Nonlinear Eigenvalue Problems: Theory and Applications

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Joint work with Amanda Hood
Why eigenvalues?
Why eigenvalues?

- The polynomial connection
- The optimization connection
- The approximation connection
- The Fourier/quadrature/special function connection
- The \textbf{dynamics} connection
Vibrations are everywhere, and so too are the eigenvalues associated with them. As mathematical models invade more and more disciplines, we can anticipate a demand for eigenvalue calculations in an ever richer variety of contexts.

– Beresford Parlett, The Symmetric Eigenvalue Problem
Why eigenvalues?

As a student in a first ODE course:

\[ y' - Ay = 0 \quad \Rightarrow \quad y = e^{\lambda t}v \quad \Rightarrow \quad (\lambda I - A)v = 0. \]

Me: “How do I compute this?”

\[ p(\lambda) = \det(\lambda I - A) = 0. \]

And then I learned better...
Eigenvalues or polynomial roots?

The cozy relationship that mathematicians enjoyed with polynomials suffered a severe setback in the early fifties when electronic computers came into general use. Speaking for myself I regard it as the most traumatic experience in my career as a numerical analyst.

— J. H. Wilkinson

Representation matters.
Why nonlinear eigenvalues?

\[ y' - Ay = 0 \quad \xrightarrow{y = e^{\lambda t} v} \quad (\lambda I - A)v = 0 \]
\[ y'' + By' + Ky = 0 \quad \xrightarrow{y = e^{\lambda t} v} \quad (\lambda^2 I + \lambda B + K)v = 0 \]
\[ y' - Ay - By(t - 1) = 0 \quad \xrightarrow{y = e^{\lambda t} v} \quad (\lambda I - A - B e^{-\lambda})v = 0 \]
\[ T\left(\frac{d}{dt}\right)y = 0 \quad \xrightarrow{y = e^{\lambda t} v} \quad T(\lambda)v = 0 \]

- Higher-order ODEs
- Dynamic element formulations
- Delay differential equations
- Boundary integral equation eigenproblems
- Radiation boundary conditions
Big and little

\[ y'' + By' + Ky = 0 \quad \overset{v = y'}{\longrightarrow} \quad \begin{bmatrix} v \\ y \end{bmatrix}' + \begin{bmatrix} B & K \\ -I & 0 \end{bmatrix} \begin{bmatrix} v \\ y \end{bmatrix} = 0 \]

\[ (\lambda^2 I + \lambda B + K)u = 0 \quad \overset{v = \lambda u}{\longrightarrow} \quad \left( \lambda I + \begin{bmatrix} B & K \\ -I & 0 \end{bmatrix} \right) \begin{bmatrix} v \\ u \end{bmatrix} = 0 \]

**Tradeoff:** more variables = more linear.
My motivation

\[ T(\omega)v \equiv \left( K - \omega^2 M + G(\omega) \right) v = 0 \]
My motivation

\[ T(\omega)v \equiv (K - \omega^2 M + G(\omega)) v = 0 \]

**Wanted:** Perturbation theory justifying a terrible estimate of \( G(\omega) \)
The big picture

\[ T(\lambda)v = 0, \quad v \neq 0. \]

where

- \( T : \Omega \to \mathbb{C}^{n \times n} \) analytic, \( \Omega \subset \mathbb{C} \) simply connected
- Regularity: \( \det(T) \neq 0 \)

Nonlinear spectrum: \( \Lambda(T) = \{ z \in \Omega : T(z) \text{ singular} \} \).

What do we want?

- Qualitative information (e.g. no eigenvalues in right half plane)
- Error bounds on computed/estimated eigenvalues
- Assurances that we know all the eigenvalues in some region
Solver strategies

\[ T(\lambda)v = 0, \quad v \neq 0. \]

A common approach:

1. **Approximate** \( T \) locally (linear, rational, etc)
2. Solve **approximate** problem.
3. Repeat as needed.

How should we choose solver parameters? What about global behavior? What can we trust? What might we miss?
$T(\lambda)v = 0, \quad v \neq 0$. 

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- $T : \Omega \rightarrow \mathbb{C}^{n \times n}$ analytic, $\Omega \subset \mathbb{C}$ simply connected
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Nonlinear spectrum: $\Lambda(T) = \{z \in \Omega : T(z) \text{ singular}\}$.

