Dynamics via Nonlinear Pseudospectra

David Bindel

Department of Computer Science
Cornell University

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The NEP Picture

\[ T(\lambda)v = 0, \quad v \neq 0. \]

where

- \( T : \Omega \rightarrow \mathbb{C}^{n \times n} \) analytic, \( \Omega \subset \mathbb{C} \) simply connected
- Regularity: \( \det(T) \neq 0 \)

Nonlinear spectrum: \( \Lambda(T) = \{ z \in \Omega : T(z) \text{ singular} \} \).

What do we want?

- Qualitative information (e.g. no eigenvalues in right half plane)
- Error bounds on computed/estimated eigenvalues
- Assurances that we know all the eigenvalues in some region

Why? Because of dynamics connections!
Why Eigenvalues?

\[ y' - Ay = 0 \quad \Rightarrow \quad y(t) = e^{\lambda t} v \rightarrow (\lambda I - A)v = 0 \]

\[ y_{k+1} - Ay_k = 0 \quad \Rightarrow \quad y_k = \lambda^k v \rightarrow (\lambda I - A)v = 0 \]

One standard use: analyze dynamics of LTI systems

- Special solutions characterizing full system
- General solutions as linear combinations of special solutions
- *Asymptotic* stability analysis and decay rates
Why Nonlinear Eigenvalues?

We want special solutions and asymptotic decay rates for

\[ y'' + B y' + K y = 0 \quad \overset{y = e^{\lambda t} v}{\rightarrow} \quad (\lambda^2 I + \lambda B + K) v = 0 \]

\[ y' - A y - B y(t - 1) = 0 \quad \overset{y = e^{\lambda t} v}{\rightarrow} \quad (\lambda I - A - B e^{-\lambda}) v = 0 \]

\[ T \left( \frac{d}{dt} \right) y = 0 \quad \overset{y = e^{\lambda t} v}{\rightarrow} \quad T(\lambda) v = 0 \]

- Higher-order ODEs
- Delay differential equations
- Boundary integral equation eigenproblems
- Radiation boundary conditions
- Dynamic element formulations
My Motivation

\[ T(\omega)v \equiv (K - \omega^2 M + G(\omega))v = 0 \]
Hidden Variables

Many real NEPs come from a decision to “hide” some state by dealing with it semi-analytically:

- Higher-order ODEs — hide extra derivatives
- Delay differential equations — hide lagged state (e.g. in delay lines)
- Boundary integral equation eigenproblems — hide domain unknowns
- Radiation boundary conditions — hide behavior outside computational domain
Linearization

Ex: Second-order ODE and quadratic eigenvalue problem

\[ y'' + Dy' + Ky = 0 \quad \rightarrow \quad \frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} - \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = 0 \]

\[ \lambda^2 y + \lambda Dy + Ky = 0 \quad \rightarrow \quad \lambda \begin{bmatrix} y \\ \lambda y \end{bmatrix} - \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix} \begin{bmatrix} y \\ \lambda y \end{bmatrix} = 0 \]

Trade nonlinearity vs size more generally:

\[ T \left( \frac{d}{dt} \right) y = 0 \quad \rightarrow \quad \frac{du}{dt} - Au = 0 \text{ and } y = Cu \]

\[ T(\lambda)y = 0 \quad \rightarrow \quad \lambda u - Au = 0 \text{ and } y = Cu \]

... but \( u \) may be infinite dimensional (e.g. DDE case).
Exact Dynamics

Laplace transforms:

\[ T \left( \frac{d}{dt} \right) y = f \quad \longrightarrow \quad T(z)Y(z) = F(z) + \text{I.C. terms} \]

\[ y(t) = \mathcal{L}^{-1}[Y](t) = \frac{1}{2\pi i} \int_{\Gamma} Y(z)e^{zt} \, dz \]

or first-order connection:

\[ T \left( \frac{d}{dt} \right) y = f \quad \longrightarrow \quad \frac{du}{dt} - Au = Bf, \quad y = Cu \]

\[ y(t) = C \exp(tA)u_0 + \int_0^t [C \exp((t - s)A)B] f(s) \, ds \]

But what do I do if I’m too lazy and ignorant to solve exactly?
Asymptotics

First approach:

- Observe $y(t) \sim \exp(\alpha t)$ where $\alpha \equiv \max_{\lambda \in \Lambda(T)} \Re(\lambda)$.
- Bound $\alpha$ somehow.
- Go explore Barcelona.

But this approach hides too much...
But this long run is a misleading guide to current affairs. In the long run we are all dead.

