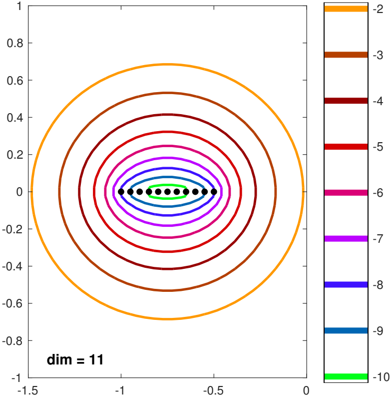
$$\exp(tA) = \frac{1}{2\pi i} \int_\Gamma (zI - A)^{-1} e^{zt}\, dz$$

Convert bounds on resolvent to bounds on $\exp(tA)$

$$\|\exp(tA)\| \le \frac{1}{2\pi} \int_\Gamma \|(zI - A)^{-1}\|\, |e^{zt}|\, d\Gamma.$$

We need “only” summarize how $\|(zI - A)^{-1}\|$ behaves.

Pseudospectra



Summarize $\|(zI - A)^{-1}\|$ with

$$\Lambda_\epsilon(A) \equiv \{z \in \mathbb{C} : \|(zI - A)^{-1}\| > \epsilon^{-1}\}$$
$$= \bigcup_{\|E\|<\epsilon} \Lambda(A + E)$$

Pseudospectral abscissa is

$$\alpha_\epsilon(A) \equiv \max_{z \in \Lambda_\epsilon(A)} \mathrm{Re}(z)$$

[Trefethen and Embree, 2005]

Pseudospectral Bounds

Set $\Gamma = \partial\Lambda_\epsilon(A)$ and L_ϵ the length of Γ. Then:

$$\|\exp(tA)\| \;\leq\; \frac{1}{2\pi}\int_\Gamma \|(zI-A)^{-1}\|\,|e^{zt}|\,d\Gamma \;\leq\; \frac{L_\epsilon}{2\pi\epsilon}\exp(t\alpha_\epsilon).$$

NB: If eigenvectors (columns of V) are normalized,

$$\kappa(V) \;\leq\; \lim_{\epsilon\to 0}\frac{L_\epsilon}{2\pi\epsilon} = \sum_j \|V^{-1}e_j\| \;\leq\; \sqrt{n}\kappa(V)$$

Can also get a lower bound: for any $\omega \in \mathbb{R}$ and $\epsilon > 0$,

$$\sup_{t\geq 0}\|\exp(-\omega t)\exp(tA)\| \geq \frac{\alpha_\epsilon - \omega}{\epsilon}.$$

Beyond First-Order Systems

Approach: Exploit same Laplace transform pairing as before

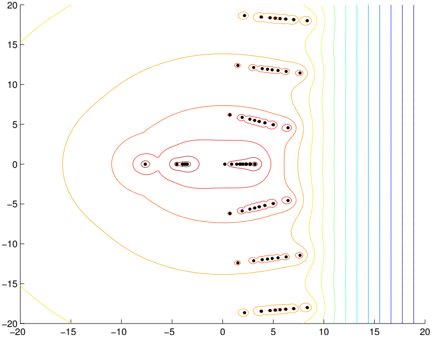
$$\exp(tA) \xrightarrow{\mathcal{L}} (zI - A)^{-1}$$
$$\Psi(t) \xrightarrow{\mathcal{L}} T(z)^{-1}$$

Here $\Psi(t) = C\exp(t\mathcal{A})B$ and $T(z)^{-1} = C(zI - \mathcal{A})^{-1}B$.

As before, to control behavior of $\Psi(t)$:

- Asymptotic stability / decay: look at spectral abscissa
- Pre-asymptotic behavior: consider "resolvent" norm $\|T(z)^{-1}\|$

Nonlinear Pseudospectra



Summarize $\|T(z)^{-1}\|$ with

$$\Lambda_\epsilon(T) \equiv \{z \in \mathbb{C} : \|T(z)^{-1}\| > \epsilon^{-1}\}$$
$$= \bigcup_{\|E\|<\epsilon} \Lambda(T+E)$$

Pseudospectral abscissa is

$$\alpha_\epsilon(T) \equiv \max_{z \in \Lambda_\epsilon(T)} \mathrm{Re}(z)$$

[Bindel and Hood, 2015]

Aside: Comparing Pseudospectra

Suppose $T, \hat{T} : \Omega \to \mathbb{C}^{n \times n}$ and

$$\|T(z) - \hat{T}(z)\| \leq \eta, \quad \forall z \in \Omega.$$

Then

$$\Lambda_\epsilon(T) \subset \Lambda_{\epsilon+\eta}(\hat{T}).$$

So we can approximate T by a polynomial $\hat{T}$ locally and bound pseudospectra (for example)... but we usually won't get all of $\mathbb{C}$.

Can also work with easier-to-compute sets (e.g. Gershgorin regions).