**Goal:** Use analyticity to *compare* and to *count*
Winding and Cauchy’s argument principle

\[ W_\Gamma(f) = \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} \, dz \]

\[ = \# \text{ zeros} - \# \text{ poles} \]
Winding, Rouché, and Gohberg-Sigal

Analytic: \( f, g : \Omega \rightarrow \mathbb{C} \)

Winding #:
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} \, dz
\]

Theorem:
- Rouché (1862):
  \(|g| < |f|\) on \(\Gamma\) \implies same # zeros of \(f, f + g\)
- Gohberg-Sigal (1971):
  \(|\|T^{-1}E\| < 1\) on \(\Gamma\) \implies same # eigs of \(T, T + E\)
Comparing NEPs

Suppose

\[ T, E : \Omega \rightarrow \mathbb{C}^{n \times n} \text{ analytic} \]
\[ \Gamma \subset \Omega \text{ a simple closed contour} \]
\[ T(z) + sE(z) \text{ nonsingular } \forall s \in [0, 1], \, z \in \Gamma \]

Then \( T \) and \( T + E \) have the same number of eigenvalues inside \( \Gamma \).

**Pf:** Constant winding number around \( \Gamma \).
Nonlinear pseudospectra

$$\Lambda_\epsilon(T) \equiv \{ z \in \Omega : \| T(z)^{-1} \| > \epsilon^{-1} \}$$
$E$ analytic, $\|E(z)\| < \epsilon$ on $\Omega_\epsilon$. Then

$$\Lambda(T + E) \cap \Omega_\epsilon \subset \Lambda_\epsilon(T) \cap \Omega_\epsilon$$

Also, if $\mathcal{U}_\epsilon$ a component of $\Lambda_\epsilon$ and $\bar{\mathcal{U}}_\epsilon \subset \Omega_\epsilon$, then

$$|\Lambda(T + E) \cap \mathcal{U}_\epsilon| = |\Lambda(T) \cap \mathcal{U}_\epsilon|$$
Pseudospectral comparison

- Most useful when $T$ is linear
- Even then, can be expensive to compute!
- What about related tools?
The Gershgorin picture (linear case)

\[ A = D + F, \quad D = \text{diag}(d_i), \quad \rho_i = \sum_j |f_{ij}| \]
Gershgorin $(+\varepsilon)$

Write $A = D + F$, $D = \text{diag}(d_1, \ldots, d_n)$. Gershgorin disks are:

$$G_i = \left\{ z \in \mathbb{C} : |z - d_i| \leq \sum_j |f_{ij}| \right\}.$$ 

Useful facts:

- Spectrum of $A$ lies in $\bigcup_{i=1}^{m} G_i$
- $\bigcup_{i \in \mathcal{I}} G_i$ disjoint from other disks $\Rightarrow$ contains $|\mathcal{I}|$ eigenvalues.

**Pf:**

$A - zI$ strictly diagonally dominant outside $\bigcup_{i=1}^{m} G_i$.

Eigenvalues of $D - sF$, $0 \leq s \leq 1$, are continuous.
Nonlinear Gershgorin

Write $T(z) = D(z) + F(z)$. Gershgorin regions are

$$G_i = \left\{ z \in \mathbb{C} : |d_i(z)| \leq \sum_j |f_{ij}(z)| \right\}.$$

Useful facts:

- Spectrum of $T$ lies in $\bigcup_{i=1}^m G_i$
- Bdd connected component of $\bigcup_{i=1}^m G_i$ strictly in $\Omega$$$\implies$$ same number of eigs of $D$ and $T$ in component
$$\implies$$ at least one eig per component of $G_i$ involved

**Pf:** Strict diag dominance test + continuity of eigs
Example I: Haderler

\[ T(z) = (e^z - 1)B + z^2A - \alpha I, \quad A, B \in \mathbb{R}^{8 \times 8} \]
Comparison to simplified problem

Bauer-Fike idea: apply a similarity!

\[ T(z) = (e^z - 1)B + z^2 A - \alpha I \]

\[ \tilde{T}(z) = U^T T(z) U \]
\[ = (e^z - 1)D_B + z^2 I - \alpha E \]
\[ = D(z) - \alpha E \]

\[ G_i = \{ z : |\beta_i(e^z - 1) + z^2| < \rho_i \} . \]
Gershgorin regions
A different comparison