— John Maynard Keynes
A Tract on Monetary Reform (1923)
Asymptotic Behavior and First-Order IVPs

Consider a first-order problem:

\[ y' = Ay + f, \quad y(0) = y_0 \]

\[ y(t) = \exp(tA)y_0 + \int_0^t \exp((t - s)A)f(s)\, ds \]

Bounds if \( A = V\Lambda V^{-1} \) and \( \| f(t) \| \leq \gamma \):

\[ \| \exp(tA) \| = \| V \exp(t\Lambda)V^{-1} \| \leq \kappa(V)\exp(t\alpha) \]

\[ \| y(t) \| \leq \kappa(V) \left( \exp(t\alpha)\| y_0 \| + \frac{\gamma}{-\alpha} (1 - \exp(t\alpha)) \right) \]

where \( \alpha = \max \Re(\lambda) \) is the spectral abscissa.
Pre-Asymptotic Behavior for IVP \textit{aka} the Hump

Simple bounds if $A = V \Lambda V^{-1}$

$$\| \exp(tA) \| = \| V \exp(t \Lambda) V^{-1} \| \leq \kappa(V) \exp(t \alpha)$$

where $\alpha = \max \text{Re}(\lambda)$. Nothing says $V$ need be well-conditioned!
The Complex Connection

General solutions to LTI problems via Laplace transforms

\[(zI - A)^{-1} = \mathcal{L} [e^{tA}] = \int_0^\infty e^{-zt} e^{tA} \, dt\]

\[\exp(tA) = \mathcal{L}^{-1} [(zI - A)^{-1}] = \frac{1}{2\pi i} \int_\Gamma (zI - A)^{-1} e^{zt} \, dz\]

for large enough \(\text{Re}(z)\) and for appropriate \(\Gamma\), e.g.:

- \(\Gamma\) a closed contour surrounding spectrum.
- \(\Gamma\) a vertical line to the right of the spectrum.
Begin from the contour integral representation:

$$\exp(tA) = \frac{1}{2\pi i} \int_{\Gamma} (zI - A)^{-1} e^{zt} \, dz$$

Convert bounds on resolvent to bounds on $\exp(tA)$

$$\| \exp(tA) \| \leq \frac{1}{2\pi} \int_{\Gamma} \| (zI - A)^{-1} \| |e^{zt}| \, d\Gamma.$$ 

We need “only” summarize how $\| (zI - A)^{-1} \|$ behaves.
Summarize \( \|(zI - A)^{-1}\| \) with

\[
\Lambda_\epsilon(A) \equiv \{ z \in \mathbb{C} : \|(zI - A)^{-1}\| > \epsilon^{-1} \} = \bigcup \Lambda(A + E)
\]

where \( \|E\| < \epsilon \)

Pseudospectral abscissa is

\[
\alpha_\epsilon(A) \equiv \max_{z \in \Lambda_\epsilon(A)} \text{Re}(z)
\]

[Trefethen and Embree, 2005]
Pseudospectral Bounds

Set $\Gamma = \partial \Lambda_{\epsilon}(A)$ and $L_\epsilon$ the length of $\Gamma$. Then:

$$\| \exp(tA) \| \leq \frac{1}{2\pi} \int_{\Gamma} \| (zI - A)^{-1} \| |e^{zt}| \, d\Gamma \leq \frac{L_\epsilon}{2\pi\epsilon} \exp(t\alpha_\epsilon).$$

NB: If eigenvectors (columns of $V$) are normalized,

$$\kappa(V) \leq \lim_{\epsilon \to 0} \frac{L_\epsilon}{2\pi\epsilon} = \sum_j \| V^{-1} e_j \| \leq \sqrt{n} \kappa(V).$$

Can also get a lower bound: for any $\omega \in \mathbb{R}$ and $\epsilon > 0$,

$$\sup_{t \geq 0} \| \exp(-\omega t) \exp(tA) \| \geq \frac{\alpha_\epsilon - \omega}{\epsilon}.$$
Beyond First-Order Systems

Approach: Exploit same Laplace transform pairing as before

\[
\exp(tA) \xrightarrow{\mathcal{L}} (zI - A)^{-1} \\
\Psi(t) \xrightarrow{\mathcal{L}} T(z)^{-1}
\]

Here \( \Psi(t) = C \exp(tA)B \) and \( T(z)^{-1} = C(zI - A)^{-1}B \).