Pseudospectral Bounds

Set $\Gamma = \partial\Lambda_\epsilon(A)$ and L_ϵ the length of Γ. Then:

$$\|\Psi(t)\| \;\leq\; \frac{1}{2\pi}\int_\Gamma \|T(z)^{-1}\|\,|e^{zt}|\,d\Gamma \;\leq\; \frac{L_\epsilon}{2\pi\epsilon}\exp(t\alpha_\epsilon).$$

But this may be useless (e.g. $L_\epsilon = \infty$) — need to be careful!

Can also get a lower bound: for any $\omega \in \mathbb{R}$ and $\epsilon > 0$,

$$\sup_{t\geq 0} \|\exp(-\omega t)\Psi(t)\| \geq \frac{\alpha_\epsilon - \omega}{\epsilon}.$$

Example: Delay Differential Equation

DDE is

$$u'(t) = Au(t) + Bu(t - \tau)$$

Characteristic function:

$$T(z) = zI - A - Be^{-\tau z}$$

Assume A symmetric, $\alpha(A) < 0$, and $\alpha(T) < 0$.

Problem: Infinitely many eigenvalues! Have to be more clever.

Sketch of Approach

- Seek a simpler reference problem ($\hat{u}' = A\hat{u}$).
- Split into reference + difference term.
- Choose a congenial contour right of both spectra.
- Bound contour integral involving difference term.

Reference Comparison

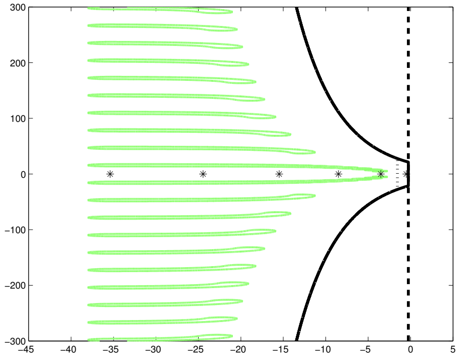
Define $R(z) = (zI - A)^{-1}$; for proper choices of Γ,

$$\Psi(t) = \exp(tA) + \frac{1}{2\pi i} \int_\Gamma [T(z)^{-1} - R(z)] e^{zt} \, dz$$

Could choose difference reference (e.g. characteristic fn for PEP).

Now we need a contour on which we can control $\|T(z)^{-1} - R(z)\|$.

Choice of Contour



Choose Γ right of $\Lambda(T)$ and $\Lambda(A)$ but in LHP:

$$\Gamma = \Gamma_\infty \cup \Gamma_0 \qquad \Gamma_\infty = \{x(y) + iy : |y| > y_0\}$$

$$x(y) = -\frac{1}{\tau}\log\left(|y|\eta\right) \qquad \Gamma_0 = \{x_0 + iy : |y| \leq y_0, x_0 = x(y_0)\}.$$

Control on Contour

Let $E(z) = T(z)^{-1} - R(z)$, contour as before:

$$\int_{\Gamma_0} \|E(z)\| \, |e^{zt}| \, d\Gamma \leq 2\exp(x_0 t) \int_0^{y_0} \|E(x_0 + iy)\| \, dy$$

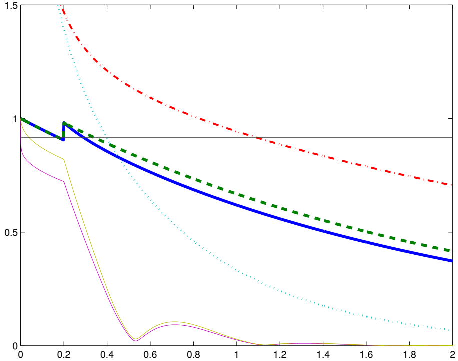
$$\int_{\Gamma_\infty} \|E(z)\| \, |e^{zt}| \, d\Gamma \leq \exp(x_0 t) \frac{C\tau}{t}$$

using boundedness of $\|E(z)\|$ on Γ together with curvature into RHP.

Bound:

$$\|\Psi(t)\| \leq \|\exp(tA)\| + e^{x_0 t} \left(I_0 + \frac{C\tau}{t} \right)$$

Choices



- Could choose vertical contour (lose $1/t$ factor in second term)
- Drop R (bigger constants, but faster decay)
- Probably many more options!

Summary

For both first-order systems and more complex problems:

- Eigenvalues describe asymptotic dynamics
- Pre-asymptotic behavior requires more information:
  - Complete eigendecomposition: Nice if you can get it.
  - Conditioning of V: A blunt tool for blunt bounds.
  - Pseudospectra and company: A sharper tool for complex bounds.
- Pseudospectra alone don't suffice — choices of contours, comparison problems, *etc* make a difference.

References

- Trefethen and Embree, *Spectra and Pseudospectra*, 2005.
- Bindel and Hood, "Localization Theorems for Nonlinear Eigenvalues," SIREV, Dec. 2015.
- Hood and Bindel, "Pseudospectral Bounds on Transient Growth for Higher Order and Delay Differential Equations," `http://arxiv.org/abs/1611.05130`.