Approximate $e^z - 1$ by a Chebyshev interpolant:

$$T(z) = (e^z - 1)B + z^2A - \alpha I$$
$$\tilde{T}(z) = q(z)B + z^2A - \alpha$$

$$T(z) = \tilde{T}(z) + r(z)B$$

Linearize $\tilde{T}$ and transform both:

$$\tilde{T}(z) \mapsto DC - zI$$
$$T(z) \mapsto DC - zI + r(z)E$$

Restrict to $\Omega_\epsilon = \{z : |r(z)| < \epsilon\}$:

$$G_i \subset \hat{G}_i = \{z : |z - \mu_i| < \rho_i\epsilon\}, \quad \rho_i = \sum_j |e_{ij}|$$
Spectrum of $\tilde{T}$
\[ \hat{G}_i \text{ for } \epsilon < 10^{-10} \]
\( \hat{G}_i \) for \( \epsilon = 0.1 \)
$\hat{G}_i$ for $\epsilon = 1.6$
Example II: Resonance problem

\[ \psi(0) = 0 \]

\[ \left( -\frac{d^2}{dx^2} + V - \lambda \right) \psi = 0 \quad \text{on } (0, b), \]

\[ \psi'(b) = i\sqrt{\lambda} \psi(b), \]
Reduction via shooting

\[ \psi(0) = 0, \]

\[ R_{0a}(\lambda) \begin{bmatrix} \psi(0) \\ \psi'(0) \end{bmatrix} = \begin{bmatrix} \psi(a) \\ \psi'(a) \end{bmatrix}, \]

\[ R_{ab}(\lambda) \begin{bmatrix} \psi(b) \\ \psi'(b) \end{bmatrix} = \begin{bmatrix} \psi(b) \\ \psi'(b) \end{bmatrix}, \]

\[ \psi'(b) = i\sqrt{\lambda}\psi(b) \]
Reduction via shooting

First-order form:

\[
\frac{du}{dx} = \begin{bmatrix} 0 & 1 \\ V - \lambda & 0 \end{bmatrix} u, \text{ where } u(x) \equiv \begin{bmatrix} \psi(x) \\ \psi'(x) \end{bmatrix}.
\]

On region \((c, d)\) where \(V\) is constant:

\[
u(d) = R_{cd}(\lambda)u(c), \quad R_{cd}(\lambda) = \exp\left((d - c) \begin{bmatrix} 0 & 1 \\ V - \lambda & 0 \end{bmatrix}\right)\]

Reduce resonance problem to 6D NEP:

\[
T(\lambda)u_{\text{all}} \equiv \begin{bmatrix} R_{0a}(\lambda) & -I & 0 \\ 0 & R_{ab}(\lambda) & -I \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(a) \\ u(b) \end{bmatrix} = 0.
\]
Consider the equation
\[
\left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda I \right) \begin{bmatrix} u \\ v \end{bmatrix} = 0
\]

Partial Gaussian elimination gives the \textbf{spectral Schur complement}
\[
(A - \lambda I - B(D - \lambda I)^{-1}C) u = 0
\]

\textbf{Idea}: Given
\[
T(\lambda) = A - \lambda I + F(\lambda),
\]
find a rational approximation \( F(\lambda) \approx B(D - \lambda I)^{-1}C \).
Expansion via rational approximation

\[ u(0) \times u(a) \times u(b) \times \cdots = 0 \]
Analyzing the expanded system

- $\hat{T}(z)$ is a Schur complement in $K - zM$
  - So $\Lambda(\hat{T})$ is easy to compute.
- Or: think $T(z)$ is a Schur complement in $K - zM + E(z)$
- Compare $\hat{T}(z)$ to $T(z)$ or compare $K - zM + E(z)$ to $K - zM$
Analyzing the expanded system

**Q:** Can we find all eigs in a region *not missing anything*?

Concrete plan ($\epsilon = 10^{-8}$)

- $T =$ shooting system
- $\hat{T} =$ rational approximation
- Find region $D$ with boundary $\Gamma$ s.t.
  - $D \subset \Omega_\epsilon$ (i.e. $\|T - \hat{T}\| < \epsilon$ on $D$)
  - $\Gamma$ does not intersect $\Lambda_\epsilon(T)$

$\implies$ Same eigenvalue counts for $T$, $\hat{T}$

$\implies$ Eigs of $\hat{T}$ in components of $\Lambda_\epsilon(T)$
  - Converse holds if $D \subset \Omega_{\epsilon/2}$

Can refine eigs of $\hat{T}$ in $D$ via Newton.
Resonance approximation
Resonance approximation
For more

*Localization theorems for nonlinear eigenvalues.*
David Bindel and Amanda Hood, SIAM Review 57(4), Dec 2015