As before, to control behavior of \( \Psi(t) \):
- Asymptotic stability / decay: look at spectral abscissa
- Pre-asymptotic behavior: consider “resolvent” norm \( \|T(z)^{-1}\| \)
Summarize $\|T(z)^{-1}\|$ with

$$\Lambda_\epsilon(T) \equiv \{z \in \mathbb{C} : \|T(z)^{-1}\| > \epsilon^{-1}\}$$

$$= \bigcup_{\|E\| < \epsilon} \Lambda(T + E)$$

Pseudospectral abscissa is

$$\alpha_\epsilon(T) \equiv \max_{z \in \Lambda_\epsilon(T)} \text{Re}(z)$$

[Bindel and Hood, 2015]
Aside: Comparing Pseudospectra

Suppose $T, \hat{T} : \Omega \rightarrow \mathbb{C}^{n \times n}$ and

$$\|T(z) - \hat{T}(z)\| \leq \eta, \quad \forall z \in \Omega.$$ 

Then

$$\Lambda_\epsilon(T) \subset \Lambda_{\epsilon+\eta}(\hat{T}).$$

So we can approximate $T$ by a polynomial $\hat{T}$ locally and bound pseudospectra (for example)... but we usually won’t get all of $\mathbb{C}$. Can also work with easier-to-compute sets (e.g. Gershgorin regions).
Pseudospectral Bounds

Set \( \Gamma = \partial \Lambda_\epsilon(A) \) and \( L_\epsilon \) the length of \( \Gamma \). Then:

\[
\| \Psi(t) \| \leq \frac{1}{2\pi} \int_\Gamma \| T(z)^{-1} \| |e^{zt}| \, d\Gamma \leq \frac{L_\epsilon}{2\pi \epsilon} \exp(t\alpha_\epsilon).
\]

But this may be useless (e.g. \( L_\epsilon = \infty \)) — need to be careful!

Can also get a lower bound: for any \( \omega \in \mathbb{R} \) and \( \epsilon > 0 \),

\[
\sup_{t \geq 0} \| \exp(-\omega t) \Psi(t) \| \geq \frac{\alpha_\epsilon - \omega}{\epsilon}.
\]
Example: Delay Differential Equation

DDE is

\[ u'(t) = Au(t) + Bu(t - \tau) \]

Characteristic function:

\[ T(z) = zI - A - Be^{-\tau z} \]

Assume \( A \) symmetric, \( \alpha(A) < 0 \), and \( \alpha(T) < 0 \).

**Problem:** Infinitely many eigenvalues! Have to be more clever.
Seek a simpler reference problem ($\hat{u}' = A\hat{u}$).

Split into reference + difference term.

Choose a congenial contour right of both spectra.

Bound contour integral involving difference term.
Define $R(z) = (zI - A)^{-1}$; for proper choices of $\Gamma$,

$$\Psi(t) = \exp(tA) + \frac{1}{2\pi i} \int_{\Gamma} [T(z)^{-1} - R(z)] e^{zt} \, dz$$

Could choose difference reference (e.g. characteristic fn for PEP).

Now we need a contour on which we can control $\|T(z)^{-1} - R(z)\|$. 
Choose $\Gamma$ right of $\Lambda(T)$ and $\Lambda(A)$ but in LHP:

$$
\Gamma = \Gamma_\infty \cup \Gamma_0 \\
\Gamma_\infty = \{ x(y) + iy : |y| > y_0 \} \\
x(y) = -\frac{1}{\tau} \log(|y|\eta) \\
\Gamma_0 = \{ x_0 + iy : |y| \leq y_0, x_0 = x(y_0) \}.
$$
Let $E(z) = T(z)^{-1} - R(z)$, contour as before:

$$
\int_{\Gamma_0} \| E(z) \| |e^{zt}| \, d\Gamma \leq 2 \exp(x_0 t) \int_0^{y_0} \| E(x_0 + iy) \| \, dy
$$

$$
\int_{\Gamma_\infty} \| E(z) \| |e^{zt}| \, d\Gamma \leq \exp(x_0 t) \frac{C_T}{t}
$$

using boundedness of $\| E(z) \|$ on $\Gamma$ together with curvature into RHP.

Bound:

$$
\| \Psi(t) \| \leq \| \exp(tA) \| + e^{x_0 t} \left( I_0 + \frac{C_T}{t} \right)
$$
- Could choose vertical contour (lose $1/t$ factor in second term)
- Drop $R$ (bigger constants, but faster decay)
- Probably many more options!
Summary

For both first-order systems and more complex problems:

- Eigenvalues describe asymptotic dynamics
- Pre-asymptotic behavior requires more information:
  - Complete eigendecomposition: Nice if you can get it.
  - Conditioning of $V$: A blunt tool for blunt bounds.
  - Pseudospectra and company: A sharper tool for complex bounds.

- Pseudospectra alone don’t suffice — choices of contours, comparison problems, etc make a difference.
References